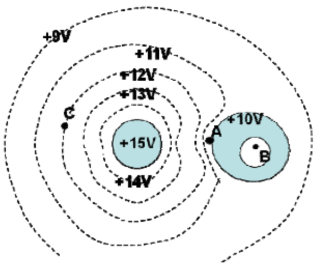
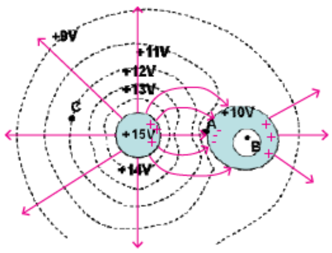
Exercise sheet #3

Problem 1. Draw some of the E-field lines between the objects in the figure below and identify the sign of the surface charge on the object at point A.

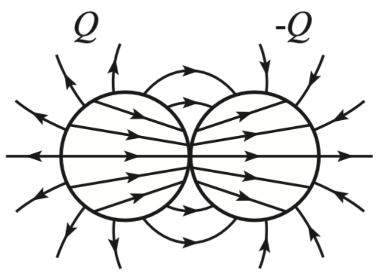


Solution: The E-field lines are perpendicular to the equipotential lines. The E-field lines are from +15V object (high potential) to the +10V object (low potential), so between the objects, the +15V object is positively charged at the surface and +10V object is negatively charged at the surface. In addition, as both objects carry net positive charge, besides the E-field lines connecting these two objects, each of them should have other E-field lines which go to infinite far place. □



Problem 2. One of two nonconducting spherical shells of radius a carries a charge Q uniformly distributed over its surface, the other carries a charge $-Q$, also uniformly distributed. The spheres are brought together until they touch. What does the electric field look like, both outside and inside the shells? How much work is needed to move them far apart?

Solution: By superposition, the electric field outside both shells is that of two point charges located at the centers. And also by superposition, the field inside each shell is that of a point charge at the center of the other shell (because a given spherical shell with uniform charge distribution produces no field in its interior). We therefore obtain the fields:



Note that the field lines inside each shell are straight. The two shells are equivalent (as far as their external fields go) to two point charges Q and $-Q$ at their centers. Therefore we may replace the spheres with these point charges. Since the centers are initially $2a$ apart, the amount of work required to move them to infinity is $\frac{1}{4\pi\epsilon_0}\frac{Q^2}{2a}$.

In more detail: Let the positive shell be labeled A, and the negative shell B. The external field of A alone is that of a point charge Q at its center. So the work required to move B to infinity in the given setup is the same as the work required in an alternative setup where A is replaced by a point charge. But the work in this case is the same as if we instead held B fixed and moved the point charge to infinity. But since the external field of B alone is that of a point charge $-Q$ at its center, we may replace B with a point charge. The work is therefore the same as in the case where we have two charges Q and $-Q$ that are initially $2a$ apart. □

Problem 3. (a) A point charge q is located at the center of a cube of edge d. What is the value of $\int \mathbf{E} \cdot d\mathbf{a}$ over one face of the cube?

(b) The charge q is moved to one corner of the cube. Now what is the value of the flux of $\mathbf{E}$ through each of the faces of the cube? (To make things well defined, treat the charge like a tiny sphere.)

Solution: (a) The total flux through the cube is q/ϵ_0, by Gauss's law. The flux through every face of the cube is the same, by symmetry. Therefore, over any one of the six faces we have $\int \mathbf{E} \cdot d\mathbf{a} = q/6\epsilon_0$.

(b) Because the field due to q is parallel to the surface of each of the three faces that touch q, the flux through these faces is zero. The total flux through the other three faces must therefore add up to $q/8\epsilon_0$, because our cube is one of eight such cubes surrounding q. Since the three faces are symmetrically located with respect to q, the flux through each must be $(1/3) * (q/8\epsilon_0) = q/24\epsilon_0$. Note: if the charge were a true point charge, and if it were located just inside or just outside the cube, then the field would not be parallel to each of the three faces that touch the given corner. The flux would depend critically on the exact location of the point charge. Replacing the point charge with a small sphere, whose center lies at the corner, eliminates this ambiguity.

□

Problem 4. Charges $2q$ and $-q$ are located on the x axis at x = 0 and x = a, respectively.

(a) Find the point on the x-axis where the electric field is zero,and make a rough sketch of some field lines.

(b) You should find that some of the field lines that start on the $2q$ charge end up on the $-q$ charge, while others head off to infinity. Consider the field lines that form the cutoff between these two cases. At what angle (with respect to the x-axis) do these lines leave the 2q charge? Hint: Draw a wisely chosen Gaussian surface that mainly follows these lines.

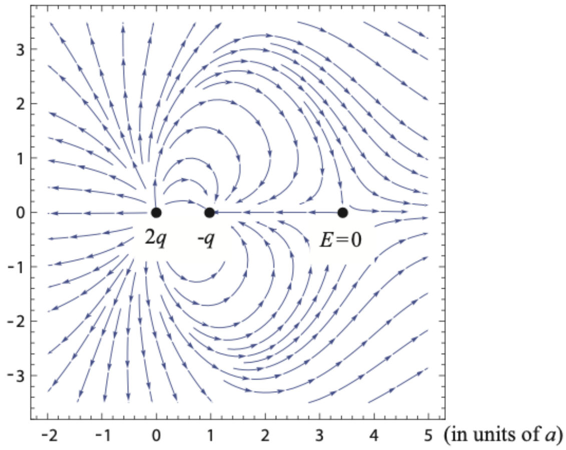
Solution: (a) You can quickly show that the desired point with $E = 0$ must satisfy $x > a$. Equating the magnitudes of the fields from the two given charges then gives.

$$\frac{2q}{4\pi\epsilon_0 x^2} = \frac{q}{4\pi\epsilon_0 (x-a)^2} \Longrightarrow 2(x-a)^2 = x^2$$

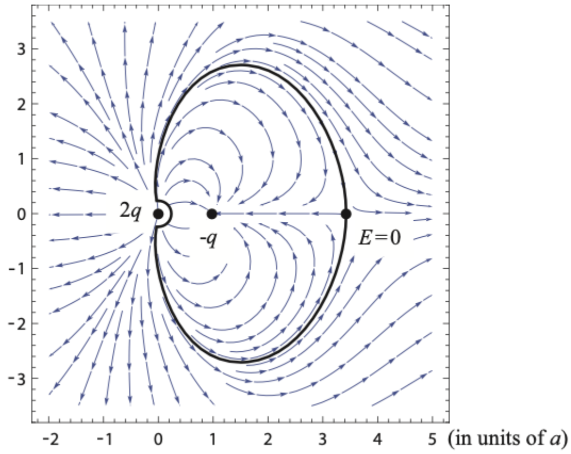
$$\Longrightarrow \sqrt{2}(x-a) = x \Longrightarrow x = \frac{\sqrt{2}a}{\sqrt{2}-1} = (2+\sqrt{2})a \approx (3.414)a$$

A few field lines,are shown below. Note that the field points in four different directions near the E = 0 point. This is consistent with the fact that the zero vector is the only vector that can simultaneously point in different directions.

(b) Consider a field line that emerges from the $2q$ charge and ends up at the x = (3.414)a point where the field is zero. (There are actually no field lines that end up right at this point, but we can pick a line infinitesimally close.) Field lines that emerge at a smaller angle (with respect to the x axis) end up at the $-q$ charge, and field lines that emerge at a larger angle end up at infinity. Consider the Gaussian surface indicated in the figure below; the surface is formed by rotating the black curve around the x axis. This surface follows the field lines except very close to the $2q$ charge, where it takes the form of a small spherical cap. The total charge enclosed within this surface is simply $-q$, so from Gauss's law there must be an electric-field flux of q/ϵ_0 pointing in to the surface. By construction, the only place where there is flux is the spherical cap, so all of the q/ϵ_0 flux must occur there. But the total flux emanating from the charge $2q$ is $2q/\epsilon_0$, so the spherical cap must represent half of the total area of a small sphere surrounding the charge $2q$. (Very close to the $2q$ charge, that charge dominates the electric field, so the field is essentially spherically symmetric.) The cap must therefore be a hemisphere, so the desired angle is 180°.



(a) Field Lines



(b) Gaussian surface

□

Problem 5. A charged soap bubble experiences an outward electrical force on every bit of its surface. Given the total charge Q on a bubble of radius R, what is the magnitude of the resultant force tending to pull any hemispherical half of the bubble away from the other half?

Solution: The field just outside the bubble is $E = \frac{Q}{4\pi\epsilon_0 R^2}$, and the field inside is zero. So the average field at the surface is $E/2$. The charge density is $\sigma = Q/4\pi R^2$. The force per unit area (that is, the pressure) is therefore:

$$P \equiv \frac{F}{A} = \sigma\frac{E}{2} = \frac{Q}{4\pi R^2}\cdot\frac{1}{2}\frac{Q}{4\pi\epsilon_0 R^2} = \frac{Q^2}{32\pi^2\epsilon_0 R^4}$$

Consider the two hemispheres defined by the horizontal great circle. The horizontal components of the forces on the different parts of the upper hemisphere cancel by symmetry, so we care only about the vertical components. The vertical component of the force on a little patch with area A is $PAcos\theta$, where θ is the angle that the plane of the patch makes with the horizontal. If we write this as $P(Acos\theta)$, we see that the vertical force on a patch equals P times the projection of the area of the patch onto the horizontal plane. Since P is constant, and since the sum of all the projections of the patches in a hemisphere is simply the great-circle area πR^2, we find that the total upward force on the hemisphere is:

$$P\cdot\pi R^2 = \frac{Q^2}{32\epsilon_0 R^2}$$

By comparison with Coulomb's force law, this has the correct units of charge$^2/(\epsilon_0\cdot\text{length}^2)$. It makes sense that it grows with Q and decreases with R.
If you don't want to use the above reasoning involving the projection, you can do an integral. Slice the sphere into rings, with θ being the angle of a ring down from the top of the sphere. The area of a ring is $da = 2\pi(Rsin\theta)(Rd\theta)$, and the vertical component of the force on the ring is $(Pda)cos\theta$. Integrating

from $\theta = 0$ to $\theta = /2$ gives the total vertical force on the upper hemisphere as:

$$\begin{aligned}\int_0^{\pi/2} (P da)\cos\theta &= \int_0^{\pi/2} \left(\frac{Q^2}{32\pi^2\epsilon_0 R^4}\right)\left(2\pi R^2 \sin\theta d\theta\right)\cos\theta \\ &= \frac{Q^2}{16\pi\epsilon_0 R^2}\int_0^{\pi/2} \sin\theta\cos\theta d\theta \\ &= \frac{Q^2}{16\pi\epsilon_0 R^2}\frac{\sin^2\theta}{2}\bigg|_0^{\pi/2} = \frac{Q^2}{32\pi\epsilon_0 R^2}\end{aligned}$$

in agreement with the above result. □