Lecture 4. The electric potential. Gauss's law revisited. Gradient & divergence revisited

Last time:

① The electric field
demo 409

$\underline{E} = \frac{\underline{F}}{q}$ mathematically

Physically: 'lines of force'

* Has a magnitude & direction at every pt in space. This can be represented by a vector field.

* Field lines: "connect the vectors"
  - Direction of $\underline{E}$ is tangent to field lines
  - $E \propto \underbrace{\frac{\#\text{lines}}{\text{area}}}_{\text{density of lines}}$ The # of lines/unit area (through a surface ⊥ to the field lines) is proportional to the magnitude of E in a given region.
  - Begin at ⊕ charges (sources) or ∞
    End at ⊖ charges (sinks) or ∞

② Gauss's law
demo 417

$$\Phi_E \equiv \int_S \underline{E} \cdot d\underline{a} = \frac{1}{\varepsilon_0} Q_{enc}$$

"The flux of $\underline{E}$ through any surface is proportional to the charge enclosed by the surface"

$$\int_S \underline{E} \cdot d\underline{a} = \frac{1}{\varepsilon_0} \sum_i q_i \quad \text{discrete distrib of } q$$

$$\int_S \underline{E} \cdot d\underline{a} = \frac{1}{\varepsilon_0} \int \rho \, dV \quad \text{continuous distrib of } q$$

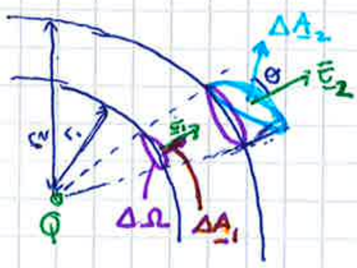
Remember $\rho \equiv \frac{dq}{dV}$

Gauss's law holds b/c
$F \propto \frac{1}{r^2}$
⊖ surface $\propto r^2$

so $F \cdot da$ does not depend on r (i.e. is a constant)
So the same # of field lines pass through any sphere, regardless of its size (or whatever shape)
↳ the same

- depends on the $\frac{1}{r^2}$ character of Coulomb's law
- holds true in general for any $F \propto \frac{1}{r^2}$

Formal derivation of Gauss's law



$\underline{E}_1$ & $\underline{E}_2$ strength of electric field at the center of area elements

$\Delta\Omega \equiv \frac{\Delta A_1}{r_1^2}$ (for purple surface) solid angle (3D) analogous to "regular" angles in 2D

for blue 'tilted' surface

$$\Delta\Omega = \frac{\Delta \underline{A}_2 \cdot \hat{r}}{r_2^2} = \frac{\Delta A_2 \cos\theta}{r_2^2}$$

↳ area of radial projection of ΔA_2 onto ⊖ of radius r_2.

$\Delta \underline{A}_1$ normal to 1st surface (oriented radially)
$\Delta \underline{A}_2$ normal to 2nd surface (makes an angle θ w/ radial)

From the picture we can see that the solid angles subtended by ΔA_2 and ΔA_1 are the same so:

$$\Delta\Omega = \frac{\Delta A_1}{r_1^2} = \frac{\Delta A_2 \cos\theta}{r_2^2} \Rightarrow \Delta A_2 \cos\theta = \frac{\Delta A_1 r_2^2}{r_1^2} \quad (*)$$

Also note that the electric field strength at the center of the surfaces ΔA_1 & ΔA_2 is:

$$E_i = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r_i^2}$$

$$\Rightarrow \frac{E_2}{E_1} = \frac{r_1^2}{r_2^2} \Rightarrow E_2 = \frac{E_1 r_1^2}{r_2^2} \quad (**)$$

Now the flux of the electric field through ΔA_1 is:

$$\Delta\Phi_1 = \underline{E}_1 \cdot \Delta\underline{A}_1 = E_1 \Delta A_1$$

And the flux through ΔA_2 is:

$$\Delta\Phi_2 = \underline{E}_2 \cdot \Delta\underline{A}_2 = E_2 \Delta A_2 \cos\theta$$ so subs E_2 from (**) and $\Delta A_2 \cos\theta$ from (*):

$$= \left(\frac{E_1 r_1^2}{r_2^2}\right)\left(\frac{\Delta A_1 r_2^2}{r_1^2}\right) = E_1 \Delta A_1 = \Phi_1$$

The electric flux through any area element subtending the same solid angle, is constant. Independent of the shape or orientation of the surface.

How should we use Gauss's law?

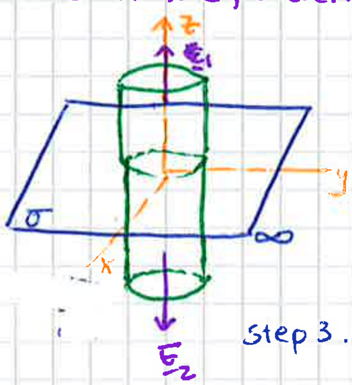
Convenient to evaluate $\underline{E}$ in systems w/ symmetry → cylindrical, planar, spherical

Which surface to draw? coaxial cylinder, "pill box", concentric sphere

"Recipe" to solve problems w/ Gauss's law:

1. ID the symmetry of the charge distribution
2. Choose a Gaussian surface on which the magnitude of $\underline{E}$ is const over portions of the surface. Determine the direction of $\underline{E}$.
3. Divide space into regions associated w/ the charge distribution. For each region calculate Q_{enc}.
4. Calculate the flux of $\underline{E}$ Φ_E through the Gaussian surface for each region.
5. Equate Φ_E w/ $\frac{Q_{enc}}{\varepsilon_0}$ & solve for E.

Let's revisit the problem of the ∞ plane of charge



Step 1. Plane → planar symmetry

Step 2. What is the direction of $\underline{E}$?
Charge is UNIFORMLY distributed, $\Rightarrow \underline{E} = E\,\hat{z}$

Choose a surface: We choose a "pill box"
NOTE that the pill box has 3 surfaces:

We drew the cylinder such that top + bottom are @ the same distance from the plane.

Step 3. We have a uniform surface charge distribution
$\sigma = \frac{Q_{enc}}{A} \Rightarrow Q_{enc} = \sigma A$ where $A = A_{top} = A_{bottom}$

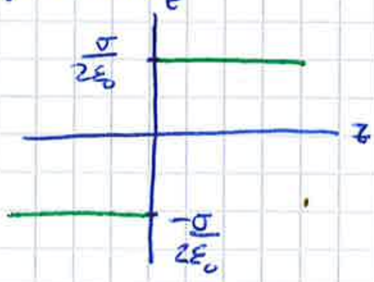
Step 4. $\Phi_E = \Phi_{top} + \Phi_{bottom} + \cancel{\Phi_{side}}^{0} = E_1 A_{top} + E_2 A_{bottom} = A(E_1 + E_2) = 2EA$

5. $\Phi_E = 2EA = \frac{Q_{enc}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$

$$\Rightarrow E = \frac{\sigma}{2\varepsilon_0}$$

(Step 6. Write down the full solution for the problem)

$$\underline{E} = \begin{cases} \frac{\sigma}{2\varepsilon_0}\hat{z} & \text{for } z > 0 \\ \frac{-\sigma}{2\varepsilon_0}\hat{z} & \text{for } z < 0 \end{cases}$$

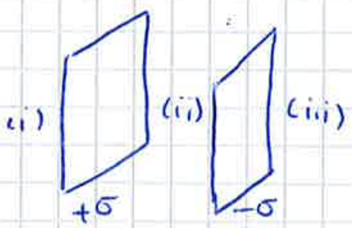


discontinuity of $\underline{E}$ across the plane.

Final note: Use of Gauss's law can be extended to "combinations of objects possessing symmetry", even if the whole arrangement is NOT symmetrical
invoke → principle of superposition

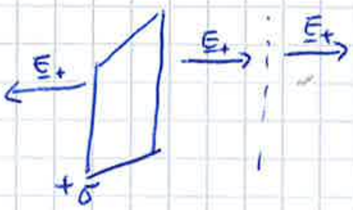
An example:

What is the $\underline{E}$ of two ∞ // planes with equal but opposite uniform charge densities?



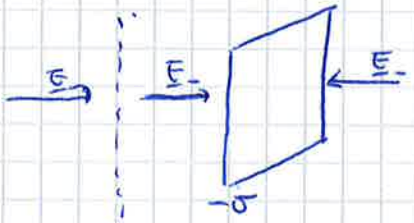
→ we have 3 regions in space (i), (ii) + (iii)

→ Field due to the plane with $+\sigma$



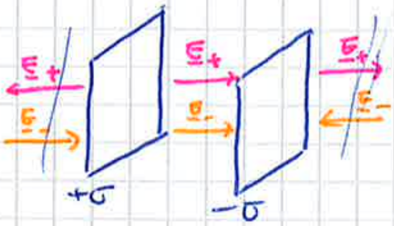
$|E_+| = \frac{\sigma}{2\varepsilon_0}$

→ Field due to the plane with $-\sigma$



$|E_-| = \frac{\sigma}{2\varepsilon_0}$

→ Applying the principle of superposition:



they cancel out except at the middle

where $|\underline{E}| = \frac{\sigma}{\varepsilon_0}$

$$\underline{E} = \frac{\sigma}{\varepsilon_0}\hat{x}$$

..ss's law in differential form

$$\int \underline{E} \cdot d\underline{a} = \frac{1}{\varepsilon_0} Q_{enc}$$ — integral equation

— give me Q I can calculate E

How do I make it into a differential equation? — give me E, I can calculate ρ

Use the divergence theorem:
(also known as Green's theorem)

$$\boxed{\int_V \underbrace{(\nabla \cdot \underline{v})}_{\text{divergence}} dV = \underbrace{\oint_S \underline{v} \cdot d\underline{a}}_{\text{Flux of } \underline{v} \text{ through a surface}}}$$ Divergence theorem

divergence
go back to lecture 1:
divergence → 'spreading out' of vectors from a pt.

Flux of $\underline{v}$ through a surface
we spent half of last lecture discussing this

Analogy with fluid flow

$$\int \text{faucets within the volume} = \oint \text{flow out through the surface}$$

"There are 2 ways of determining "how much is being produced"
- count the faucets + record how much each one spreads/puts out
- go around the boundary + measure flow at each pt + add it up.

For now we will use this theorem statement w/o a proof:

$$\int_S \underline{E} \cdot d\underline{a} = \int_V (\nabla \cdot \underline{E})\, dV = \frac{Q_{enc}}{\varepsilon_0}$$

Rewriting $Q_{enc} = \int_V \rho\, dV$

$$\Rightarrow \int_V (\nabla \cdot \underline{E})\, dV = \frac{1}{\varepsilon_0} \int_V \rho\, dV$$

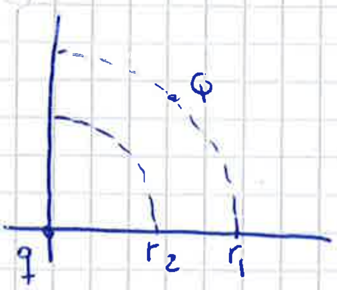
since this holds true for any volume integrands are =

$$\boxed{\nabla \cdot \underline{E} = \frac{1}{\varepsilon_0} \rho}$$

Gauss's law in differential form

Remember lecture 1? This is eq #1 from Maxwell's equations

...ctric potential



last time: How much work does it take to move Q from r_1 to r_2?

$$W_{12} = -\int_{r_1}^{r_2} \underline{F} \cdot d\underline{s}$$

Can I do something for W similar to what we did for $\underline{F}$?

$$\frac{W_{12}}{Q} = -\int_{r_1}^{r_2} \underline{E} \cdot d\underline{s}$$

In general we can define a potential difference $\Delta\varphi$ as the amount of work done per unit charge to move a test charge Q from point A to B.

$$\boxed{\Delta\varphi \equiv -\int_{P_1}^{P_2} \underline{E} \cdot d\underline{s}}$$ electric potential difference

Note that in this case we need 2 points for $\Delta\varphi$ to be meaningful. We always look @ differences. BUT we can pick one reference point, and measure differences with respect to that point. e.g. $P_1 = \infty$

$$\boxed{\varphi(\underline{r}) = -\int_{\infty}^{P_2} \underline{E} \cdot d\underline{s}}$$ electric potential

(Note: Usually the convention we pick is $\varphi(\infty) = 0$, this rule is not absolute)

Thoughts about φ:

* $[\varphi] \equiv$ Volts $= \frac{\text{Joules}}{\text{Coulomb}}$

* Potential obeys the superposition principle $\varphi = \varphi_1 + \varphi_2 + \dots$

* Don't be misled by the name 'potential', it is not the potential energy of the system. But they are related by:

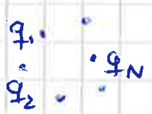
$$U = Q\varphi$$

* Why do we need to define yet another qty, isn't $\underline{E}$ sufficient?
→ φ is a scalar function ⇒ easier to work with than $\underline{F}$ or $\underline{E}$ (vector functions)
gives us another tool to derive $\underline{E}$

What are the potentials of standard charge distributions?

point charge

$$\varphi(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

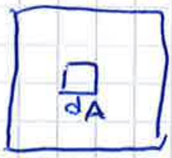


collection of charges

$$= \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{r_i}$$

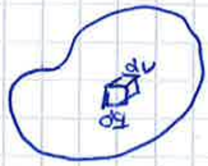
charges on a line

$$= \frac{1}{4\pi\varepsilon_0}\int_L \frac{\lambda\, dl}{r}$$



charges on a surface

$$= \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma\, dA}{r}$$



charges on a volume

$$= \frac{1}{4\pi\varepsilon_0}\int \frac{\rho\, dV}{r}$$

$$\Delta\varphi = \varphi_B - \varphi_A$$

$$= -\int_A^B \frac{q}{4\pi\varepsilon_0 r^2}\hat{r} \cdot d\underline{s}$$

$$= -\int_A^B \frac{q}{4\pi\varepsilon_0 r^2}\, dr$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right) \xrightarrow{\text{if } A=\infty} \varphi(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

Consider the pot diff $\Delta\varphi$ due to an infinitesimal charge dq

$$\Delta\varphi = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r}$$ and 'summing' integrating over all dq's in the volume

$$\varphi(\underline{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{dq}{r} = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho\, dV}{r}$$

...ving $\underline{E}$ from φ

for fun:
(For a more rigorous proof see Griffiths 2.2.4, 2.3.1)

We defined the potential as:

$$\Delta\varphi = -\int_{P_1}^{P_2} \underline{E}\cdot d\underline{s}$$

Consider the potential difference between 2 pts separated by an infinitesimal distance dr

$$d\varphi = -\int_{r}^{r+dr} \underline{E}\cdot d\underline{s} \sim -\underline{E}(\underline{r})\cdot d\underline{r} \quad (*)$$

From lecture 1 we know that for a function $\varphi(\underline{r})$ in cartesian coord. we can write:

$$d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz \equiv \underbrace{\left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\right)}_{\text{Recognize this? } \nabla\varphi \text{ (see lecture 1)}} \cdot (dx, dy, dz) = \nabla\varphi\cdot d\underline{r} \quad (**)$$

So now if we set equal (*) and (**):

$$\nabla\varphi\cdot d\underline{r} = -\underline{E}(\underline{r})\cdot d\underline{r} \Rightarrow \boxed{\underline{E} = -\nabla\varphi}$$

Note: ∇ del operator is applied to φ (scalar qty) $\xrightarrow[\text{results}]{}$ vector qty

Physically what does $\underline{E} = -\nabla\varphi$ mean?

If φ increases as a positive charge moves along some direction e.g. x
there is a non-vanishing component of $\underline{E}$ in the opposite direction. (e.g. $-E_x \neq 0$)

(analogy w/ gravity if grav potential increases when a mass is lifted by $h \Rightarrow \underline{F}$ pts downward)

Remember lecture 1? (Also see Apendix F Purcell, specifically F.2 for gradient definitions in curvilinear coordinates)

* In 1D

$$\nabla f(x) \equiv \frac{\partial f}{\partial x}\hat{x}$$

we have the derivative $\frac{df}{dx} \longrightarrow$ slope of function

the gradient describes the slope of the function & the direction of the change

* In 2D

$$\nabla f(x,y) \equiv \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y}$$

same interpretation as 1D but now we are looking at the rate of change of f in both directions.

Visualization of gradients

let's consider the potential

$$\varphi(x,y) = \sin(x)\sin(y) \Rightarrow \nabla\varphi = \cos(x)\sin(y)\hat{x} + \sin(x)\cos(y)\hat{y}$$

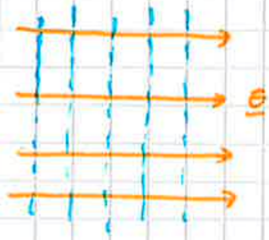
...ential surfaces

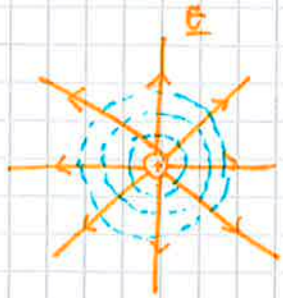
...urves characterized by a constant φ i.e $\varphi(r) = \varphi_0$ where φ_0 is a const.

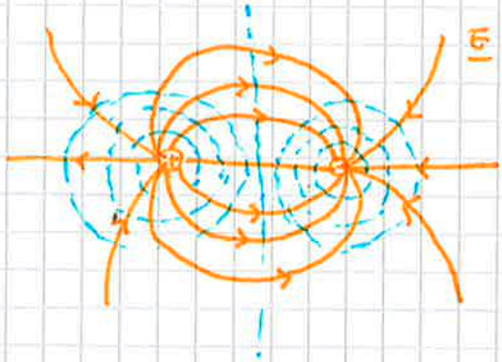
-> Equipotential lines are $\perp$ to the gradient $\Rightarrow$ they are $\perp$ to $\underline{E}$

$\Rightarrow$ field lines are $\perp$ to equipotentials

what are the equipotential curves for the following $\underline{E}$'s?







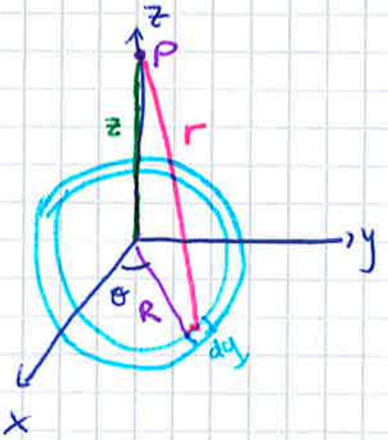
Physically what is an equipotential line?

$\varphi(\underline{r}) = \varphi_0 \leftarrow$ mathematically

* No work required to move a particle along an equipotential

"Working/operating" with the electric potential (an example)

Consider a uniformly charged ring of radius R. What is the electric potential at a distance z from the central axis? What is the electric field?



As usual let's consider a small differential element dl that carries a charge dq on the ring.

$\lambda = \frac{dq}{dl} \Rightarrow dq = \lambda dl = \lambda R d\theta$, using trig we also know that $r^2 = R^2 + z^2$

It's contribution to the electric potential is:

$$d\varphi = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda R d\theta}{\sqrt{R^2+z^2}}$$

Now if we integrate over the ring:

$$\varphi = \int d\varphi = \frac{1}{4\pi\varepsilon_0}\frac{\lambda R}{(R^2+z^2)^{1/2}}\int d\theta$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{\lambda R}{(R^2+z^2)^{1/2}}(2\pi) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{(R^2+z^2)^{1/2}}$$

since $Q = 2\pi R\lambda$ from the definition of λ

$\lambda = \frac{dQ}{dl} \Rightarrow \lambda dl = dQ$
$\Rightarrow 2\pi R\lambda = Q$

To obtain $\underline{E}$:

$$E_z = -\frac{\partial\varphi}{\partial z} = -\frac{\partial}{\partial z}\left[\frac{1}{4\pi\varepsilon_0}\frac{Q}{(R^2+z^2)^{1/2}}\right] = \frac{1}{4\pi\varepsilon_0}\frac{Qz}{(R^2+z^2)^{3/2}}$$

example: Calculate $\underline{E}$ from the electric potential!

→ Suppose φ due to a certain charge distribution can be written in cartesian coord as:

$$\varphi(x,y,z) = Ax^2y^2 + Bxyz \quad , \text{ where } A, B \text{ are constants.}$$

what is the associated $\underline{E}$?

$$E_x = -\frac{\partial \varphi}{\partial x} = -2Axy^2 - Byz$$

$$E_y = -\frac{\partial \varphi}{\partial y} = -2Ax^2y - Bxz \qquad \Rightarrow \underline{E} = (-2Axy^2 - Byz)\hat{x} - (2Ax^2y + Bxz)\hat{y} - (Bxy)\hat{z}$$

$$E_z = -\frac{\partial \varphi}{\partial z} = -Bxy$$

~~[illegible]~~

"Recipe" to solve problems involving calculating the electric potential

① Start w/the definition $d\varphi = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r}$

② Rewrite your charge element dq → $\lambda\, dl$, → $\sigma\, dA$, → $\rho\, dV$ depending on whether Q is distributed over a length, area or volume

③ Substitute dq into the expression for $d\varphi$

④ Specify an appropriate coordinate system & express the differential element (dl, dA or dV) and r in terms of the coordinates.

⑤ Rewrite $d\varphi$ in terms of the integration variable

⑥ Integrate to obtain φ

⑦ You can then calculate $\underline{E} = -\nabla\varphi$. To check your solution you can check your limits: i.e choose a point P which lies sufficiently far away from the charge distrib, in this limit, if the charge distribution is finite, $\underline{E}$ should behave as if the distrib were a pt charge & falls of $\propto \frac{1}{r^2}$