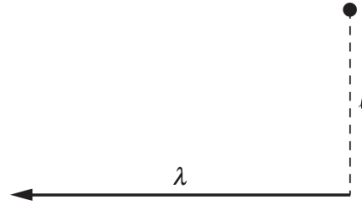


Quiz #1

Problem 1. A half-infinite line has linear charge density λ . Find the electric field at a point that is “even” with the end, a distance ℓ from it, as shown in the figure below. You should find that the field always points up at a 45° angle, independent of ℓ .



Solution: Let's parameterize the line by the angle θ shown in the figure below. A little piece of the line that subtends an angle $d\theta$ is a distance $\ell/\cos\theta$ from the given point P , and its length is $dx = d(\ell \tan\theta) = \ell d\theta/\cos^2\theta$. The magnitude of the field contribution at point P is therefore

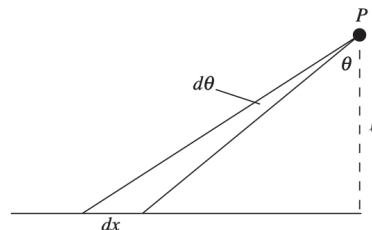
$$dE = \frac{1}{4\pi\epsilon_0} \frac{(\ell d\theta/\cos^2\theta) \lambda}{(\ell/\cos\theta)^2} = \frac{1}{4\pi\epsilon_0} \frac{\lambda d\theta}{\ell}$$

The horizontal component of this is obtained by multiplying by $\sin\theta$. The total horizontal component of the field at P is then:

$$E_x = \int_0^{\pi/2} \frac{1}{4\pi\epsilon_0} \frac{\lambda d\theta}{\ell} \sin\theta = \frac{\lambda}{4\pi\epsilon_0 \ell} \int_0^{\pi/2} \sin\theta d\theta = \frac{\lambda}{4\pi\epsilon_0 \ell}$$

Similarly, the vertical component of the field contribution in the equation for dE is obtained by multiplying by $\cos\theta$. And since $\int_0^{\pi/2} \cos\theta d\theta$ equals 1 just like the $\sin\theta$ integral, we see that E_x and E_y at point P both equal $\lambda/4\pi\epsilon_0\ell$. The field therefore points up at a 45° angle, as specified.

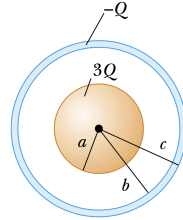
The $E_y = \lambda/4\pi\epsilon_0\ell$ result is consistent with the fact that the field from a full infinite line is $\lambda/2\pi\epsilon_0\ell$, which follows from a direct integration or a quick application of Gauss's law. By superposition, two half-infinite lines placed end-to-end yield a full infinite line, so the E_y from the latter must be twice the E_y from the former. Note that since ℓ is the only length scale in the problem, both components must be proportional to $\lambda/\epsilon_0\ell$ (assuming that they are finite). Their ratio, and hence the angle of the field, is therefore independent of ℓ . But it takes a calculation to show that the angle is 45° .



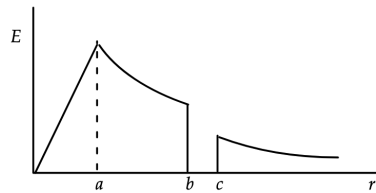
□

Problem 2. A solid insulating sphere of radius a carries a net positive charge $3Q$, uniformly distributed throughout its volume. Concentric with this sphere is a conducting spherical shell with inner radius b and outer radius c , and having a net charge $-Q$, as shown in the figure below. (i) Construct a spherical gaussian surface of radius $r > c$ and find the net charge enclosed by this surface. (ii) What is the direction of the electric field at $r > c$? (iii) Find the electric field at $r \geq c$. (iv) Find the electric field in the region with radius r where $b < r < c$ (v) Construct a spherical gaussian surface of radius r , where $b < r < c$, and find the net charge enclosed by this surface. (vi) Construct a spherical gaussian

surface of radius r , where $a < r < b$, and find the net charge enclosed by this surface. (vii) Find the electric field in the region $a < r < b$. (viii) Construct a spherical gaussian surface of radius $r < a$, and find an expression for the net charge enclosed by this surface, as a function of r . Note that the charge inside this surface is less than $3Q$. (ix) Find the electric field in the region $r < a$. (x) Determine the charge on the inner surface of the conducting shell. (xi) Determine the charge on the outer surface of the conducting shell. (xii) Make a plot of the magnitude of the electric field versus r .



Solution: (i) $q_{\text{in}} = 3Q - Q = 2Q$. (ii) The charge distribution is spherically symmetric and $q_{\text{in}} > 0$. Thus, the field is directed radially outward. (iii) For $r \geq c$, $E = \frac{1}{4\pi\epsilon_0} \frac{q_{\text{in}}}{r^2} = \frac{Q}{2\pi\epsilon_0 r^2}$. (iv) Since all points within this region are located inside conducting material, $E = 0$, for $b < r < c$. (v) $\Phi_E = \int \vec{E} \cdot d\vec{A} = 0 \Rightarrow q_{\text{in}} = \epsilon_0 \Phi_E = 0$. (vi) $q_{\text{in}} = 3Q$. (vii) For $a \leq r < b$, $E = \frac{1}{4\pi\epsilon_0} \frac{q_{\text{in}}}{r^2} = \frac{3Q}{4\pi\epsilon_0 r^2}$ (radially outward). (viii) $q_{\text{in}} = \rho V = \frac{3Q}{\frac{4}{3}\pi a^3} \frac{4}{3}\pi r^3 = 3Q \frac{r^3}{a^3}$. (ix) For $0 \leq r \leq a$, $E = \frac{q_{\text{in}}}{4\pi\epsilon_0 r^2} = \frac{3Qr}{4\pi\epsilon_0 a^3}$ (radially outward). (x) From part (iv), for $b < r < c$, $E = 0$. Thus, for a spherical gaussian surface with $b < r < c$, $q_{\text{in}} = 3Q + q_{\text{inner}} = 0$ where q_{inner} is the charge on the inner surface of the conducting shell. This yields $q_{\text{inner}} = -3Q$. (xi) Since the total charge on the conducting shell is $q_{\text{net}} = q_{\text{outer}} + q_{\text{inner}} = -Q$, we have $q_{\text{outer}} = -Q - q_{\text{inner}} = -Q - (-3Q) = 2Q$. (xii) This is shown in the figure below: $\square$



Problem 3. Imagine the xy plane, the xz plane, and the yz plane all made of metal and soldered together at the intersections. A single point charge Q is located a distance d from each of the planes. Sketch the configuration of image charges you need to satisfy the boundary conditions. What is the direction and magnitude of the force that acts on the charge Q ?

Solution: The required image charges are at the other seven corners of a cube of side $2d$, as shown in the figure below. This configuration satisfies the condition of equipotential surfaces where the planes are; the potential is zero at every point on these planes, because at any such point, $\sum(\pm Q/r) = 0$ by symmetry. Equivalently, the total electric field is perpendicular to all of the planes at every point. This can be seen by grouping the eight charges into four dipoles (with charges on opposite sides of a given plane); the field from each dipole is perpendicular to the plane at every point on the plane. By superposition, the field from all four dipoles is also perpendicular to the plane at every point on the plane.

From the symmetry of the setup, the net force on Q is directed toward (or away) from the origin (the center of the cube of side $2d$). So we need compute only the force components in that direction. There are three classes of charges:

- Three charges $-Q$ at a distance $2d$ make an angle $\cos^{-1}(1/\sqrt{3})$ with the direction toward the origin. This follows from the fact that the diagonal of the cube has length $\sqrt{3}(2d)$. Alternatively, you can find the cosine by using the two standard expressions for the dot product. We therefore have (ignoring the $4\pi\epsilon_0$),

$$F_1 = 3 \frac{Q^2}{(2d)^2} \cdot \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{4} \frac{Q^2}{d^2} \quad (\text{toward origin}).$$

- Three charges Q at a distance $2\sqrt{2}d$ make an angle $\cos^{-1}(\sqrt{2}/\sqrt{3})$ with the direction toward the origin. So

$$F_2 = 3 \frac{Q^2}{(2\sqrt{2}d)^2} \cdot \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{3}}{4\sqrt{2}} \frac{Q^2}{d^2} \quad (\text{away from origin}).$$

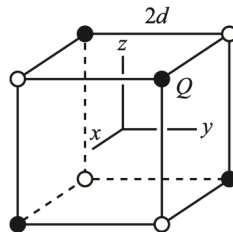
- One charge $-Q$ at a distance $2\sqrt{3}d$ is located in line with the origin. So

$$F_3 = \frac{Q^2}{(2\sqrt{3}d)^2} = \frac{Q^2}{12d^2}. \quad (\text{toward origin})$$

The total force on Q is therefore (bringing back in the $4\pi\epsilon_0$)

$$F = \frac{1}{4\pi\epsilon_0} \left(\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4\sqrt{2}} + \frac{1}{12} \right) \frac{Q^2}{d^2} \approx \frac{(0.210)}{4\pi\epsilon_0} \frac{Q^2}{d^2} \quad (\text{toward origin}).$$

□



Problem 4. If a point charge is located outside a hollow conducting shell, there is an electric field outside, but no electric field inside. On the other hand, if a point charge is located inside a hollow conducting shell, there is an electric field both inside and outside (although the external field would be zero in the special case where the shell happened to have charge exactly equal and opposite to the point charge). The situation is therefore not symmetric with respect to inside and outside. Explain why this is the case, by considering where electric field lines can begin and end.

Solution: Electrostatic field lines can begin and end only at charges or at infinity. Also, there can be no closed loops since $\text{curl } \mathbf{E} = 0$.

If the point charge is located outside the shell, the field lines can have their ends at the point charge, the shell, or infinity. There can't be any field lines inside the shell because they would have to start at one point on the shell and end at another. This would imply a nonzero potential difference between these two points, contradicting the fact that all points on the shell have the same potential.

If the point charge is located inside the shell, the field lines starting at the point charge must end up on the shell. And there can be field lines outside the shell because they can have one end on the shell and the other end at infinity.

We see that the basic difference between inside and outside is that the inside region has only one boundary (the shell), while the outside region has two (the shell and infinity). The former case therefore requires the existence of an additional termination point (a charge) if field lines are to exist. □