

Exercise sheet #14

Problem 1. A longitudinal $\mathbf{E}$ field inside a wire causes a current $\mathbf{J} = \sigma\mathbf{E}$. Since the curl of $\mathbf{E}$ is zero, this same longitudinal $\mathbf{E}$ component must also exist right outside the surface of the wire. Show that the Poynting vector flux through a cylinder right outside the wire is equal to IV , where I the current and V is the electric potential (resistance heating).

Solution: The electric field inside the wire is given by $E = J/\sigma$, where σ is the electrical conductivity of the wire. Since the curl of $\mathbf{E}$ is zero, we can draw a thin rectangular loop along the surface to show that the electric field right outside the wire is also $E = J/\sigma$ (and it points in the direction of the current, of course). The magnetic field right outside the wire points tangentially with the usual magnitude of $B = \mu_0 I / 2\pi R$, where R is the radius of the wire. $\mathbf{E}$ and $\mathbf{B}$ are perpendicular, and you can show with the right-hand rule that the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{B} / \mu_0$ points radially into the wire. So the direction is correct; the energy in the wire increases, consistent with the fact that it heats up. The magnitude of $\mathbf{S}$ equals

$$S = \frac{1}{\mu_0} EB = \frac{1}{\mu_0} \frac{J}{\sigma} \frac{\mu_0 I}{2\pi R} = \frac{JI}{2\pi R\sigma}.$$

To obtain the power flux into the wire through the surface, we must multiply by $2\pi R\ell$, where ℓ is the length of a given section of the wire. So the total energy flow per time into a length ℓ of the wire is

$$P_\ell = S \cdot 2\pi R\ell = \frac{JI}{2\pi R\sigma} 2\pi R\ell = \frac{JI}{\sigma} \ell = \frac{(I/A)I}{\sigma} \ell = I^2 \frac{\ell}{\sigma A} = I^2 \frac{\rho \ell}{A} = I^2 R,$$

where R is the resistance of the length ℓ of the wire. We have used the fact that the resistivity ρ is given by $\rho = 1/\sigma$. As desired, P_ℓ equals the rate of resistance heating in the length ℓ of the wire. P_ℓ can also be written as $I(IR) = IV$, of course, where V is the voltage drop along the length ℓ of the wire. Alternatively, we never actually had to use the J/σ form of E . A quicker method is:

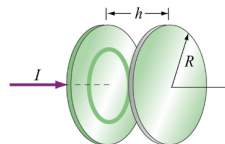
$$P_\ell = S \cdot 2\pi R\ell = \frac{1}{\mu_0} E \frac{\mu_0 I}{2\pi R} \cdot 2\pi R\ell = IE\ell = IV$$

□

Problem 2. The intensity of sunlight, at the earth, is roughly 1 kilowatt /m². How large is the magnetic field strength? Assume that the EM radiation from the sun is a plane sinusoidal wave.

Solution: We derived in class that for a plane sinusoidal wave: $\langle S \rangle = \frac{E_0 B_0}{2\mu_0} = \frac{E_0^2}{2c\mu_0} = \frac{cB_0^2}{2\mu_0}$ where $\langle S \rangle$ is the intensity (in this case $\langle S \rangle = 1$ kilowatt /m²), $c = 3 \times 10^8$ is the speed of light and $\mu_0 = 1.257 \times 10^{-6} \text{ kg} \cdot \text{m} / \text{C}^2$. Substituting these values in the equation for the intensity and solving for B_0 we get $B_0 = 0.9154 \times 10^{-6} \text{ T}$ □

Problem 3. A parallel-plate capacitor with circular plates of radius R and separated by a distance h is charged through a straight wire carrying current I , as shown in the figure below:



- Show that as the capacitor is being charged, the Poynting vector $\vec{S}$ points radially inward toward the center of the capacitor.
- By integrating $\vec{S}$ over the cylindrical boundary, show that the rate at which energy enters the capacitor is equal to the rate at which electrostatic energy is being stored in the electric field.

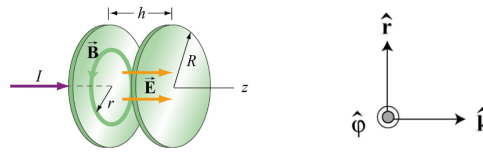
Solution: (a) Let the axis of the circular plates be the z -axis, with current flowing in the $+z$ direction. Suppose at some instant the amount of charge accumulated on the positive plate is $+Q$. The electric field is

$$\vec{E} = \frac{\sigma}{\epsilon_0} \hat{k} = \frac{Q}{\pi R^2 \epsilon_0} \hat{k}$$

According to the Ampere-Maxwell's equation, a magnetic field is induced by changing electric flux:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{enc}} + \mu_0 \epsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot d\vec{A}$$

From the cylindrical symmetry of the system, we see that the magnetic field will be circular, centered on the z -axis, i.e., $\vec{B} = B\hat{\phi}$ (see Figure below).



Consider a circular path of radius $r < R$ between the plates. Using the above formula, we obtain

$$B(2\pi r) = 0 + \mu_0 \epsilon_0 \frac{d}{dt} \left(\frac{Q}{\pi R^2 \epsilon_0} \pi r^2 \right) = \frac{\mu_0 r^2}{R^2} \frac{dQ}{dt}$$

or

$$\vec{B} = \frac{\mu_0 r}{2\pi R^2} \frac{dQ}{dt} \hat{\phi}$$

The Poynting $\vec{S}$ vector can then be written as

$$\begin{aligned} \vec{S} &= \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{1}{\mu_0} \left(\frac{Q}{\pi R^2 \epsilon_0} \hat{k} \right) \times \left(\frac{\mu_0 r}{2\pi R^2} \frac{dQ}{dt} \hat{\phi} \right) \\ &= - \left(\frac{Qr}{2\pi^2 R^4 \epsilon_0} \right) \left(\frac{dQ}{dt} \right) \hat{r} \end{aligned}$$

Note that for $dQ/dt > 0$ $\vec{S}$ points in the $-\hat{r}$ direction, or radially inward toward the center of the capacitor.

- (b) The energy per unit volume carried by the electric field is $u_E = \epsilon_0 E^2/2$. The total energy stored in the electric field then becomes

$$U_E = u_E V = \frac{\epsilon_0}{2} E^2 (\pi R^2 h) = \frac{1}{2} \epsilon_0 \left(\frac{Q}{\pi R^2 \epsilon_0} \right)^2 \pi R^2 h = \frac{Q^2 h}{2\pi R^2 \epsilon_0}$$

Differentiating the above expression with respect to t , we obtain the rate at which this energy is being stored:

$$\frac{dU_E}{dt} = \frac{d}{dt} \left(\frac{Q^2 h}{2\pi R^2 \epsilon_0} \right) = \frac{Qh}{\pi R^2 \epsilon_0} \left(\frac{dQ}{dt} \right)$$

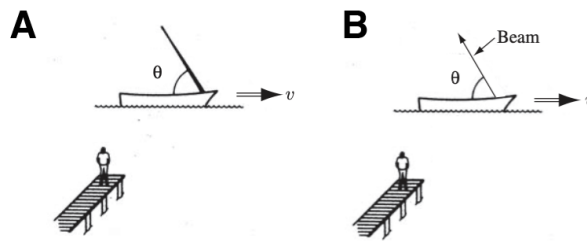
On the other hand, the rate at which energy flows into the capacitor through the cylinder at $r = R$ can be obtained by integrating $\vec{S}$ over the surface area:

$$\oint \vec{S} \cdot d\vec{A} = SA_R = \left(\frac{Qr}{2\pi^2\epsilon_0 R^4} \frac{dQ}{dt} \right) (2\pi Rh) = \frac{Qh}{\epsilon_0 \pi R^2} \left(\frac{dQ}{dt} \right)$$

which is equal to the rate at which energy stored in the electric field is changing. $\square$

Problem 4. (a) A sailboat is manufactured so that the mast leans at an angle $\bar{\theta}$ with respect to the deck. An observer standing on a dock sees the boat go by at speed v (Fig. below A). What angle does this observer say the mast makes?

(b) A spotlight is mounted on a boat so that its beam makes an angle $\bar{\theta}$ with the deck (Fig. below B). If this boat is then set in motion at speed v , what angle θ does an individual photon trajectory make with the deck, according to an observer on the dock? What angle does the beam (illuminated, say, by a light fog) make? Compare with part A.



Solution: (a) Say length of mast (at rest) is $\bar{l}$. To an observer on the boat, height of mast is $\bar{l} \sin \bar{\theta}$, horizontal projection is $\bar{l} \cos \bar{\theta}$. To observer on dock, the former is unaffected, but the latter is Lorentz contracted to $\frac{1}{\gamma} \bar{l} \cos \bar{\theta}$. Therefore:

$$\tan \theta = \frac{\bar{l} \sin \bar{\theta}}{\frac{1}{\gamma} \bar{l} \cos \bar{\theta}} = \gamma \tan \bar{\theta}, \quad \text{or} \quad \tan \theta = \frac{\tan \bar{\theta}}{\sqrt{1 - v^2/c^2}}$$

(b) $\mathcal{S}$ = dock frame; $\mathcal{S}'$ = boat frame; we need reverse transformations ($v \rightarrow -v$): $\tan \theta = -\frac{u_y}{u_x} = -\frac{\bar{u}_y/\gamma(1+\frac{v\bar{u}_x}{c^2})}{(\bar{u}_x+v)/(1+\frac{v\bar{u}_x}{c^2})} = -\frac{1}{\gamma} \frac{\bar{u}_y}{(\bar{u}_x+v)}$. In this case $\bar{u}_x = -c \cos \bar{\theta}$; $\bar{u}_y = c \sin \bar{\theta}$, so $\tan \theta = -\frac{1}{\gamma} \frac{c \sin \bar{\theta}}{(-c \cos \bar{\theta} + v)} = \frac{1}{\gamma} \left(\frac{\sin \bar{\theta}}{\cos \bar{\theta} - v/c} \right)$ [Contrast $\tan \theta = \gamma \frac{\sin \bar{\theta}}{\cos \bar{\theta}}$ in problem (a). The point is that velocities are sensitive not only to the transformation of distances, but also of times. That's why there is no universal rule for translating angles - you have to know whether it's an angle made by a velocity vector or a position vector.

That's how the velocity vector of an individual photon transforms. But the beam as a whole is a snapshot of many different photons at one instant of time, and it transforms the same way the mast does. $\square$

Problem 5. A capacitor consists of two parallel rectangular plates with a vertical separation of 2 cm . The east-west dimension of the plates is 20 cm , the north-south dimension is 10 cm . The capacitor has been charged by connecting it temporarily to a battery of 300 V . What is the electric field strength between the plates? How many excess electrons are on the negative plate? Now give the following quantities as they would be measured in a frame of reference that is moving eastward, relative to the

laboratory in which the plates are at rest, with speed $0.6c$: the three dimensions of the capacitor; the number of excess electrons on the negative plate; the electric field strength between the plates. Answer the same questions for a frame of reference that is moving upward with speed $0.6c$.

Solution: The electric field in the lab frame is $E_0 = V_0/d = (300 \text{ V})/(.02 \text{ m}) = 15,000 \text{ V/m}$. The charge density is $\sigma_0 = \epsilon_0 E_0$, so the charge on the plates is $Q_0 = \sigma_0 A_0 = \epsilon_0 E_0 A_0$. The number of excess electrons on the negative plate is therefore $N = Q_0/e = \epsilon_0 E_0 A_0/e$, which yields

$$N = \frac{\left(8.85 \cdot 10^{-12} \frac{\text{s}^2 \text{C}^2}{\text{kg m}^3}\right) (15,000 \text{ V/m}) (0.02 \text{ m}^2)}{1.6 \cdot 10^{-19} \text{ C}} = 1.66 \cdot 10^{10}.$$

In the frame F_1 moving east at $v = 0.6c$, the plates are moving west at $0.6c$. The γ factor associated with $0.6c$ is $5/4$. So the EW dimension is shrunk to $(20 \text{ cm})/\gamma = 16 \text{ cm}$. The NS dimension is unchanged, as is the vertical separation. The number of electrons on the negative plate is the same. But the area of the plates is smaller by $1/\gamma$, so σ_1 (and hence E_1) is larger by a factor γ . Therefore, $E_1 = \gamma E_0 = 18,750 \text{ V/m}$. In the frame F_2 moving upward at $v = 0.6c$, the plates are moving downward at $0.6c$. The plate dimensions are still 20 cm and 10 cm , but the vertical separation is shrunk to $(2 \text{ cm})/\gamma = 1.6 \text{ cm}$. The number of electrons is the same. And the density is the same also, so $E_2 = E_0 = 15,000 \text{ V/m}$.

□