

Exercise sheet #9

Problem 1. Particle A with charge q and mass m_A and particle B with charge $2q$ and mass m_B , are accelerated from rest by a potential difference ΔV , and subsequently deflected by a uniform magnetic field into semicircular paths. The radii of the trajectories by particle A and B are R and $2R$, respectively. The direction of the magnetic field is perpendicular to the velocity of the particles. What is their mass ratio?

Solution: The kinetic energy gained by the charges is equal to

$$\frac{1}{2}mv^2 = q\Delta V$$

which yields

$$v = \sqrt{\frac{2q\Delta V}{m}}$$

The charges move in semicircles, since the magnetic force points radially inward and provides the source of the centripetal force:

$$\frac{mv^2}{r} = qvB$$

The radius of the circle can be readily obtained as:

$$r = \frac{mv}{qB} = \frac{m}{qB} \sqrt{\frac{2q\Delta V}{m}} = \frac{1}{B} \sqrt{\frac{2m\Delta V}{q}}$$

which shows that r is proportional to $(m/q)^{1/2}$. The mass ratio can then be obtained from

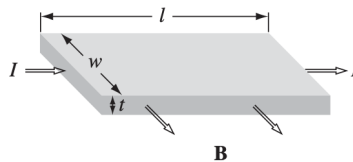
$$\frac{r_A}{r_B} = \frac{(m_A/q_A)^{1/2}}{(m_B/q_B)^{1/2}} \Rightarrow \frac{R}{2R} = \frac{(m_A/q)^{1/2}}{(m_B/2q)^{1/2}}$$

which gives:

$$\frac{m_A}{m_B} = \frac{1}{8}$$

□

Problem 2. A current I flows to the right through a rectangular bar of conducting material, in the presence of a uniform magnetic field $\mathbf{B}$ pointing out of the page (See fig. below).



- If the moving charges are positive, in which direction are they deflected by the magnetic field? This deflection results in an accumulation of charge on the upper and lower surfaces of the bar, which in turn produces an electric force to counteract the magnetic one. Equilibrium occurs when the two exactly cancel. (This phenomenon is known as the Hall effect.)
- Find the resulting potential difference (the Hall voltage) between the top and bottom of the bar, in terms of B, v (the speed of the charges), and the relevant dimensions of the bar.
- How would your analysis change if the moving charges were negative? [The Hall effect is the classic way of determining the sign of the mobile charge carriers in a material.]

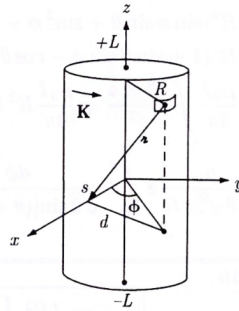
Solution: (a) If positive charges flow to the right, they are deflected down, and the bottom plate acquires a positive charge.

(b) $qvB = qE \Rightarrow E = vB \Rightarrow V = Et = vBt$, with the bottom at higher potential.

(c) If negative charges flow to the left, they are also deflected down, and the bottom plate acquires a negative charge. The potential difference is still the same, but this time the top plate is at the higher potential.

□

Problem 3. Use the Biot-Savart law to find the field inside and outside an infinitely long solenoid of radius R , with n turns per unit length, carrying a steady current I



Solution: Put the field point on the x axis, so $\mathbf{r} = (s, 0, 0)$. Then $\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{(\mathbf{K} \times \mathbf{r})}{r^3} da$; $da = R d\phi dz$; $\mathbf{K} = K \hat{\phi} = K(-\sin \phi \hat{x} + \cos \phi \hat{y})$; $\mathbf{r} = (s - R \cos \phi) \hat{x} - R \sin \phi \hat{y} - z \hat{z}$. $\mathbf{K} \times \mathbf{r} = K \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\sin \phi & \cos \phi & 0 \\ (s - R \cos \phi) & (-R \sin \phi) & (-z) \end{vmatrix} = K[(-z \cos \phi) \hat{x} + (-z \sin \phi) \hat{y} + (R - s \cos \phi) \hat{z}]$; $r^2 = z^2 + R^2 + s^2 - 2Rs \cos \phi$. The x and y components integrate to zero (z integrand is odd).

$$B_z = \frac{\mu_0}{4\pi} K R \int \frac{(R - s \cos \phi)}{(z^2 + R^2 + s^2 - 2Rs \cos \phi)^{3/2}} d\phi dz = \frac{\mu_0 K R}{4\pi} \int_0^{2\pi} (R - s \cos \phi) \left\{ \int_{-\infty}^{\infty} \frac{dz}{(z^2 + d^2)^{3/2}} \right\} d\phi,$$

$$\text{where } d^2 \equiv R^2 + s^2 - 2Rs \cos \phi. \text{ Now } \int_{-\infty}^{\infty} \frac{dz}{(z^2 + d^2)^{3/2}} = \frac{2z}{d^2 \sqrt{z^2 + d^2}} \Big|_0^{\infty} = \frac{2}{d^2}.$$

$$= \frac{\mu_0 K R}{2\pi} \int_0^{2\pi} \frac{(R - s \cos \phi)}{(R^2 + s^2 - 2Rs \cos \phi)} d\phi; (R - s \cos \phi) = \frac{1}{2R} [(R^2 - s^2) + (R^2 + s^2 - 2Rs \cos \phi)].$$

$$= \frac{\mu_0 K}{4\pi} \left[(R^2 - s^2) \int_0^{2\pi} \frac{d\phi}{(R^2 + s^2 - 2Rs \cos \phi)} + \int_0^{2\pi} d\phi \right].$$

$$\int_0^{2\pi} \frac{d\phi}{a + b \cos \phi} = 2 \int_0^{\pi} \frac{d\phi}{a + b \cos \phi} = \frac{4}{\sqrt{a^2 - b^2}} \tan^{-1} \left[\frac{\sqrt{a^2 - b^2} \tan(\phi/2)}{a + b} \right] \Big|_0^{\pi}$$

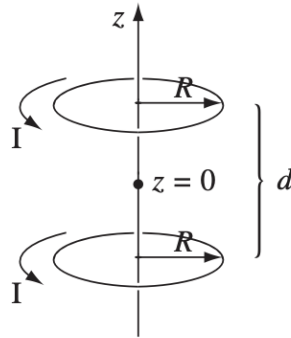
$$= \frac{4}{\sqrt{a^2 - b^2}} \tan^{-1} \left[\frac{\sqrt{a^2 - b^2} \tan(\pi/2)}{a + b} \right] = \frac{4}{\sqrt{a^2 - b^2}} \left(\frac{\pi}{2} \right) = \frac{2\pi}{\sqrt{a^2 - b^2}}. \text{ Here } a = R^2 + s^2,$$

$$b = -2Rs, \text{ so } a^2 - b^2 = R^4 + 2R^2 s^2 + s^4 - 4R^2 s^2 = R^4 - 2R^2 s^2 + s^4 = (R^2 - s^2)^2; \sqrt{a^2 - b^2} = |R^2 - s^2|.$$

$$B_z = \frac{\mu_0 K}{4\pi} \left[\frac{(R^2 - s^2)}{|R^2 - s^2|} 2\pi + 2\pi \right] = \frac{\mu_0 K}{2} \left(\frac{R^2 - s^2}{|R^2 - s^2|} + 1 \right).$$

Inside the solenoid, $s < R$, so $B_z = \frac{\mu_0 K}{2} (1 + 1) = \mu_0 K$. Outside the solenoid, $s > R$, so $B_z = \frac{\mu_0 K}{2} (-1 + 1) = 0$. Here $K = nI$, so $\mathbf{B} = \mu_0 n I \hat{z}$ (inside), and $\mathbf{0}$ (outside). □

Problem 4. The magnetic field on the axis of a circular current loop is far from uniform (it falls off sharply with increasing z). You can produce a more nearly uniform field by using two such loops a distance d apart (Fig. below).



- (a) Find the field (B) as a function of z , and show that $\partial B/\partial z$ is zero at the point midway between them ($z = 0$).
- (b) If you pick d just right, the second derivative of B will also vanish at the midpoint. This arrangement is known as a Helmholtz coil; it's a convenient way of producing relatively uniform fields in the laboratory. Determine d such that $\partial^2 B/\partial z^2 = 0$ at the midpoint, and find the resulting magnetic field at the center. [Answer: $8\mu_0 I/5\sqrt{5}R$]

Solution: (a) In the lecture we derived the field of a single loop:

$$B(z) = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z^2)^{3/2}}$$

In this case:

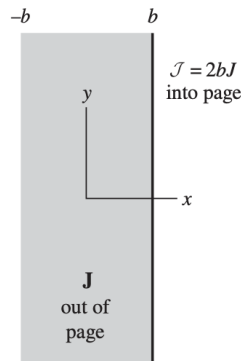
$$\begin{aligned} \mathbf{B} &= \frac{\mu_0 I R^2}{2} \left\{ \frac{1}{[R^2 + (d/2 + z)^2]^{3/2}} + \frac{1}{[R^2 + (d/2 - z)^2]^{3/2}} \right\} \\ \frac{\partial B}{\partial z} &= \frac{\mu_0 I R^2}{2} \left\{ \frac{(-3/2)2(d/2 + z)}{[R^2 + (d/2 + z)^2]^{5/2}} + \frac{(-3/2)2(d/2 - z)(-1)}{[R^2 + (d/2 - z)^2]^{5/2}} \right\} \\ &= \frac{3\mu_0 I R^2}{2} \left\{ \frac{-(d/2 + z)}{[R^2 + (d/2 + z)^2]^{5/2}} + \frac{(d/2 - z)}{[R^2 + (d/2 - z)^2]^{5/2}} \right\} \\ \left. \frac{\partial B}{\partial z} \right|_{z=0} &= \frac{3\mu_0 I R^2}{2} \left\{ \frac{-d/2}{[R^2 + (d/2)^2]^{5/2}} + \frac{d/2}{[R^2 + (d/2)^2]^{5/2}} \right\} = 0. \end{aligned}$$

(b) Differentiating again:

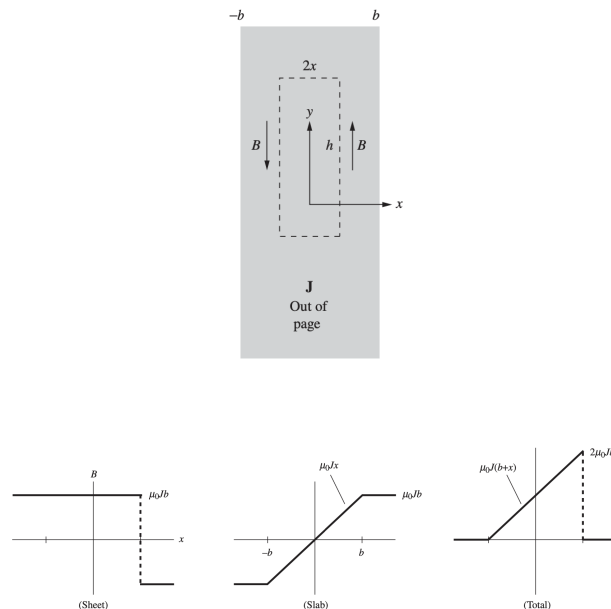
$$\begin{aligned} \frac{\partial^2 B}{\partial z^2} &= \frac{3\mu_0 I R^2}{2} \left\{ \frac{-1}{[R^2 + (d/2 + z)^2]^{5/2}} + \frac{-(d/2 + z)(-5/2)2(d/2 + z)}{[R^2 + (d/2 + z)^2]^{7/2}} \right. \\ &\quad \left. + \frac{-1}{[R^2 + (d/2 - z)^2]^{5/2}} + \frac{(d/2 - z)(-5/2)2(d/2 - z)(-1)}{[R^2 + (d/2 - z)^2]^{7/2}} \right\} \\ \left. \frac{\partial^2 B}{\partial z^2} \right|_{z=0} &= \frac{3\mu_0 I R^2}{2} \left\{ \frac{-2}{[R^2 + (d/2)^2]^{5/2}} + \frac{2(5/2)2(d/2)^2}{[R^2 + (d/2)^2]^{7/2}} \right\} = \frac{3\mu_0 I R^2}{[R^2 + (d/2)^2]^{7/2}} \left(-R^2 - \frac{d^2}{4} + \frac{5d^2}{4} \right) \\ &= \frac{3\mu_0 I R^2}{[R^2 + (d/2)^2]^{7/2}} (d^2 - R^2). \text{ Zero if } d = R, \text{ in which case} \\ B(0) &= \frac{\mu_0 I R^2}{2} \left\{ \frac{1}{[R^2 + (R/2)^2]^{3/2}} + \frac{1}{[R^2 + (R/2)^2]^{3/2}} \right\} = \mu_0 I R^2 \frac{1}{(5R^2/4)^{3/2}} = \frac{8\mu_0 I}{5^{3/2}R}. \end{aligned}$$

□

Problem 5. A volume current density $\mathbf{J} = J\hat{\mathbf{z}}$ exists in a slab between the infinite planes at $x = -b$ and $x = b$. (So the current is coming out of the page, see figure below) Additionally, a surface current density $\mathcal{J} = 2bJ$ points in the $-\hat{\mathbf{z}}$ direction on the plane at $x = b$. (a) Find the magnetic field as a function of x , both inside and outside the slab. (b) Verify that $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ inside the slab. (Don't worry about the boundaries.)



Solution: (a) The total magnetic field equals the field due to the thin sheet plus the field due to the thick slab. The field due to the thin sheet is simply $\mu_0 \mathcal{J}/2 = \mu_0(2bJ)/2 = \mu_0 Jb$. (This can be found via an Amperian loop with a side on either side of the sheet.) It points upward on the left, and downward on the right; see the step function shown below. (The direction can be found by imagining the sheet to be built up from a series of parallel wires.) To find the magnetic field due to the thick slab, consider an Amperian loop centered in the slab, as shown here:



The slab is symmetric under translations in the y direction, so the field must be independent of y . Also, the slab is symmetric under rotations by 180° around the z axis, so the y component of the field must be an odd function of x , otherwise the field wouldn't look the same after a rotation by 180° . (Additionally, you can rule out x and z components by considering the slab to be built up from wires.) The current enclosed in the Amperian loop is $I = h(2x)J$. Since only the left and right sides contribute to the line integral, we have

$$\int \mathbf{B} \cdot d\mathbf{s} = \mu_0 I \implies 2Bh = \mu_0(2xhJ) \implies B = \mu_0 Jx.$$

Outside the slab, the slab looks like a sheet (from the same Amperian argument that is used for an actual sheet). So on either side, the field has a constant value equal to the value at the boundary, namely $\pm\mu_0 Jb$. The slab's field is shown above. The total field, which is the sum of the sheet's field and the slab's field, is also shown. It equals zero outside the slab, and $\mu_0 J(b+x)\hat{\mathbf{y}}$ inside.

Alternatively, the interior field of the slab can be found by considering the two sub-slabs on either side of a given position. At position x there is a slab with thickness $b+x$ on the left which is equivalent to a sheet with surface charge density $\mathcal{J}_{\text{left}} = J(b+x)$. And likewise there is a slab with thickness $b-x$ on the right which is equivalent to a sheet with surface charge density $\mathcal{J}_{\text{right}} = J(b-x)$. The left "sheet" produces a field $\mu_0 \mathcal{J}/2 = \mu_0 J(b+x)/2$ upward, and the right "sheet" produces a field $\mu_0 \mathcal{J}/2 = \mu_0 J(b-x)/2$ downward. The net interior field of the slab is therefore $\mu_0 Jx$ upward (so if x is negative, this points downward).

(b) Inside the slab, the curl of $\mathbf{B}$ is

$$\nabla \times \mathbf{B} = \begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & \mu_0 J(b+x) & 0 \end{pmatrix} = \mu_0 J \hat{\mathbf{z}} = \mu_0 \mathbf{J},$$

as desired. Outside the slab, $\mathbf{B}$ and $\mathbf{J}$ are both zero, so $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ is trivially true. At the boundary at $x = b$, the B_y component is discontinuous, so the $\partial B_y / \partial x$ derivative in the curl is infinite. This is consistent with the fact that a nonzero $\mathcal{J}$ implies an infinite J . $\square$