

Exercise sheet #13

Problem 1. Verify that, for $\omega = kc$,

$$\begin{aligned} E(x, t) &= E_0 \cos(kx - \omega t) \\ B(x, t) &= B_0 \cos(kx - \omega t) \end{aligned}$$

satisfy the one-dimensional wave equation:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \begin{Bmatrix} E(x, t) \\ B(x, t) \end{Bmatrix} = 0$$

Solution: Differentiating $E = E_0 \cos(kx - \omega t)$ with respect to x gives

$$\frac{\partial E}{\partial x} = -kE_0 \sin(kx - \omega t), \quad \frac{\partial^2 E}{\partial x^2} = -k^2 E_0 \cos(kx - \omega t)$$

Similarly, differentiating E with respect to t yields

$$\frac{\partial E}{\partial t} = \omega E_0 \sin(kx - \omega t), \quad \frac{\partial^2 E}{\partial t^2} = -\omega^2 E_0 \cos(kx - \omega t)$$

Thus,

$$\frac{\partial^2 E}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = \left(-k^2 + \frac{\omega^2}{c^2} \right) E_0 \cos(kx - \omega t) = 0$$

where we have made use of the relation $\omega = kc$. One may follow a similar procedure to verify the magnetic field. □

Problem 2. If the electric field in free space is $\mathbf{E} = E_0(\hat{\mathbf{x}} + \hat{\mathbf{y}}) \sin[(2\pi/\lambda)(z + ct)]$ with $E_0 = 20$ volts/m, then the magnetic field, not including any static magnetic field, must be what?

Solution: The wave is traveling in the $-\hat{\mathbf{z}}$ direction, as shown by the sign in $(z + ct)$; if t increases, then z must decrease to keep the same value of $(z + ct)$. $\mathbf{B}$ is perpendicular to both this direction and to $\mathbf{E}$. So $\mathbf{B}$ must point in the $\pm(\hat{\mathbf{x}} - \hat{\mathbf{y}})$ direction. But since we know that $\mathbf{E} \times \mathbf{B}$ points in the direction of the wave's velocity, which is $-\hat{\mathbf{z}}$, we must pick the "+" sign, as you can quickly verify with the right-hand rule. The magnitude of $\mathbf{B}$ is $1/c$ times the magnitude of $\mathbf{E}$, so the desired $\mathbf{B}$ field is

$$\mathbf{B} = (E_0/c)(\hat{\mathbf{x}} - \hat{\mathbf{y}}) \sin[(2\pi/\lambda)(z + ct)]$$

With $E_0 = 20$ V/m, we have $B_0 = E_0/c = (20 \text{ V/m}) (3 \cdot 10^8 \text{ m/s}) = 6.67 \cdot 10^{-8} \text{ T}$. The amplitudes of the $\mathbf{E}$ and $\mathbf{B}$ waves are actually $\sqrt{2}$ times E_0 and B_0/c , respectively, because the magnitude of the $(\hat{\mathbf{x}} \pm \hat{\mathbf{y}})$ vectors is $\sqrt{2}$. □

Problem 3. Write out formulas for $\mathbf{E}$ and $\mathbf{B}$ that specify a plane electromagnetic sinusoidal wave with the following characteristics. The wave is traveling in the direction $-\hat{\mathbf{x}}$; its frequency is 100 megahertz (MHz), or 10^8 cycles per second; the electric field is perpendicular to the $\hat{\mathbf{z}}$ direction.

Solution: It is given that $\mathbf{E} \perp \hat{\mathbf{z}}$. And we know that $\mathbf{E} \perp \mathbf{v}$, where $\mathbf{v} \propto -\hat{\mathbf{x}}$ here. So $\mathbf{E}$ must point in the $\pm\hat{\mathbf{y}}$ direction. Let's pick $+\hat{\mathbf{y}}$. The other direction would simply change the sign of E_0 ; the sign is arbitrary, since the trig function switches signs anyway. So we have (a sine would work just as well)

$$\mathbf{E} = \hat{\mathbf{y}} E_0 \cos(kx + \omega t),$$

where $\omega = 2\pi\nu = 6.28 \cdot 10^8 \text{ s}^{-1}$ and $k = \omega/c = 2.09 \text{ m}^{-1}$. The sign inside the cosine is a " + " because the wave is traveling in the negative x direction. Since $\mathbf{E} \times \mathbf{B}$ points in the direction of $\mathbf{v}$, which is $-\hat{\mathbf{x}}$, and since $B_0 = E_0/c$, the $\mathbf{B}$ field must take the form,

$$\mathbf{B} = -\hat{\mathbf{z}} (E_0/c) \cos(kx + \omega t).$$

□

Problem 4. Consider the two oppositely traveling electric-field waves,

$$\mathbf{E}_1 = \hat{\mathbf{x}}E_0 \cos(kz - \omega t) \quad \text{and} \quad \mathbf{E}_2 = \hat{\mathbf{x}}E_0 \cos(kz + \omega t).$$

The sum of these two waves is the standing wave, $2\hat{\mathbf{x}}E_0 \cos kz \cos \omega t$.

- Find the magnetic field associated with this standing electric wave by finding the $\mathbf{B}$ fields associated with each of the above traveling $\mathbf{E}$ fields, and then adding them.
- Find the magnetic field by instead using Maxwell's equations to find the $\mathbf{B}$ field associated with the standing electric wave, $2\hat{\mathbf{x}}E_0 \cos kz \cos \omega t$.

Solution: (a) The traveling $\mathbf{B}$ fields must point in the $\pm\hat{\mathbf{y}}$ directions because they must be perpendicular to both the associated $\mathbf{E}$ field and the direction of propagation, which is $\pm\hat{\mathbf{z}}$. The magnitudes of the $\mathbf{B}$ fields are E_0/c . The signs are determined by the fact that $\mathbf{E} \times \mathbf{B}$ points in the direction of propagation. The two magnetic waves are therefore

$$\begin{aligned} \mathbf{B}_1 &= \hat{\mathbf{y}} (E_0/c) \cos(kz - \omega t) \\ \mathbf{B}_2 &= -\hat{\mathbf{y}} (E_0/c) \cos(kz + \omega t) \end{aligned}$$

The sum of these waves is $\mathbf{B} = \hat{\mathbf{y}} (2E_0/c) \sin kz \sin \omega t$.

- We use the Maxwell equation $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$ to find $\mathbf{B}$. The curl of $\mathbf{E} = 2\hat{\mathbf{x}}E_0 \cos kz \cos \omega t$ is

$$\begin{aligned} \nabla \times \mathbf{E} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2E_0 \cos kz \cos \omega t & 0 & 0 \end{vmatrix} \\ &= -\hat{\mathbf{y}} 2kE_0 \sin kz \cos \omega t. \end{aligned}$$

Setting this equal to $-\partial\mathbf{B}/\partial t$ gives $\mathbf{B} = \hat{\mathbf{y}} (2kE_0/\omega) \sin kz \sin \omega t$. But we know that $\omega/k = c$, because the $(kz - \omega t)$ factor in the waves can be written as $k(z - (\omega/k)t)$, and the coefficient of t is the speed of the wave. So the $2kE_0/\omega$ factor in $\mathbf{B}$ equals $2E_0/c$, in agreement with the result in part (a). We have ignored the constant of integration in $\mathbf{B}$ because we are concerned only with the varying part of the field. But a constant $\mathbf{B}$ field can certainly be superposed. (It must be constant in time, but it can vary with position, as long as the rest of Maxwell's equations are satisfied.)

Alternatively, you can find $\mathbf{B}$ via the Maxwell equation $\nabla \times \mathbf{B} = \mu_0\epsilon_0\partial\mathbf{E}/\partial t$ (with $\mu_0\epsilon_0 = 1/c^2$). You should check that this gives the same result.

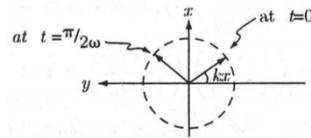
□

Problem 5. A wave propagating through a string in the z directions with the vertical $\mathbf{f}_v(z, t) = A \cos(kz - \omega t + \delta_v)\hat{\mathbf{x}}$ and horizontal components $\mathbf{f}_h(z, t) = A \cos(kz - \omega t + \delta_h)\hat{\mathbf{y}}$ is circularly polarized if the two components are of equal amplitude but out of phase by 90 degrees.

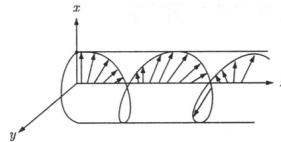
- Show that at a fixed point z , the string moves in a circle about the z axis. Does it go clockwise or counterclockwise, as you look down the axis toward the origin? How would you construct a wave circling the other way? (In optics, the clockwise case is called right circular polarization, and the counterclockwise, left circular polarization. Hint: set $\delta_v = 0, \delta_h = 90^\circ$ in the equations for $\mathbf{f}_v(z, t)$ and $\mathbf{f}_h(z, t)$.)

- (b) Sketch the string at time $t = 0$
- (c) How would you shake the string in order to produce a circularly polarized wave?

Solution: (a) Setting the phase shift between the vertical and horizontal components as suggested $\mathbf{f}_v(z, t) = A \cos(kz - \omega t) \hat{\mathbf{x}}$ and $\mathbf{f}_h(z, t) = A \cos(kz - \omega t + 90^\circ) \hat{\mathbf{y}} = -A \sin(kz - \omega t) \hat{\mathbf{y}}$. Since $f_v^2 + f_h^2 = A^2$, the vector sum $\mathbf{f} = \mathbf{f}_v + \mathbf{f}_h$ lies on a circle of radius A . To determine its direction we can look at the wave at different timepoints: At time $t = 0$, $\mathbf{f} = A \cos(kz) \hat{\mathbf{x}} - A \sin(kz) \hat{\mathbf{y}}$. At time $t = \pi/2\omega$, $\mathbf{f} = A \cos(kz - 90^\circ) \hat{\mathbf{x}} - A \sin(kz - 90^\circ) \hat{\mathbf{y}} = A \sin(kz) \hat{\mathbf{x}} + A \cos(kz) \hat{\mathbf{y}}$. Evidently it circles counterclockwise. To make a wave circling the other way, use $\delta_h = -90^\circ$.



- (b) This is what the string looks like at $t = 0$



- (c) Shake the string around in a circle instead of up and down

□