Exercise sheet #4

Problem 1. Suppose the electric field in some region is found to be $E = kr^3\hat{r}$ (where k is a constant), in spherical coordinates.

(a) Find the charge density ρ

(b) Find the total charge contained in a sphere of radius R, centered at the origin.

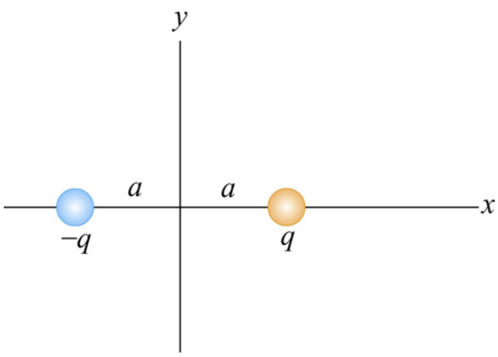
Solution. (a) From Gauss's law in differential form we have that:
$\rho = \epsilon_0 \boldsymbol{\nabla} \cdot \mathbf{E} = \epsilon_0 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot kr^3\right) = \epsilon_0 \frac{1}{r^2} k\left(5r^4\right) = 5\epsilon_0 k r^2.$

(b) By Gauss's law: $Q_{\text{enc}} = \epsilon_0 \oint \mathbf{E} \cdot d\mathbf{a} = \epsilon_0 \left(kR^3\right)\left(4\pi R^2\right) = 4\pi\epsilon_0 k R^5$

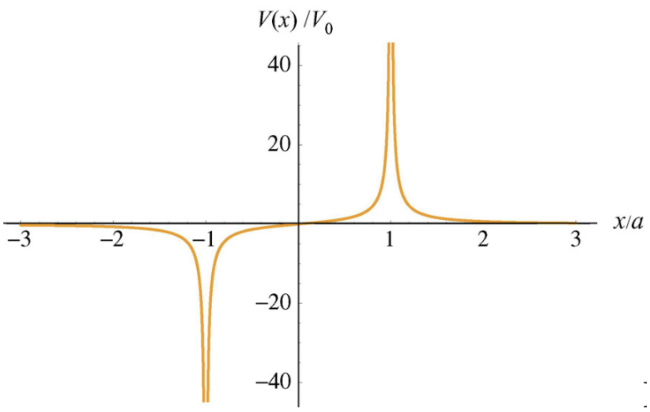
By direct integration: $Q_{\text{enc}} = \int \rho dV = \int_0^R \left(5\epsilon_0 k r^2\right)\left(4\pi r^2 dr\right) = 20\pi\epsilon_0 k \int_0^R r^4 dr = 4\pi\epsilon_0 k R^5$

□

Problem 2. Consider a system of two charges shown in the figure below. Find the electric potential at an arbitrary point on the x axis and make a plot of the electric potential as a function of x/a.



Solution: The electric potential can be found by the superposition principle. At a point on the x-axis,we have $V(x) = \frac{1}{4\pi\epsilon_0}\frac{q}{x-a} + \frac{1}{4\pi\epsilon_0}\frac{-q}{|x+a|} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|x-a|} - \frac{1}{|x+a|}\right]$. This expression can be rewritten as: $V(x)/V_0 = \frac{1}{|x/a-1|} - \frac{1}{|x/a+1|}$ where $V_0 = \frac{q}{4\pi\epsilon_0 a}$. The plot of the dimensionless electric potential as a function of x/a is depicted in the figure below.



□

Problem 3. We have two metal spheres, of radii R_1 and R_2, quite far apart from one another compared with these radii. Given a total amount of charge Q which we have to divide between the spheres, how should it be divided so as to make the potential energy of the resulting charge distribution as small as possible? When you have found the optimum division of the charge, show that with that division the potential difference between the two spheres is zero.

Solution: Since the potential is constant over the surface of a given sphere:

$$U = \frac{1}{2}\int \rho\phi dv = \frac{1}{2}\phi\int \rho dv$$

Let the charge of one sphere of radius R_1 be q, thus the charge on the sphere of radius R_2 will be $Q-q$. The potential energy of the system is then given by:

$$U(q) = \frac{q}{4\pi\epsilon_0 R_1}\cdot\frac{q}{2} + \frac{Q-q}{4\pi\epsilon_0 R_2}\cdot\frac{Q-q}{2} = \frac{1}{4\pi\epsilon_0}\left(\frac{q^2}{2R_1} + \frac{(Q-q)^2}{2R_2}\right)$$

Minimizing this function by setting the derivative with respect to q equal to zero yields:

$$\frac{dU}{dq} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{R_1} - \frac{(Q-q)}{R_2}\right) = 0 \implies q = \frac{QR_1}{(R_1+R_2)}$$

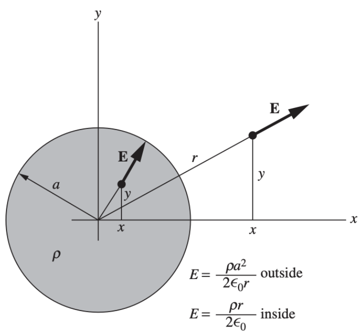
So there is a charge $q = \frac{QR_1}{(R_1+R_2)}$ on the first sphere and a charge $q = \frac{QR_2}{(R_1+R_2)}$ on the second sphere. The two terms in the equation for $\frac{dU}{dq}$ are simply the potentials of two spheres, so the condition of minimum energy is equivalent to the condition of equal potentials.

□

Problem 4. For the cylinder of uniform charge density in the figure below:

(a) Show that the expression there given for the field inside the cylinder follows from Gauss's law;

(b) Find the potential ϕ as a function of r, both inside and outside the cylinder, taking $\phi = 0$ at $r = 0$.



Solution: (a) Consider a coaxial cylinder with length ℓ and radius $r < a$. The charge contained inside is $\pi r^2\ell\rho$. The area of the cylindrical part of the surface is $2\pi r\ell$, and since $\mathbf{E}$ is perpendicular to the surface by symmetry, the flux is $2\pi r\ell E$. So Gauss's law gives the internal electric field as

$$\int \mathbf{E}\cdot d\mathbf{a} = \frac{q}{\epsilon_0} \Longrightarrow 2\pi r\ell E = \frac{\pi r^2\ell\rho}{\epsilon_0} \Longrightarrow E = \frac{\rho r}{2\epsilon_0} \quad (\text{ for } r < a)$$

We'll also need the external field for part (b). For this field, consider a cylinder of radius $r > a$. This contains a fixed amount of charge $\pi a^2\ell\rho$, so Gauss's law gives

$$\int \mathbf{E}\cdot d\mathbf{a} = \frac{q}{\epsilon_0} \Longrightarrow 2\pi r\ell E = \frac{\pi a^2\ell\rho}{\epsilon_0} \Longrightarrow E = \frac{\rho a^2}{2\epsilon_0 r} \quad (\text{ for } r > a)$$

This is the same as the field from a line of charge (namely $\lambda/2\pi\epsilon_0 r$) with linear density $\lambda = \pi a^2\rho$. Note that the internal and external fields agree at $r = a$.

(b) If $\phi = 0$ at $r = 0$, then we have

$$\text{For } r < a: \quad \phi(r) = -\int_0^r E dr = -\int_0^r \frac{\rho r dr}{2\epsilon_0} = -\frac{\rho r^2}{4\epsilon_0}$$

$$\text{For } r > a: \quad \phi(r) = -\int_0^a E dr - \int_a^r E dr$$

$$= -\frac{\rho a^2}{4\epsilon_0} - \int_a^r \frac{\rho a^2 dr}{2\epsilon_0 r} = -\frac{\rho a^2}{4\epsilon_0} - \frac{\rho a^2}{2\epsilon_0}\ln(r/a).$$

□

Problem 5. A square sheet has uniform surface charge density σ. Letting the electric potential ϕ be zero at infinite distance from the square, denote by ϕ_0 the potential at the center of the square and by ϕ_1 the potential at a corner. Determine the ratio ϕ_0/ϕ_1. The answer can be found with very little calculation by combining a dimensional argument with superposition. (Think about the potential at the center of a square with the same charge density and with twice the edge length.)

Solution: Dimensional analysis tells us that for a given charge density σ, the potential at the center of a square of side s must be proportional to Q/s, where Q is the total charge, σs^2. (This is true because the potential has the units of $q/4\pi\epsilon_0 r$, and Q is the only charge in the setup, and s is the only length scale.) Hence ϕ is proportional to $\sigma s^2/s = \sigma s$. So for fixed σ it is proportional to s. Equivalently, if we imagine increasing the size of the square by a factor f in each direction, the integral $\phi \propto \int(\sigma da)/r$ picks up a factor of f^2 in the da and a factor of f in the r, yielding a net factor of f in the numerator. Said in yet another (equivalent) way, if we imagine increasing the size of the square by a factor f in each direction, and if we look at corresponding pieces of the small and large versions, then the piece in the large version has f^2 times as much charge, but it is f times as far away from a given point, so its contribution to the potential is $f^2/f = f$ times the contribution in the small version. If we assemble 4 squares of side b, we make a square of side $2b$. The potential at the center is $4\phi_1$ (the sum of 4 corner potentials of the side b square). But from the above reasoning that $\phi \propto s$, this potential of $4\phi_1$ must also be 2 times the center potential of the side-b square. So we have $4\phi_1 = 2\phi_0$ or $\phi_0 = 2\phi_1$. Hence the desired ratio is 2. It therefore takes twice as much work to bring a charge in from infinity to the center, as it does to a corner. The above result for the square actually holds more generally for any rectangle with uniform charge density. All of the steps in the above logic are still valid, so the potential at the center is twice the potential at a corner. □