Lecture 21. Maxwell's equations

Last time:

* AC circuits: → Circuit with a sinusoidal voltage source $V(t) = V_0 \sin \omega t$
  ↳ current is given by:

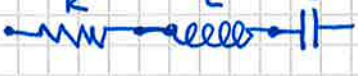
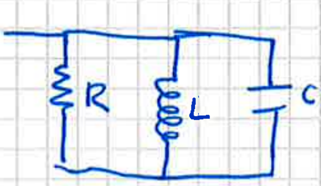
$$I(t) = I_0 \sin(\omega t - \phi) \quad \text{where } I_0 \equiv \text{amplitude}, \quad \phi \equiv \text{phase}$$

* For an AC circuit with ONLY one element:

	Resistance/ Reactance	Current amplitude	Phase angle ϕ
 R	R	$I_{R0} = \frac{V_0}{R}$	0
 L	$X_L = \omega L$	$I_{L0} = \frac{V_0}{X_L}$	$\pi/2$ current lags voltage by $90°$ $(\pi/2)$
 C	$X_C = \frac{1}{\omega C}$	$I_{C0} = \frac{V_0}{X_C}$	$-\pi/2$ current leads voltage by $90°$ $(\pi/2)$

$X_L \equiv$ inductive reactance ; $X_C \equiv$ capacitive reactance

* For an RLC circuit:

	Impedance Z	current amplitude	Phase angle
 R L C	$Z = \sqrt{R^2 + (X_L - X_C)^2}$	$I_0 = \frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}}$	$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$
 R L C	$\frac{1}{Z} = \sqrt{\frac{1}{R^2} + \left(\frac{1}{X_C} - \frac{1}{X_L}\right)^2}$	$I_0 = V_0\sqrt{\frac{1}{R^2} + \left(\frac{1}{X_C} - \frac{1}{X_L}\right)^2}$	$\phi = \tan^{-1}\left[R\left(\omega C - \frac{1}{\omega L}\right)\right]$

* The root mean square (rms) voltage and current in an AC circuit are given by

$$V_{rms} = \frac{V_0}{\sqrt{2}} \; ; \quad I_{rms} = \frac{I_0}{\sqrt{2}}$$

Today: Maxwell's equations.

So far we have encountered the following laws, specifying the div and curl of $\underline{E}$ & $\underline{B}$.

(i) $\nabla \cdot \underline{E} = \frac{1}{\varepsilon_0}\rho$; Gauss's law (ii) $\nabla \cdot \underline{B} = 0$; "No name"

(iii) $\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$; Faraday's law (iv) $\nabla \times \underline{B} = \mu_0 \underline{J}$; Ampère's law

Can you see what is wrong with these equations?

There is a mathematical inconsistence with these equations.

We have used countless times that the divergence of a curl is zero. What happens if you ~~take the~~ calculate the divergence of $(\nabla \times \underline{E})$ and $(\nabla \times \underline{B})$?

$$\nabla\cdot(\nabla\times\underline{E}) = \nabla\cdot\left(-\frac{\partial \underline{B}}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\nabla\cdot\underline{B}\right) = 0 \quad ✓$$

because $\nabla\cdot\underline{B}$ is zero

Now if we calculate the $\nabla\cdot(\nabla\times\underline{B})$:

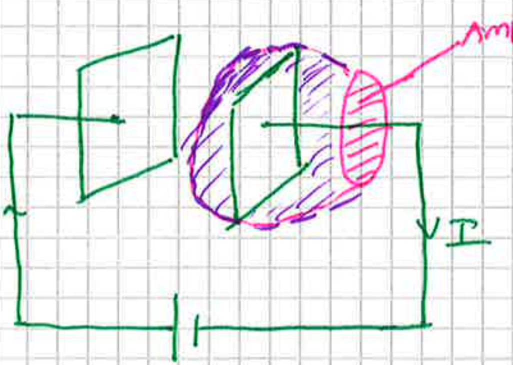
$$\nabla\cdot(\nabla\times\underline{B}) = \nabla\cdot(\mu_0\underline{J}) = \mu_0(\nabla\cdot\underline{J})$$

For steady currents it is true that $\nabla\cdot\underline{J}=0$ HOWEVER beyond magnetostatics where $\underline{J}$ is not steady this eq is NOT right.

and $\nabla\cdot\underline{J} = \frac{-\partial\rho}{\partial t} \neq 0$ if we have a non-steady current

Remember this equation is a statement of Ampère's law. What is wrong with Ampère's law when $\underline{J}$ is nonsteady?

Let's look at an example where $\underline{J}$ is nonsteady i.e. charges 'pile up somewhere'. The prototypical example of this situation is a capacitor.



Suppose we are charging up this capacitor.

In integral form Ampère's law reads:

$$\oint \underline{B}\cdot d\underline{l} = \mu_0 I_{enc} \text{ and } \oint \underline{B}\cdot d\underline{l} \overset{\text{stokes}}{=} \int_s \nabla\times(\underline{B}\cdot d\underline{a})$$

suppose I want to apply this to the loop shown in the fig.

How do I determine I_{enc}? I_{enc} is the total current passing through the loop. For the surface enclosed by the loop I is ~~now~~ flowing and $\nabla\times\underline{B} = \mu_0\underline{J}$. What happens if we consider now a surface on the capacitor gap?

~~We can~~ By stokes's theorem any surface spanning C can be used to calculate I_{enc}. So we can use the balloon shaped surface. However here, there is NO current flowing (because it includes the gap in the capacitor). However the curl of $\underline{B}$ has to be NON zero to ~~violate~~ stokes's theorem. There must be a "missing" term in the curl of $\underline{B}$ that does NOT depend on $\underline{J}$ i.e:

$$\nabla\times\underline{B} = \mu_0\underline{J} + (?)$$

Another equiv statement for this is that for non-steady currents "current enclosed by the loop" is ill-defined, since it depends on the surface we choose.

~~Would~~ In retrospect since Ampère's law was derived from Biot-Savart's law there is no surprise that it doesn't hold for nonsteady currents. However, experimentally there was no inconsistency with it, and the problem was purely mathematical. so it got "fixed" by purely theoretical arguments

How Ampère's law was "fixed"?

We need to find what is missing, i.e. what is (?) in the eq above. Let's name this term $\underline{F}$.

$$\nabla\times\underline{B} = \mu_0\underline{J} + \underline{F}$$

. What is $\underline{F}$ such that $\nabla\cdot(\nabla\times\underline{B}) = 0$?

$$\nabla\cdot(\nabla\times\underline{B}) = \nabla\cdot(\mu_0\underline{J} + \underline{F}) = 0$$

$$\Rightarrow \mu_0\nabla\cdot\underline{J} + \nabla\cdot\underline{F} = 0 \Rightarrow \mu_0\nabla\cdot\underline{J} = -\nabla\cdot\underline{F}$$

$$\Rightarrow \nabla\cdot\underline{F} = \mu_0\left(\frac{\partial\rho}{\partial t}\right) \quad (*)$$

continuity equation

This looks similar to Gauss's law. (eq i from page 1)

Let's take the time derivative from Gauss's law:

$$\frac{\partial}{\partial t}\left(\nabla\cdot\underline{E}\right) = \frac{\partial}{\partial t}\left(\frac{1}{\varepsilon_0}\rho\right) = \frac{1}{\varepsilon_0}\frac{\partial \rho}{\partial t}$$

$$\Rightarrow \nabla\cdot\frac{\partial \underline{E}}{\partial t} = \frac{1}{\varepsilon_0}\underbrace{\frac{\partial \rho}{\partial t}}_{= -\frac{\nabla\cdot\underline{F}}{\mu_0} \text{ from eq } (*)} = \frac{1}{\varepsilon_0\mu_0}\nabla\cdot\underline{F}$$

$$\therefore \nabla\cdot\frac{\partial \underline{E}}{\partial t} = \frac{1}{\varepsilon_0\mu_0}\nabla\cdot\underline{F} \Rightarrow \boxed{\underline{F} = \varepsilon_0\mu_0\left(\frac{\partial \underline{E}}{\partial t}\right)}$$

This gives us the generalized Ampere's equation:

$$\boxed{\nabla\times\underline{B} = \mu_0\underline{J} + \varepsilon_0\mu_0\left(\frac{\partial \underline{E}}{\partial t}\right)} \quad **$$

What does this mean? ~~Physically~~

- As far as magnetostatics is concerned this changes nothing since there $\frac{\partial \underline{E}}{\partial t} = 0$ so Ampere's law as we knew it holds.
- This term is hard to measure in the lab and it's not dominant in ordinary experiments. HOWEVER, it's really important when discussing the propagation of electromagnetic waves.
- This change is "nice" from a theoretical standpoint since makes things symmetric

$\boxed{\text{A changing electric field, induces a magnetic field}}$

Of course ~~now~~ these arguments are not physically sound, they are only suggestive. The real confirmation of this came after Hertz experiments on EM waves.

Maxwell called this term that was added a displacement current:

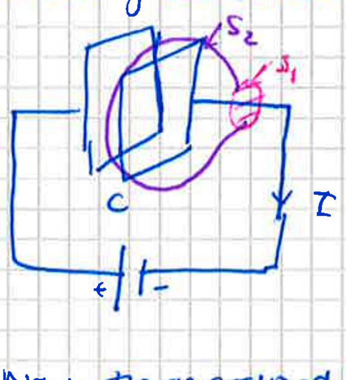
$$\boxed{\underline{J}_d \equiv \varepsilon_0\frac{\partial \underline{E}}{\partial t}}$$

With this the generalized Ampère's law becomes: $\nabla\times\underline{B} = \mu_0\underline{J} + \mu_0\underline{J}_d = \mu_0\left(\underline{J} + \underline{J}_d\right)$

In this case $\underline{J}_d$ is NOT an actual current: physically it does NOT describe charges flowing through some region.

But it 'acts' as a current in the sense that if we have change in $\underline{E}$ we can treat its effect as if due to a real current $\underline{I}_d$.

Let's see if we can solve the paradox in our circuit using this ~~displacem~~ definition. For clarity we'll draw again the circuit:



If the capacitor's plates are close together we can approximate the electric field ~~out~~ between them as:

$$E = \frac{1}{\varepsilon_0}\sigma = \frac{1}{\varepsilon_0}\frac{Q}{A}$$ where Q is the charge on the plate & A its area.

So between plates:

$$\frac{\partial E}{\partial t} = \frac{1}{\varepsilon_0 A}\frac{dQ}{dt} = \frac{1}{\varepsilon_0 A} I \quad (\text{☼})$$

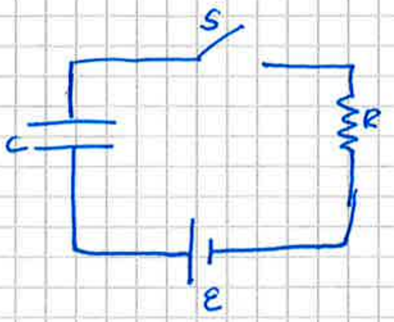
Now, the generalized Ampère equation (**) reads in integral form:

$$\oint \underline{B}\cdot d\underline{l} = \mu_0 I_{enc} + \mu_0\varepsilon_0\int\left(\frac{\partial \underline{E}}{\partial t}\right)\cdot d\underline{a}$$

For our capacitor if we choose the flat surface S_1 then $\underline{E} = 0$ and $I_{enc} = I$. If we choose the "balloon surface" S_2 then $I_{enc} = 0$ but $\int\left(\frac{\partial \underline{E}}{\partial t}\right)\cdot d\underline{a} = \frac{I}{\varepsilon_0}$ using eq ☼: we fixed the inconsistency!

The importance of displacement currents.

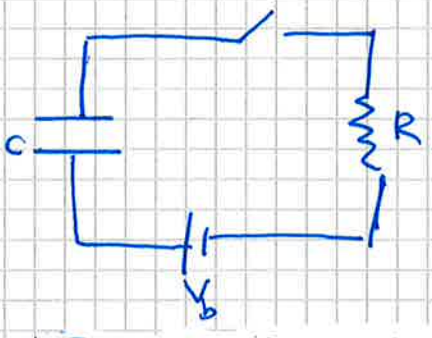
Remember where we studied circuits that had capacitors in them?



We said we had the same current flowing through ALL circuit elements.

- How is it possible if there is no current flowing through the capacitor?!
- → Displacement currents "continues" the "real" current across the capacitor ensuring the validity of Kirchhoff's laws.

EXAMPLE: Calculate $\underline{B}$ inside the plates of the following circuit!



Consider an RC circuit connected to a direct current voltage source supplying V_b. What is the magnetic field inside the plates of the capacitor? Assume you have cylindrical plates of radius a.

① First calculate $\underline{E}$ between the capacitor plates:

$$\underline{E} = \frac{1}{\varepsilon_0}\sigma = \frac{1}{\varepsilon_0}\frac{Q(t)}{A}$$ where Q is the charge & A the area of the plates

$$= \frac{1}{\varepsilon_0}\frac{Q(t)}{\pi a^2}$$

subs A for cylindrical plates of radius a

2) From the last expression we can calculate the displacement current density:

$$\underline{J}_d = \varepsilon_0 \frac{\partial E}{\partial t} = \frac{\varepsilon_0}{\varepsilon_0 \pi a^2}\underbrace{\frac{\partial Q}{\partial t}}_{\substack{I \\ \text{by definition}}} = \frac{I(t)}{\pi a^2}$$

3) Remembering that for a capacitor:

$$I(t) = \frac{V_b}{R} e^{-t/RC} \Rightarrow J_{\underline{d}} = \frac{1}{\pi a^2}\frac{V_b}{R} e^{-t/RC}$$

4) Now the magnetic field inside the plate is given by Ampère's law (the generalized form):

$$\oint \underline{B}\cdot d\underline{l} = \mu_0 \cancel{I_{enc}}^{0} + \mu_0 \int \underline{J}_d \cdot d\underline{a} = \mu_0 \int \underline{J}_d \cdot d\underline{a} = \frac{\mu_0}{\pi a^2}\cdot\frac{V_b}{R}e^{-t/RC}\underbrace{(\pi r^2)}_{\int da}$$

$$\Rightarrow B = \frac{\mu_0 V_b e^{-t/RC}}{\underbrace{(2\pi r)}_{\int dl} R\,(\pi a^2)}\pi r^2 =$$

$$\therefore B(t) = \frac{\mu_0 r V_b e^{-t/RC}}{2\pi a^2 R}$$

With the "fix" in Ampère's law we have finally arrived at Maxwell's equations as we know them:

$$\boxed{\begin{array}{ll} \text{(i) } \nabla\cdot\underline{E} = \frac{1}{\varepsilon_0}\rho & \text{(iii) } \nabla\times\underline{E} + \frac{\partial \underline{B}}{\partial t} = 0 \\ \text{(ii) } \nabla\cdot\underline{B} = 0 & \text{(iv) } \nabla\times\underline{B} - \mu_0\varepsilon_0\frac{\partial \underline{E}}{\partial t} = \mu_0 \underline{J} \end{array}}$$

- together with Lorentz's Force law $\underline{F} = q(\underline{E} + \underline{v}\times\underline{B})$ they contain ALL the physics you've learned in this course.
- Writing the equations using this ~~notation~~ form suggested by Griffiths emphasizes that the sources for $\underline{E}$ & $\underline{B}$ are charges and currents.
- If you think about it the sources for the terms $\frac{\partial \underline{E}}{\partial t}$ and $\frac{\partial \underline{B}}{\partial t}$ are also charges & currents.

We can re-write ~~the~~ Maxwell's equations in integral form as:

$$\boxed{\begin{array}{ll} \text{(i) } \oint \underline{E}\cdot d\underline{a} = \frac{Q}{\varepsilon_0} & \text{(iii) } \oint \underline{E}\cdot d\underline{l} = -\frac{d\Phi_B}{dt} \\ \text{(ii) } \oint \underline{B}\cdot d\underline{a} = 0 & \text{(iv) } \oint \underline{B}\cdot d\underline{l} = \mu_0 I + \mu_0\varepsilon_0\frac{d\Phi_E}{dt} \end{array}}$$

What is the physical meaning of these equations?

(i) Electric flux through a closed surface is proportional to the charge enclosed.

(ii) The total magnetic flux through a closed surface is zero $\Leftrightarrow$ Non existence of magnetic monopole.

(iii) Changing magnetic flux produces an electric field.

(iv) Electric current and changing electric flux produces a magnetic field.

Are there any consequences we can predict from these equations or is it just a nice way to summarize everything we knew before? These equations actually predict the existence of electromagnetic waves.

Maxwell's equations in vacumm

In vacumm (i.e $\rho = 0$ and $\underline{J} = 0$) Maxwell's equations become:

(i) $\nabla\cdot\underline{E} = 0$ (iii) $\nabla\times\underline{E} = -\frac{\partial \underline{B}}{\partial t}$

(ii) $\nabla\cdot\underline{B} = 0$ (iv) $\nabla\times\underline{B} = \mu_0\varepsilon_0\frac{\partial \underline{E}}{\partial t}$

In this form we can appreciate that except for a minus sign, the equations for $\underline{B}$ & $\underline{E}$ are symmetric.

An $\underline{E}$ varying in time will create $\underline{B}$ and viceversa.

Let's analyze the solutions for these equations:

To solve them, we need to uncouple them i.e show equations that only contain $\underline{E}$ or $\underline{B}$. We will do this by taking the curl of equations (iii) and (iv) and using the rest of the equations accordingly as needed.

Let's start taking the curl of eq (iii):

$$\nabla\times(\nabla\times\underline{E}) = \nabla(\nabla\cdot\underline{E}) - \nabla^2\underline{E} = -\nabla^2\underline{E}$$ since in vacuum $\nabla\cdot\underline{E} = 0$ (eq i)

product rule vi p. 21

Taking the curl of the right hand side of that equation:

$$\nabla\times\left(\frac{-\partial \underline{B}}{\partial t}\right) = \frac{-d}{dt}\left(\nabla\times\underline{B}\right) = \frac{-d}{dt}\left(\mu_0\varepsilon_0\frac{\partial \underline{E}}{\partial t}\right)$$ using eq. IV

Equating both sides we get:

$$\boxed{\nabla^2 \underline{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \underline{E}}{\partial t^2}}$$

Now we repeat the procedure and calculate the curl of both sides of equation (iv):

$$\nabla \times (\nabla \times \underline{B}) = \nabla(\nabla \cdot \underline{B}) - \nabla^2 \underline{B} = -\nabla^2 \underline{B} \quad \text{since } \nabla \cdot \underline{B} = 0$$

And the right hand side:

$$\nabla \times \left(\mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t}\right) = \mu_0 \varepsilon_0 \frac{d}{dt}\left(\nabla \times \underline{E}\right) = \mu_0 \varepsilon_0 \frac{d}{dt}\left(-\frac{\partial B}{\partial t}\right) = -\mu_0 \varepsilon_0 \frac{\partial^2 B}{\partial t^2}$$

Equating both sides we get:

$$\boxed{\nabla^2 \underline{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \underline{B}}{\partial t^2}}$$

Do these equations remind you of something?

$$\boxed{\nabla^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}}$$ The wave equation.

where $f = f(x \pm vt)$ is a solution to the wave equation is any function with "well behaved" derivatives.

What does it mean that $\underline{B}$ & $\underline{E}$ satisfy the wave equation?

> Maxwell's equations imply that empty space supports the propagation of electromagnetic waves traveling at speed:
>
> $$v = \frac{1}{(\varepsilon_0 \mu_0)^{1/2}} = 3 \times 10^8 \, m/s = c \quad \text{speed of light}$$

We will come back to this later. For now let's examine now eto Maxwell's eq - look in matter.

Maxwell's equations in matter

We have already arrived at a complete description of Maxwell's equations. However, as we've seen ~~its convenient~~ when looking at electric & magnetic fields in matter it can be convenient to write things in terms of the auxillary fields $\underline{H}$ & $\underline{D}$.

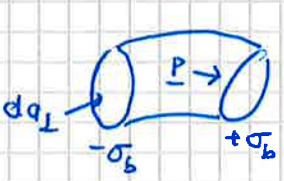
We know that in the static case, electric polarization $\underline{P}$ produces a bound charge density

$$\rho_b = -\nabla \cdot \underline{P}$$

Similarly, magnetization $\underline{M}$ produces a bound current:

$$\underline{J}_b = \nabla \times \underline{M}$$

To generalize the static case we need to consider that any change in electric polarization will involve a flow of (bound charge that we will call $\underline{J}_p$.



$$dI = \frac{\partial \sigma_b}{\partial t} da_\perp = \frac{\partial P}{\partial t} da_\perp \Rightarrow \frac{\partial \underline{P}}{\partial t} \equiv \underline{J}_p \quad \text{polarization current}$$

This polarization current has nothing to ~~do~~ do with the bound current due to magnetization, which is due to magnetic dipoles induced due to the spin & orbital motion of e^-.

By contrast $\underline{J}_p$ is the result of the movement of charge when the polarization changes e.g. if $\underline{P}$ points to the right ~~...~~ and is increasing then each + charge moves a bit to the right & each − charge a bit to the left, giving rise to $\underline{J}_p$.

We can check that by adding $\underline{J}_p$ we still fulfill conservation of charge given by the continuity equation:

$$\nabla \cdot \underline{J}_p = \nabla \cdot \frac{\partial \underline{P}}{\partial t} = \frac{\partial}{\partial t}\left(\nabla \cdot \underline{P}\right) = -\frac{\partial \rho_b}{\partial t}$$

definition of $\underline{P}$ definition of bound charge

~~What~~ Do we need to consider an extra source when we have a changing magnetization? No, changing magnetization does not lead to analogous accumulation of charge or current!

If we look at the expression $\underline{J}_b = \nabla \times \underline{M}$ we can see that $\underline{J}_b$ varies in response to changes in $\underline{M}$, but we have no extra accumulation of charge.

Now that we have all of our ~~sources~~ possible sources we can write the total charge density of a system as:

$$\rho = \rho_f + \rho_b = \rho_f - \nabla \cdot \underline{P}$$

and we have now an additional contribution to the current density due to a changing $\underline{P}$:

$$\underline{J} = \underline{J}_f + \underline{J}_b + \underline{J}_p = \underline{J}_f + \nabla \times \underline{M} + \frac{\partial \underline{P}}{\partial t}$$

With these we can rewrite Maxwell's equations as follows:

$$\nabla \cdot \underline{E} = \frac{1}{\varepsilon_0}\left(\rho_f - \nabla \cdot \underline{P}\right) \Rightarrow \nabla \cdot \underline{D} = \rho_f$$

remembering $\underline{D} = \varepsilon_0 \underline{E} + \underline{P}$

And Ampère's law with Maxwell's term becomes:

$$\nabla \times \underline{B} = \mu_0\left(\underline{J}_f + \nabla \times \underline{M} + \frac{\partial \underline{P}}{\partial t}\right) + \mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t}$$

$$\Rightarrow \nabla \times \underline{H} = \underline{J}_f + \frac{\partial \underline{D}}{\partial t}$$

remembering that $\underline{H} \equiv \frac{1}{\mu_0}\underline{B} - \underline{M}$ + using $\underline{D} = \varepsilon_0 \underline{E} + \underline{P}$

The remaining equations don't involve ρ or $\underline{J}$ so they don't need to be rewritten, so finally we have that in matter:

$$\boxed{\begin{array}{ll} (i)\ \nabla \cdot \underline{D} = \rho_f & (iii)\ \nabla \times \underline{E} = -\dfrac{\partial \underline{B}}{\partial t} \\ (ii)\ \nabla \cdot \underline{B} = 0 & (iv)\ \nabla \times \underline{H} = \underline{J}_f + \dfrac{\partial \underline{D}}{\partial t} \end{array}}$$

and for this set of eq to be complete we need to include the constitutive relations for $\underline{H}$ + $\underline{D}$, which will depend on the nature of the material. For linear media we have:

$$\underline{P} = \varepsilon_0 \chi_e \underline{E} \quad \text{and} \quad \underline{M} = \chi_m \underline{H}$$

Remember $\varepsilon \equiv \varepsilon_0(1 + \chi_e)$

so

$$\underline{D} = \varepsilon \underline{E} \quad \text{and} \quad \underline{H} = \frac{1}{\mu}\underline{B}$$

In these equations you can see why we call Maxwell's fix to Ampère's law displacement current, it's because we ~~w~~ defined $\underline{D}$ as the electric displacement.

The equivalent of these equations in integral form will be:

(i) $\oint_S \underline{D} \cdot d\underline{a} = Q_{fenc}$

(ii) $\oint_S \underline{B} \cdot d\underline{a} = 0$

over any closed surface S

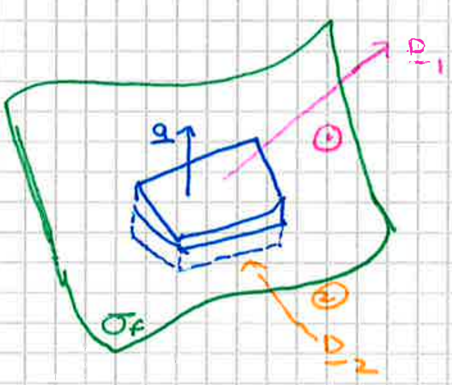
(iii) $\oint_P \underline{E} \cdot d\underline{l} = -\frac{d}{dt}\int_S \underline{B} \cdot d\underline{a}$

(iv) $\oint_P \underline{H} \cdot d\underline{l} = I_{fenc} + \frac{d}{dt}\int_S \underline{D} \cdot d\underline{a}$

over any closed surface S bounded by the closed loop P

As we have seen before $\underline{E}$, $\underline{B}$, $\underline{D}$ and $\underline{H}$ will be discontinuous at the boundary between 2 different media (because the values for μ & ε will change) or at a surface that carries a charge density σ or a current density K. Being discontinuous means that their values will "change abruptly" from one side of the boundary to the other.

Using Maxwell's equations in this form we can derive explicitly the form of these discontinuities; let's consider the following material:



Let's consider a Gaussian "pill box" extending into the material on either side of the boundary.

Applying eq. (i) using this pillbox we obtain:

$$\underline{D}_1 \cdot \underline{a} - \underline{D}_2 \cdot \underline{a} = \underbrace{\sigma_f a}_{\text{this is } Q_{fenc}}$$

remember the definition of σ

The component of $\underline{D}$ that is perpendicular to the interface is discontinuous in the amount:

$$\boxed{D_1^{\perp} - D_2^{\perp} = \sigma_f}$$

If we apply the same reasoning to equation (ii) we get:

$$\boxed{B_1^{\perp} - B_2^{\perp} = 0}$$

Now applying eq (iii) for an amperian loop across the surface:

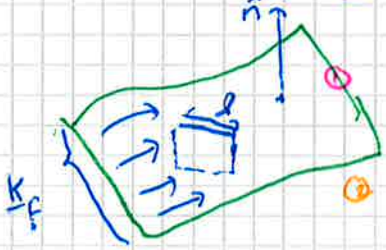
$$\underline{E}_1 \cdot \underline{l} - \underline{E}_2 \cdot \underline{l} = -\frac{d}{dt}\underbrace{\int_S \underline{B} \cdot d\underline{a}}$$

flux vanishes as the width of the loop goes to zero:

$$\Rightarrow \boxed{E_1^{\parallel} - E_2^{\parallel} = 0}$$

And the ~~[illegible]~~ eq (iv) for the amperian loop gives us:

$H_1 \cdot \underline{l} - H_2 \cdot \underline{l} = I_{fenc}$ where I_{fenc} is the current passing through the amperian loop.



Let ~~[illegible]~~ $\hat{n}$ be the unit vector normal to the surface, pointing from $2 \to 1$ $\Rightarrow$ $\hat{n} \times \underline{l}$ will be normal to the amperian loop

then $I_{fenc} = \underset{\text{remember def of } K_f}{K_f} \cdot (\hat{n} \times \underline{l}) = (K_f \times \hat{n}) \cdot \underline{l}$

so with this we have that

$$\underline{H}_1 \cdot \underline{l} - \underline{H}_2 \cdot \underline{l} = (\underline{K}_f \times \hat{n}) \cdot \underline{l}$$

$$\Rightarrow \boxed{H_1^{\parallel} - H_2^{\parallel} = \underline{K}_f \times \hat{n}}$$

In summary:

$$D_1^{\perp} - D_2^{\perp} = \sigma_f \qquad \underline{E}_1^{\parallel} - \underline{E}_2^{\parallel} = \underline{0}$$

$$B_1^{\perp} - B_2^{\perp} = 0 \qquad \underline{H}_1^{\parallel} - \underline{H}_2^{\parallel} = \underline{K}_f \times \hat{n}$$

Or in plain english: c

- The component of $\underline{D}$ $\perp$ to the interface is discontinuous by σ_f
- The parallel components of $\underline{H}$ are discontinuous by an amount proportional to the free surface current density
- the $\perp$ component of B & $\parallel$ component of E are continuous at the boundary.

We can write these equations for linear media as:

(i) $\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = \sigma_f$ (iii) $\underline{E}_1^{\parallel} - \underline{E}_2^{\parallel} = \underline{0}$

(ii) $B_1^{\perp} - B_2^{\perp} = 0$ (iv) $\frac{1}{\mu_1} \underline{B}_1^{\parallel} - \frac{1}{\mu_2} \underline{B}_2^{\parallel} = \underline{K}_f \times \hat{n}$

And in particular if we have no free charge or currents at the interface:

(i) $\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = 0$ (iii) $\underline{E}_1^{\parallel} - \underline{E}_2^{\parallel} = \underline{0}$

(ii) $B_1^{\perp} - B_2^{\perp} = 0$ (iv) $\frac{1}{\mu_1} \underline{B}_1^{\parallel} - \frac{1}{\mu_2} \underline{B}_2^{\parallel} = \underline{0}$

these equations are the basis for the theory of reflection & refraction.