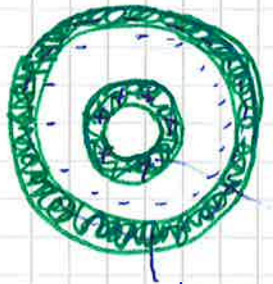
Lecture 7. Capacitors & capacitance

Last time:

Conductors (materials w/ free $\bar{e}$)
- → $\underline{E}$ inside a conductor is zero
- → $\underline{E}$ field lines are always ⊥ to the surface of a conductor
- → $\underline{E}$ at the surface is $E = \frac{\sigma}{\varepsilon_0}$
- → Conductors are equipotential surfaces.

How is this info useful when solving problems?

- $\underline{E}_{inside} = 0$ + Gauss's law are useful to solve problems involving charge distributions inside conductors.



+Q

draw Gaussian surface here

example: 2 concentric cylindrical shells
we deposit a charge +Q in the inner shell.

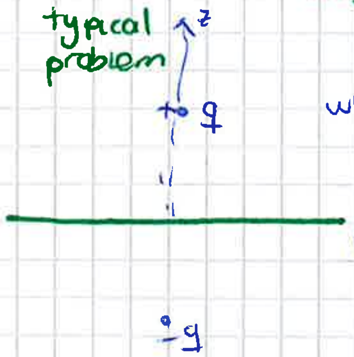
what is $\underline{E}$ between the 2 shells? what are the induced charges on the surfaces?

because $\underline{E}_{inside} = 0 \Rightarrow \oint \underline{E} \cdot d\underline{a} = 0 = Q_{enc}$

and $Q_{enc} = +Q + q_{ind} \Rightarrow q_{ind} = -Q$
on the inner surface of the outer cylinder

if the conductor is electrically neutral ⇒ +Q must be induced on the outer-most surface.

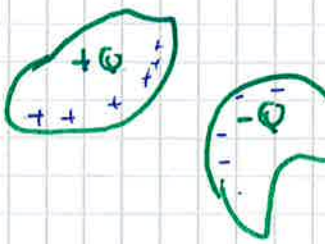
Method of images → We can solve a problem by changing it for a completely new one as long as it satisfies the BC of φ of the original problem & I don't add any charges in the region I am solving for $\underline{E}$.
uniqueness of solution is guaranteed by the uniqueness theorem.



what is $\underline{E}$ above the conductor?
- add image charge & write potential for point charges.

Today: Capacitors and capacitance

Suppose we have 2 conductors (fixed in space)



- We know that $\varphi = \varphi_0$, const over the surface of a conductor. So we can define the potential difference between them:

$$V = \varphi_+ - \varphi_- = -\int_{(-)}^{(+)} \underline{E} \cdot d\underline{s}$$

how do I get information about $\underline{E}$?

- We don't know how charge distributes itself over the conductors
- Calculating $\underline{E}$ is a nightmare for complicated shapes.

what is $\underline{E}$ due to the conductors?

...that $\underline{E}$ is proportional to Q:

$$\underline{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho}{r^2}\, dV \hat{r}$$ $\underline{E} \propto Q$

this means if I change ρ, $\underline{E}$ will change by the same amount." or proportionally.
Guaranteed by the second uniqueness theorem.
($\underline{E}$ is uniquely determined if the total charge on each conductor is given)

Since $\underline{E}$ is proportional to Q, also is V. The constant of proportionality is called the capacitance of the arrangement:

Naming the constant of proportionality $\frac{1}{C}$ so $\boxed{C \equiv \frac{Q}{V}}$ Geometrical quantity that depends on the sizes, shapes & separation of 2 conductors.
$\Rightarrow Q = CV$

physically: capacitance is a measure of the capacity to store electric charge for a given potential difference.

We just defined:
* C is the capacitance of the arrangement/system
* Capacitor: system of 2 oppositely charged conductors

Notes on capacitance & capacitors:
- Capacitance is intrinsically a positive quantity

$$C \equiv \frac{Q}{V} \quad \text{where } V = \underbrace{\varphi_+}_{\text{conductor w charge } Q} - \underbrace{\varphi_-}_{\text{conductor w/ charge } -Q}$$

- Units of capacitance

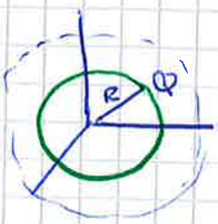
$$[C] = \frac{\text{Coulomb}}{\text{Volt}} \equiv \text{Farad (F)}$$

1 Farad is really large & not a practical unit
- Typically capacitors have a capacitance in the range of pF – μF
10^{-12} – 10^{-6}

· We can also define the capacitance of an individual conductor. In this case the "second conductor" w/negative charge is an imaginary spherical shell of ∞ radius surrounding the conductor. This contributes nothing to the field and the capacitance is given by:

$$C = \frac{Q}{V} \quad \text{where } V = \varphi_+ - \varphi_\infty$$

Example: Isolated sphere
Conductive sphere of radius R in $(0,0,0)$ w/charge Q:



This is a capacitor w/ the second "virtual conductor" at ∞
What is the capacitance?

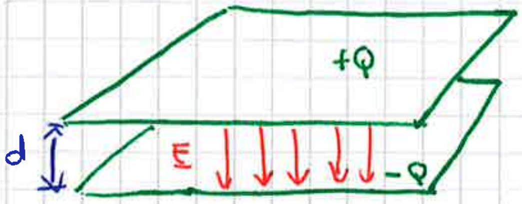
$$V = \varphi_R - \varphi_\infty = \frac{Q}{4\pi\varepsilon_0 R} \Rightarrow \frac{Q}{V} = 4\pi\varepsilon_0 R \therefore C_{sphere} = 4\pi\varepsilon_0 R$$

we draw a gaussian surface

$$\oint \underline{E}\cdot d\underline{a} = E_r A = E_r(4\pi r^2) = \frac{Q}{\varepsilon_0}$$

$$\Rightarrow E_r = \frac{Q}{4\pi\varepsilon_0 r^2} \Rightarrow \varphi(r) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r}$$

Typical capacitor: Parallel plates

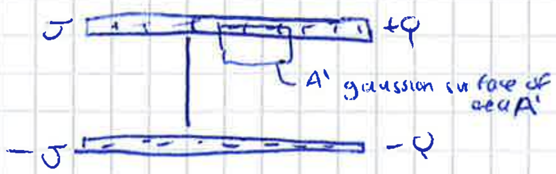


- Two parallel plates, each of area A, separated by a distance d.

take $A \gg d^2$ ⇒ neglect edge effects.

What is the capacitance of this system?

To find the capacitance we need to first find $\underline{E}$



$$\oint \underline{E} \cdot d\underline{A} = \frac{Q_{enc}}{\varepsilon_0}$$

$$\underbrace{EA' = \frac{Q_{enc}}{\varepsilon_0}}_{} = \frac{\sigma A'}{\varepsilon_0} \Rightarrow Q_{enc} = EA'\varepsilon_0 \quad (*)$$

we also derived this using the superposition principle in lecture 4.

if we have E now we can calculate V:

$$V = \varphi_+ - \varphi_- = -\int_{(-)}^{(+)} \underline{E} \cdot d\underline{s} = +Ed \quad (**)$$

path of integration chosen which follows the E lines

distance between plates

$$\therefore \boxed{C = \frac{Q}{V} = \frac{\varepsilon_0 A}{d}}$$

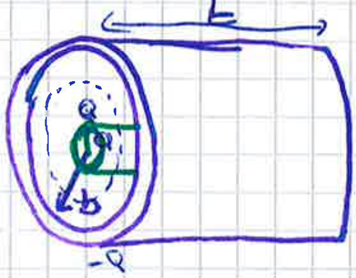
using (*) and (**)

- depends on A & d (geometry)
- increases linearly with A → larger plates can hold more charge
- Inversely proportional to d → the smaller the value of d, the smaller the potential difference V for a fixed Q.

What is E outside the capacitor? Zero

Demo 464 Verify this relation

Another example: Cylindrical capacitor



Consider a solid cylindrical conductor of radius a, surrounded by a coaxial cylindrical shell of radius b

Length of both cylinders is L with $L \gg b - a$ ⇒ neglect edge effects.

Capacitor is charged such that the inner cylinder has charge +Q while the outer one has charge −Q.

What is the capacitance?

First: compute E everywhere.

Cylindrical symmetry → choose a cylindrical surface. ( ) in the diagram.

$$\oint E \cdot dA = EA = E(2\pi r l) = \frac{\lambda l}{\varepsilon_0} \Rightarrow E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

because $\lambda = \frac{Q}{l}$

Notice: $E \neq 0$ only in the region $a < r < b$, if $r < a$ $q_{enc} = 0$ since charge in a conductor is at its surface

For $r > b$ $q_{enc} = \lambda l - \lambda l = 0$ b/c we have that the gaussian surface encloses both Q & −Q.

- Now we can calculate V from $\underline{E}$:

$$V = \varphi_b - \varphi_a = -\int_a^b E_r\,dr = \frac{-\lambda}{2\pi\varepsilon_0}\int_a^b \frac{dr}{r} = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right)$$

this is $\underline{E}\cdot d\underline{s}$ for a cylindrical geometry

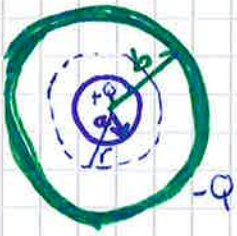
integration path along the direction of $\underline{E}$ lines.

Finally:

$$C = \frac{Q}{V} = \frac{\lambda L}{\lambda \ln(b/a)/2\pi\varepsilon_0} = \boxed{\frac{2\pi\varepsilon_0 L}{\ln(b/a)}}$$

again this only depends on geometrical factors i.e. L, b, a.

Another example: spherical capacitor



Spherical capacitor of 2 concentric spherical shells of radii a & b
Inner shell has a charge +Q, outer shell a charge -Q
What is the capacitance of this configuration?

- First calculate $\underline{E}$. $E \neq 0$ only in the region $a < r < b$, following the same arguments as before.

Drawing the gaussian surface we can obtain:

$$\oint \underline{E}\cdot d\underline{a} = E_r A = E_r(4\pi r^2) = \frac{Q}{\varepsilon_0} \Rightarrow E_r = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$$

Now with this we can calculate the potential difference:

$$V = \varphi_b - \varphi_a = -\int_a^b \underline{E}\cdot d\underline{s} = -\int_a^b E_r\,dr = \frac{-Q}{4\pi\varepsilon_0}\int_a^b \frac{dr}{r^2} = \frac{-Q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{-Q}{4\pi\varepsilon_0}\left(\frac{b-a}{ab}\right)$$

$$\therefore\ C = \frac{Q}{V} = \boxed{4\pi\varepsilon_0\left(\frac{ab}{b-a}\right)}$$

once again we see this only depends on the geometry of the system.

Now check this out. In the limit where $b \to \infty$ this becomes equivalent to our 1st example i.e. a single sphere.

~~if $b\to\infty$~~ Rewriting

$$C = 4\pi\varepsilon_0\left(\frac{ab}{b-a}\right) = 4\pi\varepsilon_0\frac{a}{\left(1-\frac{a}{b}\right)}$$

we can see that for $b \to \infty$

$$\Rightarrow C \sim 4\pi\varepsilon_0 a = 4\pi\varepsilon_0 R$$

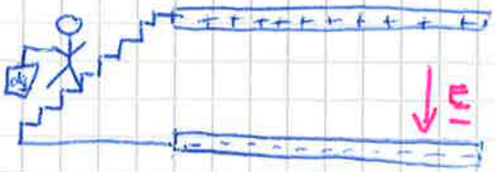
which is the result we got last time.

gy stored in a capacitor

We start initially w/ no charge on our capacitor

in each plate there are many ⊖ & ⊕ charges but they balance out so overall the charge is zero.

Now imagine we have a "magic bucket" that we use to carry charge from one plate to the other:



we start at the bottom plate & bring some charge dq to the top plate & come back. Repeat a few times... at the end of this process there is charge Q on the top & −Q on the bottom.

We have now 'built up' charge in the capacitor, creating an electric field where there was none initially.

How much work does it take to add more charge +dq to the top plate?

Let's assume that at some instant in time, the charge on the top plate is +q, so the potential difference between the two plates is

$V = \frac{q}{C}$, how much work do we need to do to dump another bucket of charge dq on the top plate?

$$dW = V\,dq$$

by definition of V

If at the end of the process the total charge is +Q then the total work done in the process is:

$$W = \int_0^Q dq\, V = \int_0^Q dq\, \frac{q}{C} = \frac{1}{2}\frac{Q^2}{C}$$

from the definition of capacitance

this is the potential energy of the system.

$$\therefore \boxed{U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QV = \frac{1}{2}CV^2}$$

this is the energy stored in a capacitor.

Energy density of the electric field

We can think of the energy stored in a capacitor as being stored in the electric field itself. Let's have a look @ how much energy is stored in a ∥ plate capacitor:

we have that $U = \frac{1}{2}CV^2 = \frac{1}{2}\frac{\varepsilon_0 A}{d}\underbrace{(Ed)^2}_{\text{this is } V} = \frac{1}{2}\varepsilon_0 E^2 \underbrace{(Ad)}_{\text{volume between plates}}$

b/c for a ∥ plate capacitor $C = \frac{\varepsilon_0 A}{d}$

...is we can define an **energy density** (energy / volume)

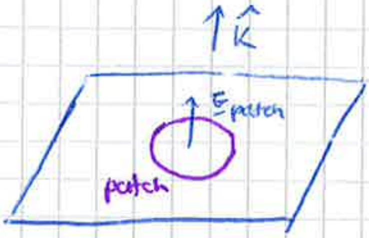
$$u_E \equiv \frac{U_E}{Volume} = \frac{1}{2}\varepsilon_0 E^2$$

Alternate derivation of energy stored in a capacitor:
We can also obtain the energy stored in a capacitor from the point of view of external work.

Since the plates are oppositely charge $\Rightarrow$ there must be a $\underline{F}$ applied to maintain constant separation between them.

What is the magnitude of the $\underline{F}$, experienced by the plate due to the other plate?

what is the n$\underline{F}$ experienced by a "patch" of conductor w/ uniform charge density?



we can think of the total electric field above the surface as:

$\underline{E} = \underline{E}_{patch} + \underline{E}'$ where $\underline{E}'$ is the field due to the rest of the charge distribution

- By Newton's 3rd law: Patch can't excert a $\underline{F}$ on itself. so the $\underline{F}$ it feels must be due to the rest of the charge. (i.e to $\underline{E}'$)

- First let's look at $\underline{E}_{patch}$

$$\underline{E}_{patch} = \begin{cases} \dfrac{\sigma}{2\varepsilon_0}\hat{k}, & z>0 \\ -\dfrac{\sigma}{2\varepsilon_0}\hat{k}, & z<0 \end{cases}$$

we derived this already for the plane.

With this we can rewrite the total $\underline{E}$ as:

going back to ☺ + subs E_{patch}

$$\underline{E}_{above} = \left(\frac{\sigma}{2\varepsilon_0}\right)\hat{k} + \underline{E}'$$

$$\underline{E}_{below} = -\frac{\sigma}{2\varepsilon_0}\hat{k} + \underline{E}'$$

$$\underset{\text{adding both eqts}}{\Rightarrow} \underline{E}_{above} + \underline{E}_{below} = 2\underline{E}' \Rightarrow \underline{E}' = \frac{1}{2}\left(\underline{E}_{above} + \underline{E}_{below}\right)$$

In the case of a conductor:

$$\underline{E}_{above} = \frac{\sigma}{\varepsilon_0}\hat{k}$$

$$\underline{E}_{below} = 0$$

$$\Rightarrow \underline{E}' = \frac{1}{2}\left(\frac{\sigma}{\varepsilon_0}\right)\hat{k}$$

So the $\underline{F}$ on the conductor is $\underline{F} = q\underline{E}' = (\sigma A)\dfrac{\sigma}{2\varepsilon_0}\hat{k} = \dfrac{\sigma^2 A}{2\varepsilon_0}\hat{k}$

because $\sigma = \frac{q}{A}$

...e can go back to the original problem:

We just found that a small patch of charge $\Delta q = \sigma \Delta A$ experiences an attractive force:

$$\Delta \underline{F} = \frac{\sigma^2 (\Delta A)}{2\varepsilon_0}$$

if the total area of the plate is A, then the 'external agent' must excert a force

$$F_{ext} = \frac{\sigma^2 A}{2\varepsilon_0}$$ to pull the 2 plates apart

We can rewrite this expression as

$$F_{ext} = \frac{\varepsilon_0}{2} E^2 A$$ considering that $E = \frac{\sigma}{\varepsilon_0}$ in the region between the plates.

So the total amount of work done externally to separate the plates by a distance $\underline{d}$ is then:

$$W_{ext} = \int \underline{F}_{ext} \cdot d\underline{s} = F_{ext} d = \frac{\varepsilon_0 E^2 A}{2} d$$

since $W_{ext} = U_E$ ← potential energy

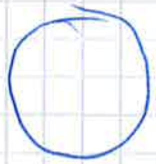
Following our definition of

$$u_E \equiv \frac{U_E}{\text{Volume}} = \frac{W_{ext}}{Ad} = \frac{\varepsilon_0 E^2 A d}{2 A d} = \boxed{\frac{\varepsilon_0 E^2}{2}}$$ same as before

Example: Energy stored in a spherical shell (~~[illegible]~~ optional discussion)

Find the energy stored in a metallic spherical shell of radius a and charge Q.
We had calculated in previous lectures that for a spherical shell of radius a:

$$\underline{E} = \begin{cases} \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} &, r > a \\ 0 &, r < a \end{cases}$$



We can use the expression we just derived for the energy density:

$$u_E = \frac{\varepsilon_0 E^2}{2} = \frac{Q^2}{32\pi^2 \varepsilon_0 r^4}$$ for $r > a$ and zero for $r < a$.

We have the density but now we need to integrate over the entire region of space from $r = a$ to $r = \infty$. Due to symmetry reasons we choose spherical coordinates with $dV = 4\pi r^2 dr$:

$$U_E = \int_a^\infty \left(\frac{Q^2}{32\pi^2 \varepsilon_0 r^4}\right) 4\pi r^2 dr = \frac{Q^2}{8\pi\varepsilon_0}\int_a^\infty \frac{dr}{r^2} = \frac{Q^2}{8\pi\varepsilon_0 a} = \frac{1}{2} Q V$$

where $V = \frac{Q}{4\pi\varepsilon_0 a}$ is the potential on the surface of the shell if $V(\infty) = 0$

· verify that the energy of the system is equal to the work done charging the sphere:

- Suppose at some instant the sphere has a charge q & it's at a potential $\varphi = \dfrac{q}{4\pi\varepsilon_0 a}$
- The work required to add an additional charge dq to the system is:

$dW = \varphi\, dq$ so if we integrate to obtain the total work:

$$W = \int dW = \int \varphi\, dq = \int_0^Q dq \left(\frac{q}{4\pi\varepsilon_0 a}\right) = \frac{Q^2}{8\pi\varepsilon_0 a}$$

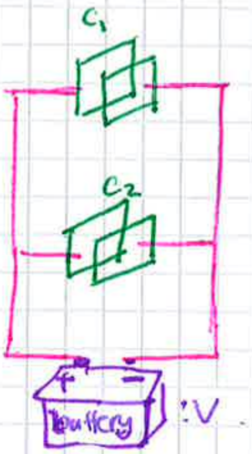
, which is the same as we got by integrating the energy density.

Capacitors in electric circuits.

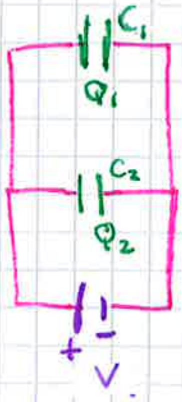
Demo 464

Parallel connection

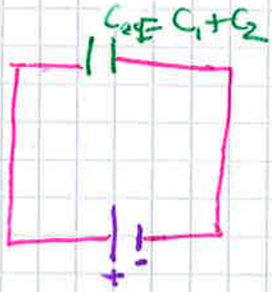
Suppose we have two capacitors C_1 & C_2 w/ charges Q_1 & Q_2 respectively. What is their capacitance if they are connected in parallel?



we would draw this as:



what is the equivalent capacitance?



left plates are connected to the ⊕ terminal
⇒ they are positively charged
right plates are connected to the ⊖ terminal
⇒ they are negatively charged.

We have that the potential difference V across each capacitor is:

$$C_1 = \frac{Q_1}{V} \quad \text{and} \quad C_2 = \frac{Q_2}{V} \quad (*)$$

What will be the capacitance C_{eq} if they are replaced by a single capacitor?

If the total charge supplied by the battery is Q so then:

$$Q = Q_1 + Q_2 \underset{\text{From } (*)}{=} C_1 V + C_2 V = (C_1 + C_2)(V)$$

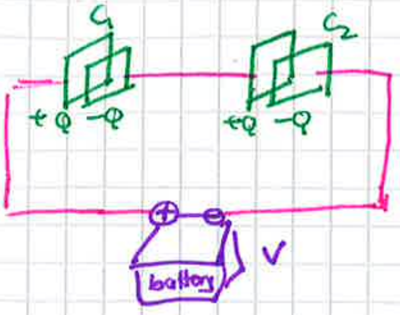
So the equivalent capacitance:

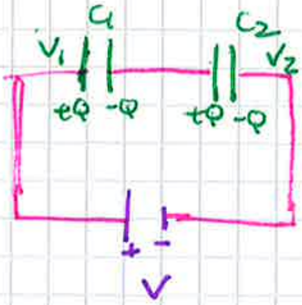
$$C_{eq} = \frac{Q}{V} = C_1 + C_2$$ & in general for N capacitors:

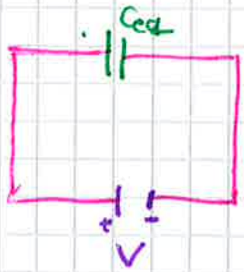
$$\boxed{C_{eq} = C_1 + C_2 + C_3 + \dots C_N = \sum_{i=1}^{N} C_i}$$ connection in parallel

connection

Now we have 2 capacitors connected in series:







Initially C_1 & C_2 are uncharged. Then we apply V to both. For the outer-most plates of both C_1 & C_2 they get a positive or negative charge, b/c they are connected directly to the battery terminals.

What about the inner plates? They get an induced charge of $-Q$ & $+Q$ respectively.

Then the potential differences across capacitors C_1 & C_2 will be:

$$V_1 = \frac{Q}{C_1} \quad \text{and} \quad V_2 = \frac{Q}{C_2}$$

And the total potential difference will be the sum of these potentials

$$\Rightarrow V = V_1 + V_2$$

In general the total potential difference across any # of capacitors in series is equal to the Σ of potential differences across capacitors $\Rightarrow C_{eq} = \frac{Q}{V}$ & since potential differences just add as stated:

$$\frac{Q}{C_{eq}} = \frac{Q}{C_1} + \frac{Q}{C_2}$$

So the equiv. capacitance will be:

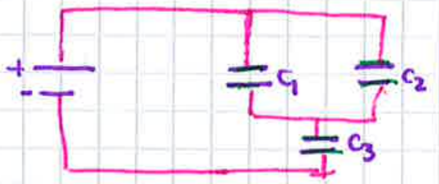
$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

and in general for N capacitors:

$$\boxed{\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \dots \frac{1}{C_N} = \sum_{i=1}^{N} \frac{1}{C_i}}$$

connection in series.

what happens if we have both elements in series and in parallel?



- First solve eq. cap for C_1 & C_2 (parallel connection)
- then add it in series to C_3

This is what this looks like:

where $C_{12} = C_1 + C_2$ (*) Now we need

$$\frac{1}{C_{123}} = \frac{1}{C_{12}} + \frac{1}{C_3} \quad (**)$$

Substituting the expression ⊛ into ⊛⊛ .

$$\boxed{C_{123} = \frac{C_{12}\,C_3}{C_{12}+C_3} = \frac{(C_1+C_2)\,C_3}{C_1+C_2+C_3}}$$