Lecture 22. Electromagnetic waves

Last time:

* Maxwell's equations (w/ Maxwell's fix to Ampère's law):

(i) $\nabla \cdot \underline{E} = \frac{1}{\epsilon_0}\rho$ (iii) $\nabla \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = 0$

(ii) $\nabla \cdot \underline{B} = 0$ (iv) $\nabla \times \underline{B} - \mu_0 \epsilon_0 \frac{\partial \underline{E}}{\partial t} = \mu_0 \underline{J}$

* Uncoupling Maxwell's equations (in vacuum) we found: (Remember in vacumm ρ & $\underline{J} = 0$)

$\nabla^2 \underline{E} = \mu_0 \epsilon_0 \frac{\partial^2 \underline{E}}{\partial t^2}$; $\nabla^2 \underline{B} = \mu_0 \epsilon_0 \frac{\partial^2 \underline{B}}{\partial t^2}$ Maxwell's equations satisfy the wave equation.

$$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

∴ Empty space supports the propagation of EM waves traveling at speed:

$$v = \frac{1}{(\epsilon_0 \mu_0)^{1/2}} = 3 \times 10^8 m/s \equiv c \quad \text{speed of light}$$

Today: Properties of waves. What are electromagnetic waves?

Given the wave equation

$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$ we can verify that a function $f = f(x \pm vt)$ is a solution for this eq:

Let $u = x \pm vt$ $\Rightarrow \frac{\partial u}{\partial x} = 1$ and $\frac{\partial u}{\partial t} = \pm v$. We will do it for the 1D case $\left(\frac{\partial^2 f}{\partial x^2} = \frac{1}{v}\frac{\partial^2 f}{\partial t^2}\right)$

Using the chain rule, the first two partial derivatives w/ respect to x are:

$$\frac{\partial f(u)}{\partial x} = \frac{\partial f}{\partial u}\cancelto{1}{\frac{\partial u}{\partial x}} = \frac{\partial f}{\partial u}$$

$$\Rightarrow \frac{\partial^2 f(u)}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u}\right) = \underbrace{\frac{\partial^2 f}{\partial u^2}\cancelto{1}{\frac{\partial u}{\partial x}}}_{\text{chain rule}} = \frac{\partial^2 f}{\partial u^2} \quad (*)$$

Remember the chain rule: If f depends on u and u depends on x:

$$\frac{df}{dx} = \frac{df}{du}\cdot\frac{du}{dx}$$

We can also calculate the time derivatives:

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial u}\cancelto{\pm v}{\frac{\partial u}{\partial t}} = \pm v \frac{\partial f}{\partial u}$$

$$\Rightarrow \frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left(\pm v \frac{\partial f}{\partial u}\right) = \pm v \frac{\partial^2 f}{\partial u^2}\cancelto{\pm v}{\frac{\partial u}{\partial t}} = v^2 \frac{\partial^2 f}{\partial u^2} \quad (**)$$

Now, we have that the wave eq. in 1D:

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$ subs (*) and (**) in this eq:

$$\frac{\partial^2 f}{\partial u^2} = \frac{1}{v^2}\left(v^2 \frac{\partial^2 f}{\partial u^2}\right) \Rightarrow \text{identity}$$ ∴ $f = f(x \pm vt)$ is a solution to the wave equation.

The wave equation is an example of a linear differential equation, so if f_1 and f_2 are solutions to this equation, then $f = f_1 \pm f_2$ is also a solution

So

In its most general form the 1D wave equation can be written as:

$$\frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial x^2}$$

and has a general solution:

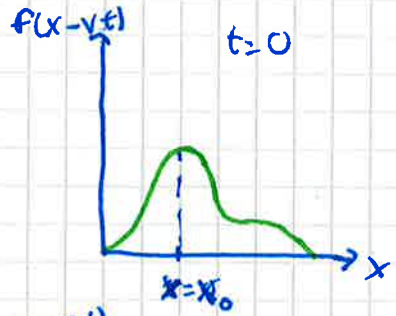
$$\psi(x,t) = f(x-vt) + g(x+vt)$$

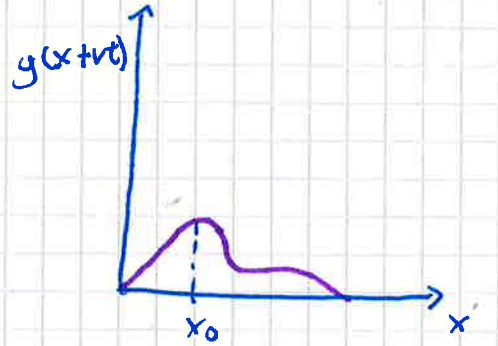
Viewing x as a position variable & t as time:

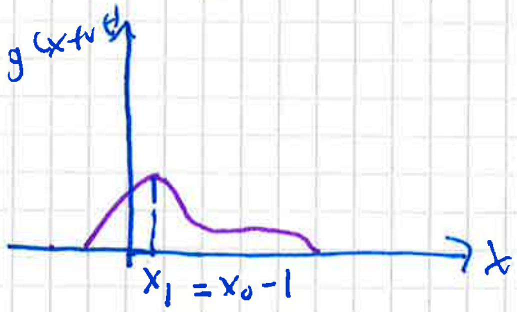
$f(x-vt)$
- wave moving with velocity v in the $+x$ direction
- The entire profile of f as a function of x at $t=0$ will be shifted uniformly towards $+x$ by an amount v at $t=1$

$g(x+vt)$
- wave moving with velocity v in the $-x$ direction
- The entire profile of g as a function of x at $t=0$ will be shifted uniformly towards $-x$ by an amount v at $t=1$

Because f & g are arbitrary functions the traveling waves they describe don't even need to be periodic, they could be entirely irregular. Moreover there is no requirement f & g have any relationship to each other.







travelling wave in the $+x$ direction with velocity v

traveling wave in the $-x$ direction with velocity v

Example: A standing wave (Demo 210)

A special case of the general solution is a situation where we have:

- $f(x,t) = A_1 \sin(k_1 x - \omega_1 t)$ and

$y(x,t) = A_2 \sin(k_2 x + \omega_2 t)$ where $k_i = \frac{2\pi}{\lambda_i} = \frac{2\pi f_i}{v_i} = \frac{\omega_i}{v_i} \equiv$ wavenumber

For simplicity we will assume:

$A_1 = A_2 = 1$
$k_1 = k_2 = k$
$\omega_1 = \omega_2 = \omega$

with this we have:

$f(x,t) = \sin(kx - \omega t)$
$g(x,t) = \sin(kx + \omega t)$

the solution to the wave equation becomes:

$$\psi(x,t) = \sin(kx - \omega t) + \sin(kx + \omega t)$$

Using the identity:

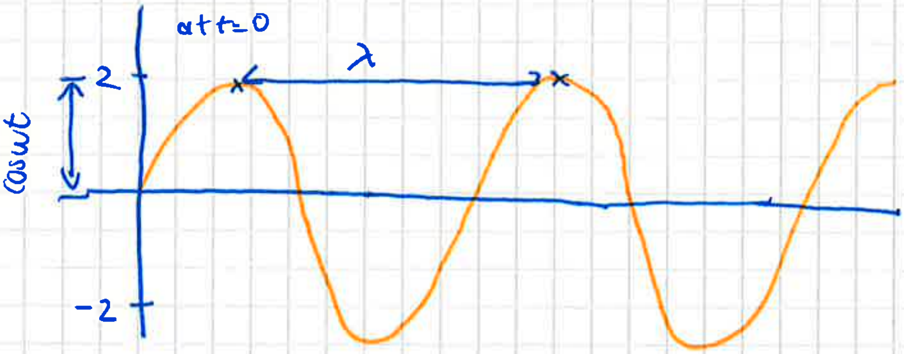
$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

We can write the expression for ψ as:

$$\psi(x,t) = 2\sin kx \cos\omega t$$

a standing wave : the time evolution of the wave's profile in x is an oscillation in amplitude, with the wave pattern not moving in either direction.

For a standing wave the nodes (i.e points where $\psi = 0$) are stationary in time, while in a traveling wave they are moving at velocity $v = \frac{\omega}{k} = \frac{\lambda\omega}{2\pi}$



the amplitude of the wave ~~way~~ $\cos\omega t$ will change as a function of t but nodes will remain constant.

This standing wave is the result of the superposition of two equal waves traveling in opposite directions.

Plane waves

So far we have been discussing solutions to the 1D wave equation. However, the wave equations ~~that~~ for E & B that we found by uncoupling Maxwell's equations are in 3D.

$$\nabla^2 \underline{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \underline{E}}{\partial t^2} \quad ; \quad \nabla^2 \underline{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \underline{B}}{\partial t^2}$$ where $\nabla^2 = \left(\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2}\right)$ in cartesian coord.

We found that solutions to the wave equation are given by linear combinations of functions of the form:

$$\underset{1D}{f(x \pm vt)} \Rightarrow \underset{3D}{f(\underline{r} \pm v\hat{k}t)}$$ where $\hat{k}$ is the propagation direction of the wave.

~~Here~~ To facilitate the physical interpretation ~~of~~ we have ~~add~~ $\underline{k}$, the wavevector, ~~to~~ in the argument of f:

argument of f: $\underline{r} \pm v\hat{k}t$

adding the wavevector:

$$\underline{k}\cdot(\underline{r} \pm v\hat{k}t) = \underline{k}\cdot\left(\underline{r} \pm \frac{\omega}{k}\hat{k}t\right) = \underline{k}\cdot\underline{r} \pm \omega t$$ where $\omega = vk$ and $\omega = \frac{2\pi v}{\lambda}$

So then we have that solutions to the wave equation have the form:

$f(\underline{k}\cdot\underline{r} \pm \omega t)$ which are periodic functions.

Fourier theorem: Any periodic function can be expressed as a linear combination of sin & cos functions.

sin and cos are the building blocks of all waves

Plane waves can be written in the most general form as:

$$\underline{E} = \underline{E}_0 \sin(\underline{k}\cdot\underline{r} - \omega t) = \underline{E}_0 \sin(k_x x + k_y y + k_z z - \omega t)$$

$$\underline{B} = \underline{B}_0 \sin(\underline{k}\cdot\underline{r} - \omega t) = \underline{B}_0 \sin(k_x x + k_y y + k_z z - \omega t)$$

where
$\underline{k}$ = wavevector
$|\underline{k}|$ = wavenumber
$\hat{k}$ = propagation direction

Let's unpack a bit more what these expressions mean.

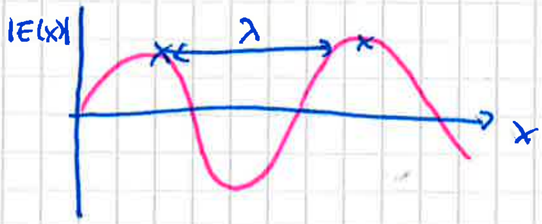
For simplicity we will choose a coordinate system such that the wave vector $\underline{k}$ is oriented // to the x axis, in this case the plane wave solution for $\underline{E}$ becomes:

$$\underline{E} = \underline{E}_0 \sin(k_x x - \omega t)$$

How does this wave vary in space and time?

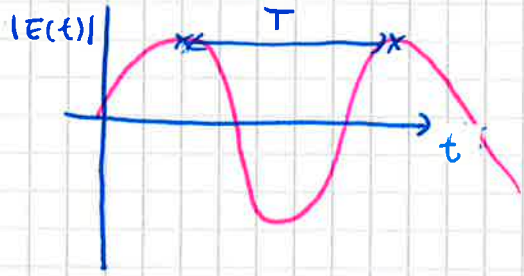
Let's consider only the spatial variation of the wave, e.g. $t=0$. In this case:

$$\underline{E} = \underline{E}_0 \sin(k_x x)$$



Now if we consider the time variation; e.g. $x=0$:

$$\underline{E} = \underline{E}_0 \sin(\cancel{\omega t})$$



So the relations between variables are:

$$\omega = \frac{2\pi}{T} = 2\pi\nu \quad ; \quad \omega = ck \quad ; \quad \lambda\nu = c$$

where $\nu \equiv$ frequency $= \frac{1}{T}$
and we have used the fact that for light the speed of propagation (v) is c, the speed of light.

We know that plane waves satisfy the wave equation but is ANY plane wave a possible EM wave? ~~...~~ for it to be an EM wave it must satisfy Maxwell's equations.

What types of plane waves satisfy Maxwell's equations?

(i) $\nabla\cdot\underline{E} = 0$ Let's calculate this for our general solution

$$\nabla\cdot\underline{E} = \nabla\cdot\left[\underline{E}_0 \sin(k_x x + k_y y + k_z z - \omega t)\right]$$

$$= (E_{0x}k_x + E_{0y}k_y + E_{0z}k_z)\cos(k_x x + k_y y + k_z z - \omega t)$$

$$= \underline{k}\cdot\underline{E}_0 \cos(\underline{k}\cdot\underline{r} - \omega t)$$

what conditions need to be met in order for this to be zero?

$\nabla \cdot \underline{E} = 0$ when $\underline{K} \cdot \underline{E}_0 = 0$ $\Rightarrow$ $\boxed{\underline{E} \text{ is } \perp \text{ to the wave's propagation direction}}$

(ii) $\nabla \cdot \underline{B} = 0$

$\nabla \cdot \underline{B} = \underline{k} \cdot \underline{B}_0 \cos(\underline{k} \cdot \underline{r} - \omega t)$ so

$\nabla \cdot \underline{B} = 0$ when $\underline{k} \cdot \underline{B}_0 = 0$ $\Rightarrow$ $\boxed{\underline{B} \text{ is } \perp \text{ to the wave's propagation direction}}$

(iii) $\nabla \times \underline{E} = \mu_0 \varepsilon_0 \dfrac{\partial \underline{B}}{\partial t}$

The time derivative does not change the direction of B (only its magnitude) $\Rightarrow$ $\boxed{\underline{E} \perp \underline{B}}$

because we have a plane wave this is not allowed.

(iv) $\nabla \times \underline{B} = \varepsilon_0 \mu_0 \dfrac{\partial \underline{E}}{\partial t}$ $\Rightarrow$ $\underline{E} \perp \underline{B}$

same conclusion as (iii)

$\therefore$ $\underline{k} \perp \underline{B} \perp \underline{E} \perp \underline{k}$

EM plane waves are transverse since $\underline{E}$ & $\underline{B}$ are perpendicular to the direction of propagation

If we actually calculate this we will get another constraint linking the amplitudes of B & E:

$\nabla \times \underline{B} = \mu_0 \varepsilon_0 \dfrac{\partial \underline{E}}{\partial t}$. Let's calculate the right hand side of this equation first:

$$\frac{1}{c}\frac{\partial \underline{E}}{\partial t} = \mu_0 \varepsilon_0 \underline{E}_0 \cos(\underline{k} \cdot \underline{r} - \omega t)(-\omega) = -\mu_0 \varepsilon_0 \omega \underline{E}_0 \cos(\underline{k} \cdot \underline{r} - \omega t) = -\underline{E}_0 \mu_0 \varepsilon_0 c k \cos(\underline{k} \cdot \underline{r} - \omega t)$$
$$= -\underline{E}_0 (\mu_0 \varepsilon_0)^{1/2} k \cos(\underline{k} \cdot \underline{r} - \omega t)$$

And the left hand side; using that $\nabla \times (f\underline{A}) = f(\nabla \times \underline{A}) - \underbrace{\underline{A} \times (\nabla f)}_{+\nabla f \times \underline{A}}$ and the fact that $B_0 = \text{const}$:

$$\nabla \times \underline{B} = \nabla \times [\underline{B}_0 \sin(k_x x + k_y y + k_z z - \omega t)]$$
$$= \underbrace{(\nabla \times \underline{B}_0) \sin(\underline{k} \cdot \underline{r} - \omega t)}_{0 \text{ since } B_0 = \text{const}} + \nabla \sin(k_x x + k_y y + k_z z - \omega t) \times \underline{B}_0$$
$$= (k_x \hat{x} + k_y \hat{y} + k_z \hat{z}) \cos(k_x x + k_y y + k_z z - \omega t) \times \underline{B}_0$$
$$= (\underline{k} \times \underline{B}_0) \cos(\underline{k} \cdot \underline{r} - \omega t) \quad **$$

Now equating both sides * & ** :

$$(\underline{k} \times \underline{B}_0) \cos(\underline{k} \cdot \underline{r} - \omega t) = -(\mu_0 \varepsilon_0)^{1/2} k \cos(\underline{k} \cdot \underline{r} - \omega t)(\underline{E}_0)$$
$$\Rightarrow +(\underline{k} \times \underline{B}_0) = -\underline{E}_0 (\mu_0 \varepsilon_0)^{1/2} k$$
$$\Rightarrow (\hat{k} \times \underline{B}_0) = -\underline{E}_0 (\mu_0 \varepsilon_0)^{1/2}$$

$\Leftrightarrow$ $\underline{k}$, $\underline{E}$ and $\underline{B}$ are right handed orthogonal vectors.

and in terms of their magnitude:

$$|\hat{k} \times \underline{B}_0| = |-\underline{E}_0 (\mu_0 \varepsilon_0)^{1/2}| \Rightarrow \boxed{E_0 = \frac{\pm B_0}{(\mu_0 \varepsilon_0)^{1/2}}}$$

and using $\mu_0 \varepsilon_0 = \dfrac{1}{c^2}$

$$\boxed{E_0 = \pm c B_0}$$

Doing a similar analysis for equation (iii) would yield:

$$E_0 = vB_0 \Rightarrow \boxed{v = \pm c}$$

Demo 248. Measurement of the speed of light

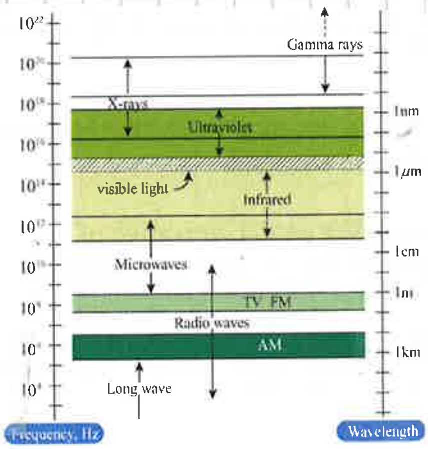
Conclusions:

- Planar electromagnetic waves are transverse (direction of $\underline{E}$ & $\underline{B}$ is $\perp$ to the direction of propagation $\underline{k}$)
- $\underline{E} \perp \underline{B} \perp \underline{k}$; $\underline{k}, \underline{E}, \underline{B}$ are right handed orthogonal vectors
- EM waves propagate at the speed of light c
- The relationship between the magnitudes of $\underline{E}$ & $\underline{B}$ is:

$$E_0 = \pm c B_0$$

- Planar EM waves are referred to as "monochromatic", different frequencies will correspond to different "colors"

You have for sure seen a poster or "schematic" of the EM wave spectrum. Now you can look at it in new light.



Demo 577 microwaves

To see nice illustrations of plane EM waves see Purcell p. 439

Polarization of EM waves

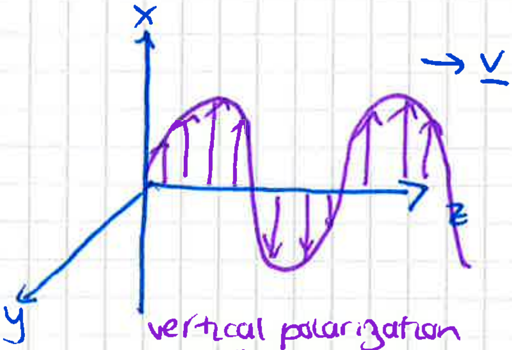
We have discussed that EM waves are transverse since their direction of propagation is perpendicular to the direction of $\underline{E}$ & $\underline{B}$. ~~So~~ their direction of propagation is $\perp$ to the planes in which $\underline{E}$, and $\underline{B}$ oscillate.

Polarization is a property of transverse waves that especifies the ~~direction of oscillation~~ geometrical orientations of the oscillations.

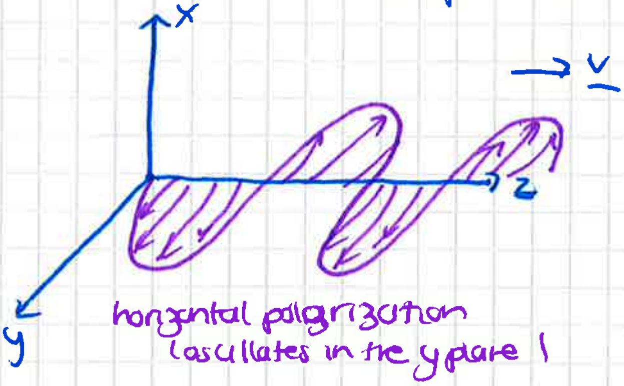
Let's go back to ~~demo 240~~ a simple demo. Propagation of waves on a string.

The wave can be transversal but oscillate either in the horizontal or vertical direction:

For nice schematics see Griffiths p. 392

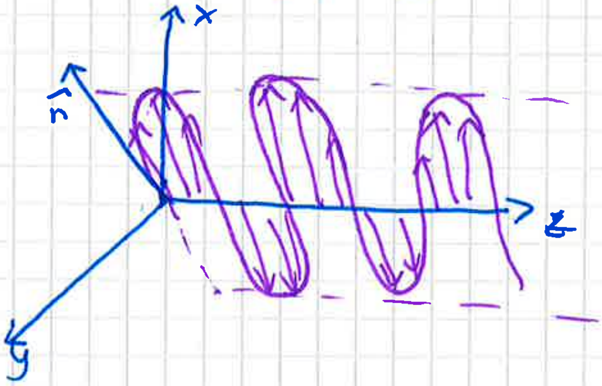


vertical polarization (oscillates in the x plane



horizontal polarization (oscillates in the y plane!

and in general it can oscillate in any plane described by the polarization vector $\hat{n}$.



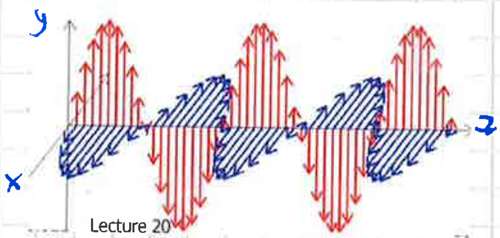
$\hat{n}\cdot\hat{z}=0$ because we have transverse waves.

Polarization of EM waves

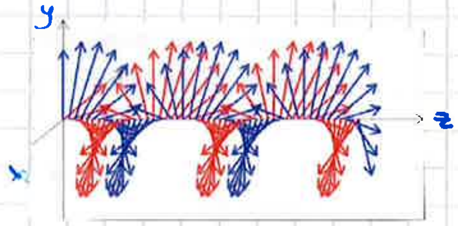
If the directions of $\underline{E}_0$ and $\underline{B}_0$ are constant in time, the wave is linearly polarized.

If the directions of $\underline{B}_0$ + $\underline{E}_0$ rotate at a frequency ω, the wave is circularly polarized. We call this circular polarization since $\underline{E}$ + $\underline{B}$ describe circles over time.

linearly polarized



circularly polarized.



$\underline{E}=E_0\cos(kx-\omega t)\hat{y}$

$\underline{B}=B_0\cos(kx-\omega t)\hat{z}$

$\underline{E}=E_0\hat{x}\sin(kz-\omega t)+E_0\hat{y}\cos(kz-\omega t)$

$\underline{B}=B_0\hat{y}\sin(kz-\omega t)-B_0\hat{x}\cos(kz-\omega t)$

We only need to know either B or E to determine the polarization, by convention the direction of polarization comes from $\underline{E}$.

Polarization of random light

Light coming from a bulb, the sun, etc is not polarized. It is the superposition of many plane waves, each one with its own polarization.

$$E_{random}=\sum_i E_0(\hat{x}\cos\theta_i+\hat{y}\sin\theta_i)\cos(kz-\omega t)$$

We can polarize light using a polarizer. One example of a polarizer are your sunglasses. Polarizing ~~lee~~ polarized glasses reduce the glare from sunlight reflecting off surfaces like water or roads.

There's also polaroids: sheet of plastic embedded ~~inside~~ with organic molecules ~~each~~ aligned in one direction. ⇒ They can carry current in one direction.

When linearly polarized light hits the material

- If $\vec{E}$ is aligned with the orientation of the molecules charges will move to due to it and a current will be generated ~~...~~. $\bar{e}$ moving in this current will collide w other particles so light is stopped".
- If $\vec{E}$ is $\perp$ to the orientation of the molecules ("preferred direction") charges won't move anywhere → all light will go through.

∴ Polaroids are transparent to light polarized // to their preferred direction and opaque to light polarized in the direction $\perp$ to their preferred direction.

What happens when light is polarized in a direction in between the preferred direction & $\perp$?

For example, if the light is polarized along the $\hat{x}$ axis and the polaroid is oriented at an angle θ:

$$\vec{E} = E_0 \cos(kz - \omega t)\,\hat{x}$$

$$\hat{p} = \hat{x}\cos\theta + \hat{y}\sin\theta$$

light will go through partially since $\vec{E}$ has a component // to the preferred direction of the polaroid.

The field that comes out of the polaroid will be an overlap between the incoming $\vec{E}$ & the polaroid's orientation.

$$|E_{out}| = \vec{E}\cdot\hat{p} = E_0\cos(kz-\omega t)(\hat{x}\cos\theta + \hat{y}\sin\theta)\cdot\hat{x} = E_0\cos\theta\cos(kz-\omega t)$$

$$E_{out} = |\vec{E}\cdot\hat{p}|\,\hat{p}$$ parallel to the polaroid's orientation

∴ Polaroids reduce the amplitude of linearly polarized light by $\cos\theta$ (where θ is the ∠ between ~~the~~ $\vec{E}$ & the polaroids orientation) and rotates the orientation of $\vec{E}$ by θ

For light with a random polarization (e.g. from a bulb, the sun)

$$E_{random} = \sum_i E_0(\hat{x}\cos\theta_i + \hat{y}\sin\theta_i)\cos(kz-\omega t)$$

will become linearly polarized when it passes through a polaroid. If the polaroid is oriented parallel to the x axis:

$$E_{out} = \sum_i E_0(\hat{x}\cos\theta_i)\cos(kz-\omega t) = E_0\hat{x}\cos(kz-\omega t)\sum_i\cos\theta$$

∴ Polaroids can be used to produce linearly polarized light
intensity of light will be reduced.

Demo 265 . Polarizers