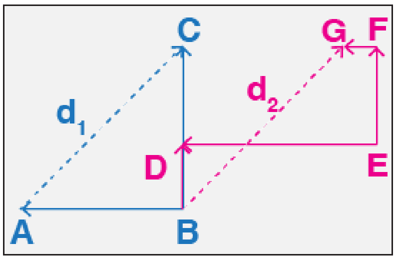
Exercise sheet #1

Problem 1. Starting from a point A, a person bicycles 10 miles due east to point B, stops for lunch, sells his bicycle, and then walks 10 miles due north to point C. Another person starts from B, walks 4 miles due north and 12 miles due east and then, feeling tired, and having brought along a surplus of travellers' checks, buys a car and drives 6 miles due north and 2 miles due west, ending at point G in the pouring rain. What is the magnitude of the displacement that each person undergoes?

Solution. Drawing the displacements:



The displacements of the first person are:

$$\overline{AB} = 10\hat{e_x} \text{ and } \overline{BC} = 10\hat{e_y}$$

$$\Rightarrow \overline{d_1} = \overline{AB} + \overline{BC} = 10\hat{e_x} + 10\hat{e_y}$$

where:

$$|\overline{d_1}| = \overline{d_1} \cdot \overline{d_1} = (10\hat{e_x} + 10\hat{e_y}) \cdot (10\hat{e_x} + 10\hat{e_y}) = 10\sqrt{2}$$

For the second person:

$$\overline{BD} = 4\hat{e_y}, \quad \overline{DE} = 12\hat{e_x}, \quad \overline{EF} = 6\hat{e_y}, \quad \overline{FG} = -2\hat{e_x}$$

$$\Rightarrow \overline{d_2} = \overline{BD} + \overline{DE} + \overline{EF} + \overline{FG} = 10\hat{e_x} + 10\hat{e_y}$$

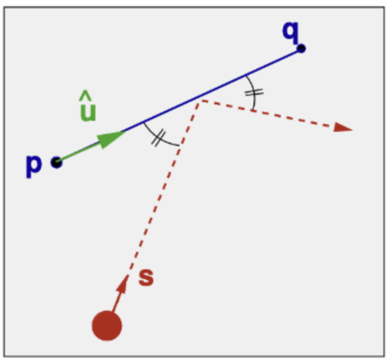
where:

$$|\overline{d_2}| = \overline{d_2} \cdot \overline{d_2} = (10\hat{e_x} + 10\hat{e_y}) \cdot (10\hat{e_x} + 10\hat{e_y}) = 10\sqrt{2}$$

□

Problem 2. A pinball moving in a plane with velocity **s** bounces, in a purely elastic impact, from a baffle whose endpoints are **p** and **q**. What is the velocity vector after the bounce?

Solution. Drawing the diagram of the collision:



Rather than imposing a new coordinate frame, it is best to refer to a natural frame that involves the direction of the new velocity $\hat{u}$ as a principal direction:

$$\hat{u} = \frac{q-p}{|q-p|}$$

The vector component of s along $\hat{u}$ is $s_{\parallel} = (s \cdot \hat{u})\hat{u}$, so the vector component perpendicular is just s minus the $s_{\parallel}$.

$$s_{\perp} = s - (s \cdot \hat{u})\hat{u}$$

Physics tells us that after the impact, the component of velocity in the direction of the baffle is unchanged and the component normal to the baffle is reversed:

$$s_{after} = (s_{\parallel} - s_{\perp}) = (s \cdot \hat{u})\hat{u} - (s - (s \cdot \hat{u})\hat{u}) = 2(s \cdot \hat{u})\hat{u} - s$$

□

Problem 3. Prove that $[\mathbf{A} \times (\mathbf{B} \times \mathbf{C})] + [\mathbf{B} \times (\mathbf{C} \times \mathbf{A})] + [\mathbf{C} \times (\mathbf{A} \times \mathbf{B})] = 0$. Under what conditions does $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$?

Solution: Notice that:

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = -\mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = -\mathbf{A}(\mathbf{B} \cdot \mathbf{C}) + \mathbf{B}(\mathbf{A} \cdot \mathbf{C})$$

Using this expression we can substitute the cross products for dot products and obtain:

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})+\mathbf{B}\times(\mathbf{C}\times\mathbf{A})+\mathbf{C}\times(\mathbf{A}\times\mathbf{B}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})+\mathbf{C}(\mathbf{A}\cdot\mathbf{B})-\mathbf{A}(\mathbf{C}\cdot\mathbf{B})+\mathbf{A}(\mathbf{B}\cdot\mathbf{C})-\mathbf{B}(\mathbf{C}\cdot\mathbf{A}) = 0$$

If $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ then:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) - (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = -\mathbf{B} \times (\mathbf{C} \times \mathbf{A}) = \mathbf{A}(\mathbf{B} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) = 0$$

If this is zero, then either $\mathbf{A}$ is parallel to $\mathbf{C}$ (including the case in which they point in *opposite* directions, or one is zero), or else $\mathbf{B} \cdot \mathbf{C} = \mathbf{B} \cdot \mathbf{A} = 0$, in which case $\mathbf{B}$ is perpendicular to $\mathbf{A}$ and $\mathbf{C}$ (including the case $\mathbf{B} = 0$).

Thus: $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \Leftrightarrow$ either $\mathbf{A}$ is parallel to $\mathbf{C}$, or $\mathbf{B}$ is perpendicular to $\mathbf{A}$ and $\mathbf{C}$. □

Problem 4. Prove that the two-dimensional rotation matrix preserves dot products. (That is, show that $\bar{A}_y \bar{B}_y + \bar{A}_z \bar{B}_z = A_y B_y + A_z B_z$). What constraints must the elements (R_{ij}) of the three-dimensional rotation matrix satisfy, in order to preserve the length of $\mathbf{A}$ (for all vectors $\mathbf{A}$) ?

$$\text{Rotation matrix applied to A:} \quad \mathbf{RA} = \begin{pmatrix} cos\phi & sin\phi \\ -sin\phi & cos\phi \end{pmatrix} \begin{pmatrix} A_y \\ A_z \end{pmatrix}$$

Solution:

$$\begin{aligned}
\bar{A}_y \bar{B}_y + \bar{A}_z \bar{B}_z = {} & (\cos\phi A_y + \sin\phi A_z)(\cos\phi B_y + \sin\phi B_z) + \\
& (-\sin\phi A_y + \cos\phi A_z)(-\sin\phi B_y + \cos\phi B_z) \\
& = \cos^2\phi A_y B_y + \sin\phi\cos\phi(A_y B_z + A_z B_y) + \\
& \sin^2\phi A_z B_z + \sin^2\phi A_y B_y - \sin\phi\cos\phi(A_y B_z + A_z B_y) + \cos^2\phi A_z B_z \\
& = (\cos^2\phi + \sin^2\phi) A_y B_y + (\sin^2\phi + \cos^2\phi) A_z B_z \\
& = A_y B_y + A_z B_z
\end{aligned}$$

In order to preserve lengths:

$$(\bar{A}_x)^2 + (\bar{A}_y)^2 + (\bar{A}_z)^2 = \sum_{i=1}^{3} \bar{A}_i \bar{A}_j = \sum_{i=1}^{3} (\sum_{j=1}^{3} R_{ij} A_j)(\sum_{k=1}^{3} R_{ik} A_k) = \sum_{j,k} (\sum_i R_{ij} R_{ik}) A_j A_k$$

This equals:

$$A_x^2 + A_y^2 + A_z^2 \quad \text{provided} \quad \sum_{i=1}^{3} R_{ij} R_{ik} = \begin{cases} 1 & if\ j = k \\ 0 & if\ j \neq k \end{cases}$$

Moreover, if R is to preserve lengths for *all* vectors $\mathbf{A}$, then this condition is not only *sufficient* but also *necessary*. For example suppose $\mathbf{A} = (1, 0, 0)$. Then $\sum_{j,k}(\sum_i R_{ij} R_{ik}) A_j A_k = \sum_i R_{i1} R_{i1}$ and this must equal 1 (since we want $\bar{A_x}^2 + \bar{A_y}^2 + \bar{A_z}^2 = 1$). Likewise, $\sum_{i=1}^{3} R_{i2} R_{i2} = \sum_{i=1}^{3} R_{i3} R_{i3} = 1$. To check the case $j \neq k$, choose $\mathbf{A} = (1, 1, 0)$.
Then we want: $2 = \sum_{j,k}(\sum_i R_{ij} R_{ik}) A_j A_k = \sum_i R_{i1} R_{i1} + \sum_i R_{i2} R_{i2} + \sum_i R_{i1} R_{i2} + \sum_i R_{i2} R_{i1}$.
But we already know that the first two sums are both 1; the third and fourth are *equal*, so $\sum_i R_{i1} R_{i2} = \sum_i R_{i2} R_{i1} = 0$, and so on the other unequal combinations of j, k.

In matrix notation: $\tilde{R} R = 1$, where $\tilde{R}$ is the transpose of R. □

Problem 5. (a) If $\mathbf{A}$ and $\mathbf{B}$ are two vector functions, what does the expression $(\mathbf{A} \cdot \nabla)\mathbf{B}$ mean? That is, what are its x,y and z components, in terms of the Cartesian components of $\mathbf{A}$,$\mathbf{B}$ and ∇?
(b)Compute $(\hat{\mathbf{r}} \cdot \nabla)\hat{\mathbf{r}}$ for $\mathbf{r} = \frac{\mathbf{r}}{r} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\sqrt{x^2+y^2+z^2}}$
(c)Evaluate $(\mathbf{v_a} \cdot \nabla)\mathbf{v_b}$ for the functions $\mathbf{v}_a = x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$ and $\mathbf{v}_b = xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3xz\hat{\mathbf{z}}$

Solution: Part (a)

$$\begin{aligned}(\mathbf{A} \cdot \nabla)\mathbf{B} = &\left(A_x \frac{\partial B_x}{\partial x} + A_y \frac{\partial B_x}{\partial y} + A_z \frac{\partial B_x}{\partial z}\right)\hat{\mathbf{x}} + \left(A_x \frac{\partial B_y}{\partial x} + A_y \frac{\partial B_y}{\partial y} + A_z \frac{\partial B_y}{\partial z}\right)\hat{\mathbf{y}} \\ &+ \left(A_x \frac{\partial B_z}{\partial x} + A_y \frac{\partial B_z}{\partial y} + A_z \frac{\partial B_z}{\partial z}\right)\hat{\mathbf{z}}\end{aligned}$$

Part (b) $\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\sqrt{x^2+y^2+z^2}}$. Let's just do the x component.

$$\begin{aligned}[(\hat{\mathbf{r}} \cdot \nabla)\hat{\mathbf{r}}]_x &= \frac{1}{\sqrt{}}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\frac{x}{\sqrt{x^2+y^2+z^2}} \\ &= \frac{1}{r}\left\{x\left[\frac{1}{\sqrt{}} + x\left(-\frac{1}{2}\right)\frac{1}{(\sqrt{})^3}2x\right] + yx\left[-\frac{1}{2}\frac{1}{(\sqrt{})^3}2y\right] + zx\left[-\frac{1}{2}\frac{1}{(\sqrt{})^3}2z\right]\right\} \\ &= \frac{1}{r}\left\{\frac{x}{r} - \frac{1}{r^3}\left(x^3 + xy^2 + xz^2\right)\right\} = \frac{1}{r}\left\{\frac{x}{r} - \frac{x}{r^3}\left(x^2 + y^2 + z^2\right)\right\} = \frac{1}{r}\left(\frac{x}{r} - \frac{x}{r}\right) = 0\end{aligned}$$

Same goes for the other components. Hence: $(\hat{\mathbf{r}} \cdot \nabla)\hat{\mathbf{r}} = \mathbf{0}$.

Part (c)

$$\begin{aligned}(\mathbf{v}_a \cdot \nabla)\,\mathbf{v}_b &= \left(x^2\frac{\partial}{\partial x} + 3xz^2\frac{\partial}{\partial y} - 2xz\frac{\partial}{\partial z}\right)(xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3xz\hat{\mathbf{z}}) \\ &= x^2(y\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + 3z\hat{\mathbf{z}}) + 3xz^2(x\hat{\mathbf{x}} + 2z\hat{\mathbf{y}} + 0\hat{\mathbf{z}}) - 2xz(0\hat{\mathbf{x}} + 2y\hat{\mathbf{y}} + 3x\hat{\mathbf{z}}) \\ &= \left(x^2y + 3x^2z^2\right)\hat{\mathbf{x}} + \left(6xz^3 - 4xyz\right)\hat{\mathbf{y}} + \left(3x^2z - 6x^2z\right)\hat{\mathbf{z}} \\ &= x^2\left(y + 3z^2\right)\hat{\mathbf{x}} + 2xz\left(3z^2 - 2y\right)\hat{\mathbf{y}} - 3x^2z\hat{\mathbf{z}}\end{aligned}$$

□

Problem 6. Prove that the divergence of a curl is always zero.

Solution.

$$\begin{aligned}\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{v}) &= \frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)\\ &= \left(\frac{\partial^2 v_z}{\partial x\partial y}-\frac{\partial^2 v_z}{\partial y\partial x}\right)+\left(\frac{\partial^2 v_x}{\partial y\partial z}-\frac{\partial^2 v_x}{\partial z\partial y}\right)+\left(\frac{\partial^2 v_y}{\partial z\partial x}-\frac{\partial^2 v_y}{\partial x\partial z}\right)=0,\ \text{by equality of cross-derivatives.}\end{aligned}$$

□

Problem 7. Prove that the curl of a gradient is always zero.

Solution:

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}t)=\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} & \frac{\partial t}{\partial z}\end{vmatrix}=\hat{\mathbf{x}}\left(\frac{\partial^2 t}{\partial y\partial z}-\frac{\partial^2 t}{\partial z\partial y}\right)+\hat{\mathbf{y}}\left(\frac{\partial^2 t}{\partial z\partial x}-\frac{\partial^2 t}{\partial x\partial z}\right)+\hat{\mathbf{z}}\left(\frac{\partial^2 t}{\partial x\partial y}-\frac{\partial^2 t}{\partial y\partial x}\right)$$

$= 0$, by equality of cross-derivatives.

□