Lecture 18. Electromagnetic Induction. Faraday's law of induction.

Last time:

* Ampère's law for magnetized materials:

$$\nabla \times \underline{H} = \underline{J}_f \qquad \text{where } \underline{H} \equiv \frac{1}{\mu_0}\underline{B} - \underline{M} \text{ is an 'auxiliary field'}$$

$$\oint \underline{H} \cdot d\underline{l} = I_{f\,enc}$$

* Linear media:

$\underline{M} = \chi_m \underline{H}$ where χ_m is the magnetic susceptibility

$\underline{B} = \mu \underline{H}$ where μ is the permeability $\mu = \mu_0(1+\chi_m)$

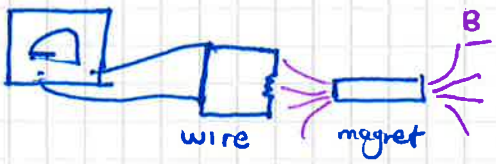
* Ferromagnets: → spins of unpaired electrons tend to align with each other.
  → magnetization of material depends on history
  hysteresis

Today: Electromagnetic induction and Faraday's law

Demo 226. Electromagnetic induction

In 1831 Faraday conducted a series of experiments that can be summarized as follows: (for an account of the original experiments see Purcell).

For all cases we have a loop of wire and a magnet. We connect a galvanometer to the wire to measure current.

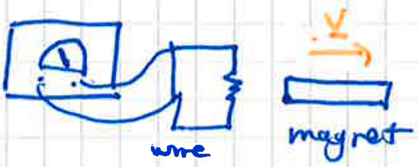


What happens:
If we have a wire next to a magnet?

NOTHING

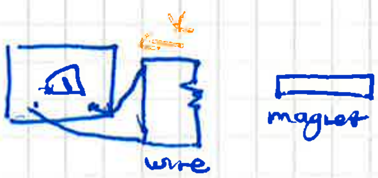
Since we know that currents are sources of $\underline{B}$, people wondered if $\underline{B}$ could also generate a current, but in this case we can't detect any current on the loop if we have a magnet next to it.

But wait, if now we move the magnet relative to the wire:



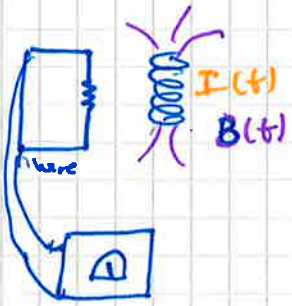
This produces a current in the wire!

Now if I leave the magnet fixed but move the wire:



This also produces a current in the wire

Now if instead of the magnet we have a source of varying magnetic field.
(you can imagine a coil w/ a variable current)



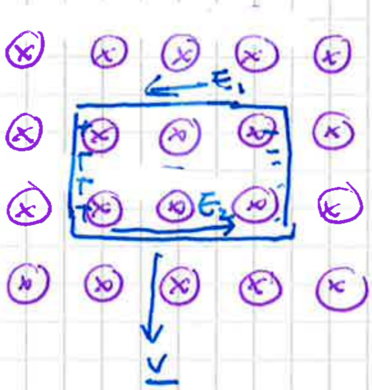
This also produces a current in the wire!

What can we conclude from these experiments?

* A constant magnetic field does NOT induce a current
* A changing magnetic field will act as an emf (electromotive force) and induce a current in the wire.

Why is there a current in the wire?

Moving a loop in a uniform magnetic field.



⊗ $\underline{B}$ out of the page

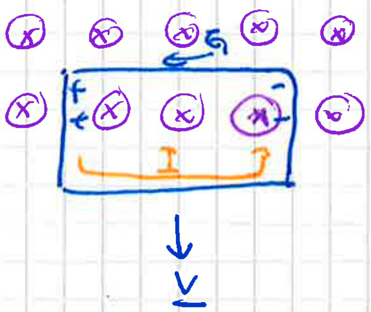
If we move the loop $\perp$ to $\underline{B}$, there is a non-zero x prod between $\underline{v}$ and $\underline{B}$ so the e^- on the wire will experience a Lorentz $\underline{F}$

$$\underline{F} = q\,(\underline{v} \times \underline{B})$$

$\underline{E}_1$ will be created due to Lorentz force, this will drive separation of charges

which will then generate $\underline{E}_2 = -\underline{E}_1$

What happens if $\underline{B}$ is non-uniform?



⊗ $\underline{B}$ out of the page

~~We move~~ We move the loop at a velocity $\underline{v}$

Above: $B_1 = B_0$ uniform field

Below: $B = 0$

Again, the Lorentz $\underline{F}$ will induce movement of charges: E_1 which results in a separation of charges on the wire.

There is no opposing $\underline{F}$ in the bottom so charges flow → current $\underline{I}$

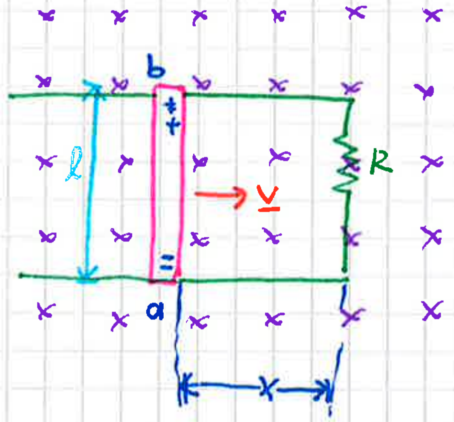
electromagnetic induction

THIS IS THE END OF ELECTROSTATICS.

Notice that $\oint \underline{E} \cdot d\underline{l} \neq 0$ or $\nabla \times \underline{E} \neq 0$

Motional emf

Consider a conducting bar sliding along 2 frictionless conducting rails connected by a resistance R. The bar is moving through a uniform magnetic field which points into the page.



Bar sliding along rails a distance ℓ apart

Rails connected by resistance R

Separation between rails ℓ

Bar moving at a velocity $\underline{v}$ in a uniform magnetic field

x is the distance from the bar to the edge of the loop.

What happens as the bar moves?

Particles on the bar with a positive charge $(q > 0)$ will experience a magnetic force:

$$\underline{F}_B = q\underline{v} \times \underline{B}$$

This force will push the positively charged particles to the top of the bar.

⇒ There will be an accumulation of negative charges at the bottom.

⇒ Separation of charges will give rise to an electric field $\underline{E}$ inside the bar, which will produce a downward electric force $\underline{F}_e$

$$\underline{F}_e = q\underline{E}.$$

The presence of this $\underline{E}$ is due to a potential difference between both ends of the bar:

$$\varphi_{ab} = \varphi_a - \varphi_b \underset{\substack{\uparrow \\ \text{potential} \\ \text{difference is} \\ \text{equivalent to an} \\ \text{emf}}}{=} \mathcal{E} \underset{\substack{\uparrow \\ \text{this is how} \\ \text{we defined it} \\ \text{in Lecture 12} \\ \text{work done/unit charge}}}{=} \frac{dW}{dQ} = \frac{1}{q} W(- \rightarrow +) = \frac{1}{q}\int \underline{F}_e \cdot d\underline{s} \underset{\substack{\uparrow \\ \text{at} \\ \text{equilibrium} \\ \underline{F}_e + \underline{F}_B = 0 \\ \Rightarrow \underline{F}_e = -\underline{F}_B}}{=} \frac{1}{q}\int \underline{F}_B \cdot d\underline{s}$$

Since $\mathcal{E}$ arises from the motion of the conductor, this potential difference is known as:

$$\boxed{\mathcal{E} = \oint \underline{F}_B \cdot d\underline{s}}$$ Motional emf

In the case of the loop we have:

$$\mathcal{E} = \oint \frac{\underline{F}_B}{q} \cdot d\underline{s} = \frac{F_B}{q}\int ds = \frac{qvB}{q}\ell = B\ell v$$ ☺

Now, there is a nice way of expressing the motional emf in terms of flux of $\underline{B}$.

This motivates us to define the magnetic flux through a loop as:

$$\boxed{\Phi_B \equiv \int \underline{B} \cdot d\underline{a}}$$ magnetic flux through a loop

We have established that for the bar we have: (Eq. ⑮)

$$\mathcal{E} = B\ell v = \boxed{B\ell \frac{dx}{dt}} \quad *$$

We can rewrite this equation in terms of the magnetic flux:

$$\Phi_B = \int \underline{B} \cdot d\underline{a} = B\int da = B\ell x \Rightarrow \frac{\partial \Phi_B}{\partial t} = \boxed{B\ell \frac{dx}{dt}} = B\ell(-v) \quad **$$

$\frac{dx}{dt}$ is negative because flux decreases as the bar moves

So from equations $*$ and $**$ we can see that the terms in $\square$ are the same so:

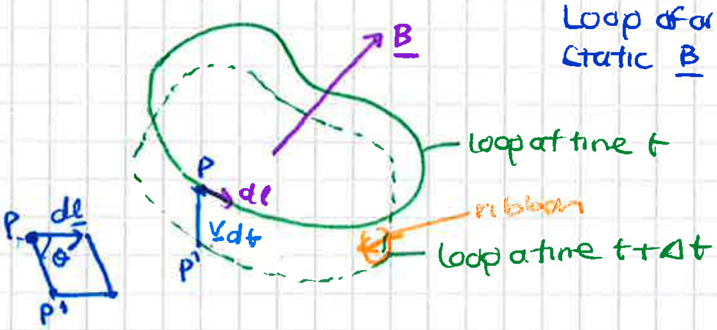
$$\boxed{\mathcal{E} = -\frac{\partial \Phi_B}{\partial t}}$$ Flux rule for motional emf: Helpful to calculate induced emf.

Here we have made explicit the minus sign. This minus sign indicates that the direction of the induced current is such to oppose a change in flux in B. → "electromagnetic inertia"

This flux rule applies for any loop moving in arbitrary directions in non-uniform magnetic fields, the loop doesn't even need to maintain a fixed shape.

General proof of flux rule for motional emf

Let's consider a loop of arbitrary shape moving with velocity v through a static magnetic field.



Loop of arbitrary shape moving at $\underline{v}$
Static $\underline{B}$

At time t the flux through the loop is: $\Phi_B = \int_S \underline{B} \cdot d\underline{a}$

How does it change when $t \to t + dt$?

$$d\Phi_B = \Phi_B(t+dt) - \Phi_B(t) = \Phi_{\text{ribbon}} = \int_{\text{ribbon}} \underline{B} \cdot d\underline{a}$$

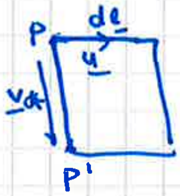
What is $d\underline{a}$ on the ribbon?

$$d\underline{a} = (\underline{v}\,dt) \times d\underline{\ell} = (\underline{v} \times d\underline{\ell})\,dt$$ so subs this in the last equation:

remember the geometric interpretation of the x product?

$$\frac{d\Phi_B}{dt} = \oint \underline{B} \cdot (\underline{v} \times d\underline{\ell}) \quad ☆$$

Let $\underline{u}$ be the velocity of charge down the wire and remembering $\underline{v}$ is the velocity of the wire:



$\underline{w} = \underline{v} + \underline{u}$ is the resultant velocity of a charge at P.

Since $\underline{u}$ is parallel to $d\underline{l}$ it won't contribute to the xproduct $\underline{v} \times d\underline{l}$, so we can swap $\underline{v}$ for $\underline{w}$ in equation ☆:

$$\frac{d\Phi_B}{dt} = \oint \underline{B} \cdot (\underline{w} \times d\underline{l})$$

Now using the identity: $\underline{a} \cdot (\underline{b} \times \underline{c}) = (\underline{a} \times \underline{b}) \cdot \underline{c}$ we can rewrite the integrand as:

$$\underline{B} \cdot (\underline{w} \times d\underline{l}) = -(\underline{w} \times \underline{B}) \cdot d\underline{l}$$

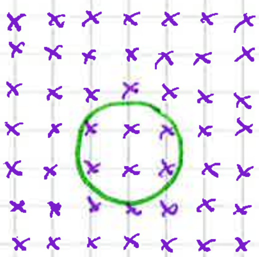
so:

$$\frac{d\Phi_B}{dt} = -\oint \underbrace{(\underline{w} \times \underline{B})}_{\text{this is actually the magnetic force/unit charge}} \cdot d\underline{l} = -\oint \underline{F}_B \cdot d\underline{l} = -\mathcal{E}$$

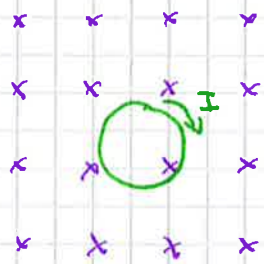
$$\therefore \quad \mathcal{E} = -\frac{d\Phi_B}{dt} \text{ holds in general}$$

From the flux rule we can see that ~~mot~~ motional emf can be induced by:

$$\left(\text{Remember } \Phi_B = \int \underline{B} \cdot d\underline{a} \right)$$

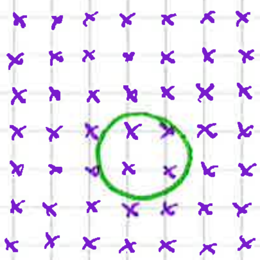


$\underline{B}$

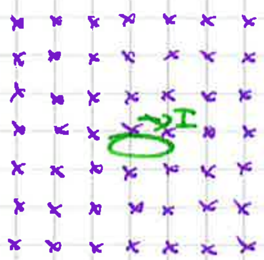


$\underline{B}' < \underline{B}$

① Changing the magnitude of B with time

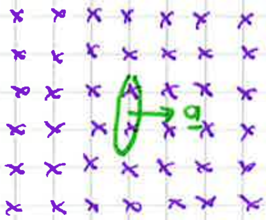


area = a



area = a' < a

② Changing the area enclosed by the loop

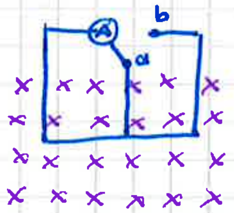


③ Changing the angle between $\underline{B}$ and $\underline{a}$

Disclaimer: The flux rule is a nice shortcut for calculating motional emfs. It doesn't contain new physics, just Lorentz's law.

HOWEVER, it can lead to error or ambiguity if you are NOT careful. The flux rule assumes that you have a single loop. The loop can stretch or distort continuously but beware of arrangements that allow multiple paths and switches

Example:



When we move the switch from $a \to b$ in this circuit we have twice the area, so we ~~should~~ have twice the flux of $\underline{B}$, but there is no motional emf. No current is measured in the ammeter.

Thoughts on the sign of the flux rule:

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

this sign determines the direction of the induced current (Remember $V = RI$, $V \to \mathcal{E}$)

The direction of the induced current is determined by Lenz's law:

$\boxed{\text{The induced current opposes changes in flux of } \underline{B}}$

(i.e. the induced current produces magnetic fields which tend to oppose the change in magnetic flux that induces such currents)

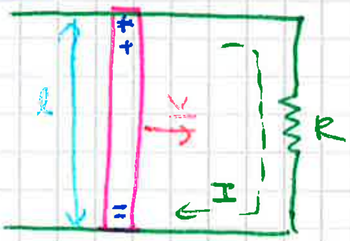
$\Leftrightarrow$

$\boxed{\text{Nature abhors a change in flux}}$

Physical interpretation of Lenz's law:

Let's think again about our moving bar:

(x) uniform $\underline{B}$ into the page



When current flows in a magnetic field it will feel a force

Lentz's law: ~~the force~~ the magnetic force will try to slow down the bar.

If I is clockwise: It creates a $\underline{B}$ pointing into the page $\Rightarrow \underline{I} \times \underline{B}$ points to the left, so it tries to slow down the motion of the bar.

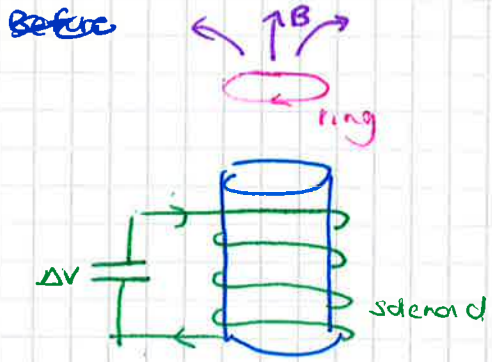
Induction is an "inertial" phenomenon. A conducting loop "likes" to maintain a constant flux, if you change it the loop responds by sending a current around in such a direction as to frustrate your efforts.

Lentz's law just tells us the direction of the current flow.

Demo 608 Jumping ring

EXAMPLE: Practical use of Lenz's law

Why does the ring "jump"?



Before turning the current on the solenoid on

$$\Phi_{ring} = 0$$

After we turn on the current on the solenoid there is $\underline{B}$, which generates an upward flux.

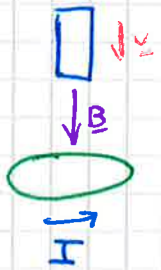
This change in flux generates an emf which induces a current in the ring.

The direction of the current according to Lenz's law is such that the field it generates will ~~cancel~~ tend to cancel the new flux

$\Rightarrow$ Current is opposite in direction to the current in the solenoid.

Remember from Lecture B that opposite currents repel? $\therefore$ Ring "jumps" or flies off.

EXAMPLE. A bar magnet is moving (with the north pole pointing down) towards a conductor loop, in which direction does the induced current flow?



magnet with north pole pointing downwards means $\underline{B}$ points downwards.

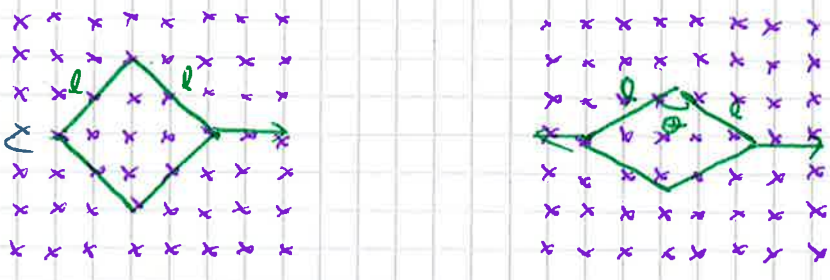
As the magnet approaches, the magnetic flux through the loop increases.

The induced current ~~should~~ will generate B that opposes this change i.e B needs to point upward

$\Rightarrow$ I induced moves clockwise

Right hand rule: If your thumb points in the direction of $\underline{B}$, then your fingers curl around in the direction of I.

EXAMPLE: Loop changing area. A square loop with length ℓ on each side is placed in a uniform $\underline{B}$ pointing into the page. During the time interval Δt, the loop is pulled from its 2 edges and turned into a rhombus. Assuming the resistance of the loop is R, find the average induced current & its direction.



We can use the flux rule to calculate the induced emf & from there we can get the current using Ohm's law.

Flux rule: $-\dfrac{\partial \Phi_B}{\partial t} = -\dfrac{\Delta \Phi}{\Delta t} = \mathcal{E}$

$\Phi_B = \int \underline{B} \cdot da = B \int da = B \cdot A$ ($\uparrow$ B uniform) $\Rightarrow -\dfrac{\Delta \Phi}{\Delta t} = \dfrac{-B \Delta A}{\Delta t} = \mathcal{E}$

All we have to do is calculate ΔA and Δt.

$\Delta A = A_f - A_i$ what are the initial and final areas?

$A_i = \ell^2$

$A_f = \ell^2 \sin\theta$, remember the geom. interpretation of the xproduct
$A = |\underline{\ell}_1 \times \underline{\ell}_2| = \ell_1 \ell_2 \sin\theta$ area of a parallelogram defined by the vectors $\underline{\ell}_1$ and $\underline{\ell}_2$.

$$\Rightarrow \frac{\Delta A}{\Delta t} = \frac{\ell^2 \sin\theta - \ell^2}{\Delta t} = -\left[\frac{\ell^2(1-\sin\theta)}{\Delta t}\right]$$

[Remember $\frac{B\Delta A}{\Delta t} = \frac{\Delta \Phi_B}{\Delta t}$ change of flux in time.]

$$\therefore \varepsilon = B\left[\frac{\ell^2(1-\sin\theta)}{\Delta t}\right]$$

Here $\frac{\Delta A}{\Delta t}$ is negative, so flux is decreasing

Then we use Ohm's law to determine the induced current:

$$I = \frac{\varepsilon}{R} = \frac{B\ell^2(1-\sin\theta)}{\Delta t R}$$

In which direction does the current flow? Since $\frac{\Delta A}{\Delta t} < 0$ the flux through the loop, into the page decreases

So the current needs to flow clockwise to generate B into the page that compensates the loss of flux.

If this makes your head hurt, you can also memorize the following conventions:

Φ_B	$\frac{d\Phi_B}{dt}$ (or $\frac{\Delta\Phi_B}{\Delta t}$)	ε	I
+	+	−	−
+	−	+	+
−	+	−	−
−	−	+	+

where
⊕ I counterclockwise
⊖ I clockwise
We always define a positive direction for $\underline{A}$.

Circled in: In the case of the last problem we had

Φ_B negative since $\underline{B}$ was into the page and $\underline{A}$ out

Remember: We always define A in a positive direction

$\frac{\Delta\Phi_B}{\Delta t}$ negative

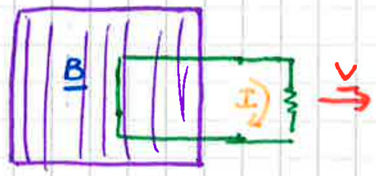
ε positive

$\therefore$ I flows in the ⊕ direction, counterclockwise.

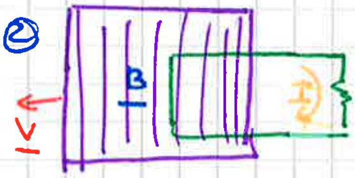
We have taken a huge detour to understand the origins of the induced currents Faraday observed. Let's now formalize our understanding.

Faraday's law:

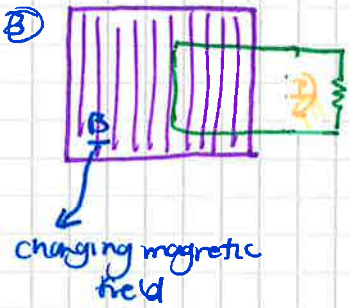
Recall Faraday's experiments (simplified). We have a magnet & a loop.

①


Pull loop to the right through $\underline{B}$ → induced current

②


Pull magnet to the left. Don't move loop → induced current

③


Don't move anything but have $\underline{B}$ that changes in time → induced current.

What is the origin of the induced current in all these cases?

Experiment 1: Motional emf drives the current. $\mathcal{E} = -\dfrac{d\Phi}{dt} = -\dfrac{d}{dt}\int \underline{B}\cdot d\underline{a} = -\int \dfrac{\partial \underline{B}}{\partial t}\cdot d\underline{a}$ (*)

Experiment 2: The same emf as in experiment 1 arises, since it's the "relative" motion of the magnet & the wire that matters. This makes sense in the light of special relativity. But Faraday knew nothing about it so this reciprocity at this point was a coincidence.

Think about it: If the loop moves, we saw that the Lorentz force (magnetic) sets up the emf, but if the loop is stationary, the force can't be magnetic.

Stationary charges experience NO magnetic force $F_B = q(\underline{v}\times\underline{B})$

What could exert a force on stationary charges? F_E electric field. But where is it in this case? Faraday had an ingenious idea:

$\boxed{\text{A changing magnetic field induces an electric field}}$

It is this induced electric field that accounts for the emf that drives the induced current in this case. As Faraday found empirically:

$$\mathcal{E} = \oint \underline{E}\cdot d\underline{l} = \frac{-d\Phi}{dt} \quad (**)$$

Equating the expressions for $\mathcal{E}$ (*) and (**):

$$\oint \underline{E}\cdot d\underline{l} = -\int \frac{\partial \underline{B}}{\partial t}\cdot d\underline{a}$$ Faraday's law in integral form

Which we can convert into differential form by applying Stokes's theorem:

$$\boxed{\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}} \quad \text{Faraday's law}$$

This reduces ~~to~~ $\oint \underline{E} \cdot d\underline{l} = 0$ for constant $\underline{B}$ (static case) as it should.

Experiment 3. We can apply Faraday's law an see that if we have a changing B we will have an induced electric field, which will ~~drive a current~~. give rise to an emf $\frac{-d\Phi}{dt}$ that will drive a current.

So in general, no matter the mechanism, if the magnetic flux through a loop changes, an emf

$$\mathcal{E} = \frac{-d\Phi}{dt} \quad \text{will appear in the loop.}$$

As observed in all 3 experiments described.

The induced electric field

Faraday's law is a generalization of our old equation $\nabla \times \underline{E} = 0$ in the case we have time dependent quantities.

If $\underline{E}$ is due exclusively to changing $\underline{B}$ we have:

$$\nabla \cdot \underline{E} = 0 \quad , \quad \nabla \times \underline{E} = \frac{-\partial \underline{B}}{\partial t}$$

(if there were charges sources of E we would have $\nabla \cdot \underline{E} = \frac{1}{\epsilon_0}\rho$)

This is mathematically identical to magnetostatics:

$$\nabla \cdot \underline{B} = 0 \quad , \quad \nabla \times \underline{B} = \mu_0 \underline{J}$$

So we could use the same equations we had for magnetostatics but ~~now~~ writing $-\left(\frac{\partial B}{\partial t}\right)$ instead of $\mu_0 \underline{J}$, and $\underline{E}$ instead of $\underline{B}$:

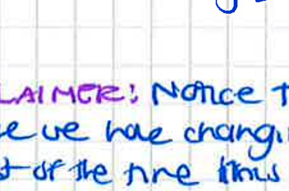
$$\underline{E} = -\frac{1}{4\pi}\int \frac{(\partial \underline{B}/\partial t) \times \hat{r}}{r^2}\, d\tau = \frac{-1}{4\pi}\frac{\partial}{\partial t}\int \frac{\underline{B} \times \hat{r}}{r^2}\, d\tau$$

and ~~using~~ if symmetry allows we can also use Ampere's law:

$$\oint \underline{E} \cdot d\underline{l} = \frac{-d\Phi}{dt} \quad \left(\text{instead of } \oint \underline{B} \cdot d\underline{l} = \mu_0 I_{enc}\right)$$

DISCLAIMER: Notice that we are using rules derived for magnetostatics in ~~the~~ a case where we have changing magnetic fields! Our solutions will only be an approximation, most of the time ~~this~~ will be a pretty good approximation unless $\underline{B}$ changes extremelly rapid or we're far away from the source. This regime is called quasistatic.

EXAMPLE: A uniform $\underline{B}$ ~~field~~ pointing straight up fills the shaded region shown below. ~~What is the induced electric field?~~ If $\underline{B}$ is changing in time what is the induced electric field?



Let's draw an Amperian loop of radius s & apply Faraday's law:

$$\oint \underline{E} \cdot d\underline{l} = \frac{-d\Phi}{dt}$$

$$\oint \underline{E} \cdot d\underline{l} = E(2\pi s) \quad \text{and} \quad \frac{-d\Phi}{dt} = -\frac{d}{dt}\left(B(t)\,\pi s^2\right) = -\pi s^2 \frac{dB}{dt}$$

$$\Rightarrow E(2\pi s) = -\pi s^2 \frac{dB}{dt}$$

$$\Rightarrow \boxed{\underline{E} = -\frac{s}{2}\frac{dB}{dt}\,\hat{\phi}}$$

$\underline{E}$ points in the circumferential direction, just like a magnetic field inside a straight wire carrying a uniform $\underline{J}$. here $\underline{B}(t)$ is eq to $\underline{J}$, the source of the field.