Lecture 12. Electromotive force. Circuits and Kirchhoff's rules

Last time:

* Electric current

$I = \frac{dQ}{dt}$ in terms of current density $I = \int_S \underline{J} \cdot d\underline{A}$

* Electric current density

$\underline{J} \equiv qn\underline{v} \equiv \rho \underline{v}$ where n is the # of charges / unit volume and $\underline{v}$ is the vel of those carriers
ρ in this case is the resistivity

* Continuity equation:

$$\nabla \cdot \underline{J} + \frac{\partial \rho}{\partial t} = 0$$

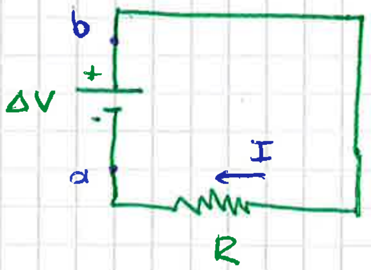
* Ohm's law:

$\underline{J} = \sigma \underline{E}$ microscopic where $\sigma = \frac{nq^2\tau}{m}$ conductivity ; $\rho = \frac{m}{nq^2\tau}$ resistivity

$V = RI$ macroscopic $R \equiv \frac{L}{\sigma A}$ resistance

Today: EMF. The electromotive force

Consider a circuit made of a battery and a resistor with resistence R:



Let the potential difference between the points a & b be $\Delta V = V_b - V_a > 0$

If a charge Δq is moved from ~~a lead~~ point a through the battery, its potential energy is increased by:

$$\Delta U = \Delta q \, \Delta V$$

On the other hand as the charge moves ~~again~~ across the resistor R, its potential energy is decreased due to collisions with the atoms in the resistor, so upon returning to a, the potential energy of Δq remains unchanged.

The rate of energy loss through the resistor is given by:

$$P = \frac{\Delta U}{\Delta t} = \left(\frac{\Delta q \, \Delta V}{\Delta t} \right) = I \Delta V$$, which is precisely the power supplied by the battery.

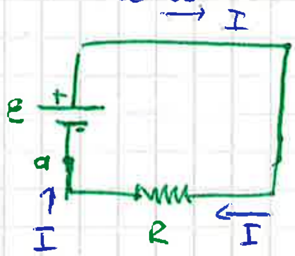
So we need to supply electrical energy in order to maintain a constant current in a closed circuit. The source of energy is referred to as the electromotive force $\mathcal{E}$.

$$\boxed{\mathcal{E} \equiv \frac{dW}{dq}}$$ work done to move a unit charge in a direction of higher potential. $[\mathcal{E}] = \text{Volt}$

No Thoughts on EMF:

* We can think of it as a 'charge pump' that moves charges from regions ~~of~~ of lower potential to higher one
* Batteries, solar cells, thermocouples are examples of sources of $\mathcal{E}$
* It is NOT a force

Now that we have defined the emf we can redraw our circuit:



- Assuming the battery has no internal resistence $\Delta V = \mathcal{E}$ emf.
- When crossing from the negative to the positive terminal, the potential increases by $\mathcal{E}$.
- As we cross the resistor, the potential decreases by an amount IR (b/c of Ω's law) and the potential energy is converted into thermal energy (i.e the resistor warms up).

- Assuming that the connecting wire carries no resistance, upon completing a loop, the net change in potential difference is zero.

$$\mathcal{E} - IR = 0$$, this is because no work is done moving a charge around in a closed loop

$$W = -q\oint \vec{E} \cdot d\vec{s} = 0$$

conservative nature of the electrostatic field.

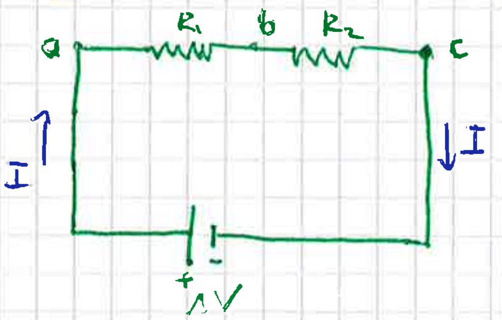
This is known as Kirchhoff's second rule:
Around any closed loops, the sum of emf and potential drops is zero.

this is eq to the statement that electrostatic field is conservative.

! Important conventions:
* we indicate emf with the symbol: (long line is the positive terminal, short line is the negative terminal)
* In circuits current flows from + to − which is counterintuitive in terms of electrons, but its the convention.

Solving circuits: Resistors in series and in parallel.

Solving a circuit means determining the currents and voltages everywhere



Two resistors R_1 & R_2 are connected in series through a voltage source ΔV

Because the current is conserved the same I is flowing through each resistor.

The voltage drop from pt a to pt c is the sum of the voltage drops across individual resistors:

$$\Delta V = IR_1 + IR_2 = I(R_1 + R_2) \quad *$$

The two resistors can be replaced by one equivalent R_{eq} with an identical voltage drop

$$\Delta V = I R_{eq} \quad **$$ so from eq * and ** : ⟶

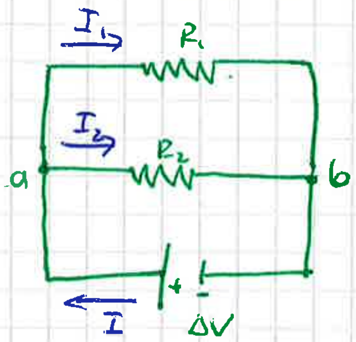
$R_{eq} = R_1 + R_2$

$$R_{eq} = R_1 + R_2$$

We can extend the above argument for N resistors connected in series:

$$\boxed{R_{eq} = R_1 + R_2 + \ldots = \sum_{i=1}^{N} R_i}$$ for resistors in series

Now let's consider resistors connected in parallel:



* We have two resistors R_1 and R_2 that are connected in parallel across a voltage source ΔV

* By current conservation, the current I that passes through the voltage source must be divided into currents I_1 & I_2 that pass through R_1 & R_2 respectively.

The potential across each resistor is the same:

$$\Delta V = \Delta V_1 = \Delta V_2$$

and each resistor satisfies Ohm's law:

$$\Delta V = R_1 I_1 \quad \text{and} \quad \Delta V = R_2 I_2$$

by current conservation:

$$I = I_1 + I_2 = \frac{\Delta V}{R_1} + \frac{\Delta V}{R_2} = \Delta V\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

so the two resistors in // can be replaced by the eq resistor:

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$$

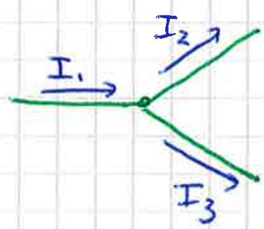
We can extend the above argument for N resistors connected in parallel:

$$\boxed{\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots = \sum_{i=1}^{N} \frac{1}{R_i}}$$ for resistors in parallel

Kirchhoff's circuit rules

All you need to know to solve a circuit

① Rule #1: Junction rule.



At any point where there is a junction between various current carrying branches by current conservation, the sum of the currents into the node must equal the sum of the currents out of the node

$$\boxed{\sum I_{in} = \sum I_{out}}$$

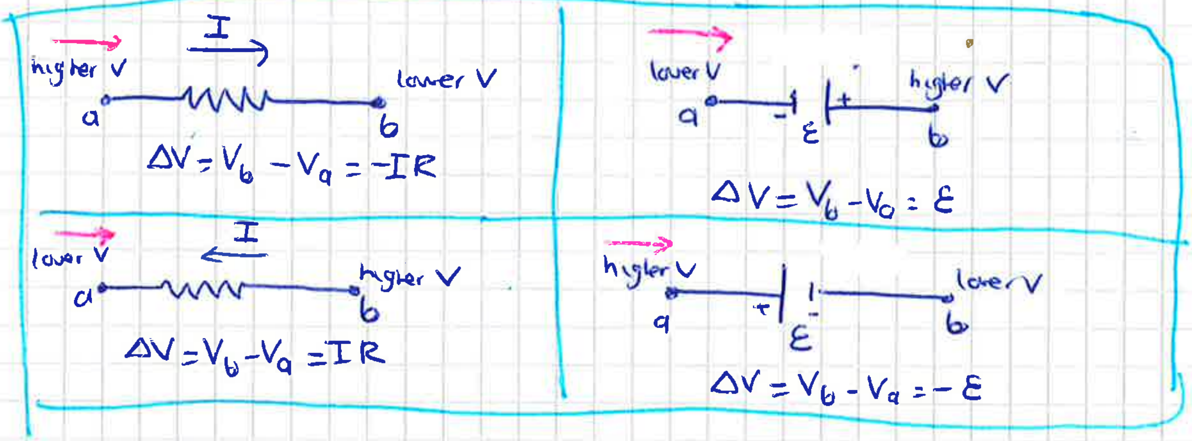
In this particular example: $I_1 = I_2 + I_3$

② Rule #2. Loop Rule

The sum of the voltage drops ΔV, across any circuit elements that form a closed circuit is zero. (We already derived this above) so we have:

$$\boxed{\sum_{\text{closed loop}} \Delta V = 0}$$

How do we determine ΔV across different circuit elements? We use Ohm's law, but here is a cheatsheet of the conventions for ΔV:



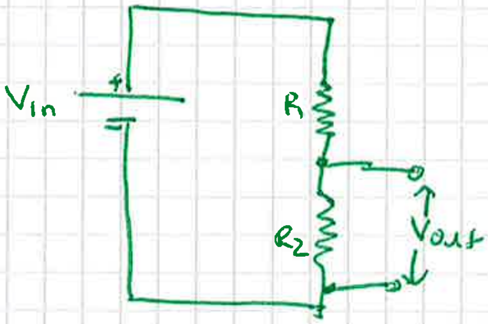
pink arrow → indicates the travel direction

the choice of the travel direction is arbitrary, same eq. is obtained independent if the loop is traversed clockwise or anticlockwise.

Demo 460 Kirchhoff's rules

EXAMPLE: Voltage divider

Solve the voltage divider circuit i.e what is V_{out} and I?



We first apply the loop rule: $\sum_{\text{closed loop}} \Delta V = 0$

$$\Rightarrow V_{in} + V_1 + V_2 = 0$$

$$\Rightarrow V_{in} - R_1 I_1 - R_2 I_2 = 0 \quad \text{and} \quad I_1 = I_2 \text{ same current across the loop.}$$

↑ the minus sign comes from observing the conventions we established and noting that V_{out} across R_2 will be less than V_{in}

so we're going from higher V to lower V

$-IR$

$$V_{in} = R_1 I + R_2 I$$

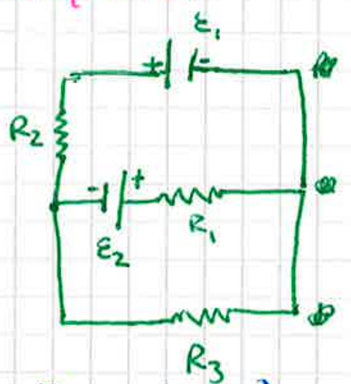
$$\Rightarrow I = \frac{V_{in}}{R_1 + R_2}$$

so now we can calculate V_{out} using Ω's law:

Now that we have the current, we can calculate V_{out}, the voltage difference across R_2.

$$V_{out} = I R_2 = \left(\frac{V_{in}}{R_1 + R_2}\right) R_2$$

Example: A multiloop circuit



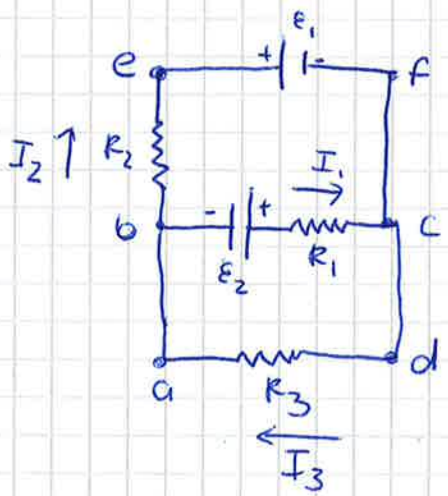
We have two sources of emf: ε_1 and ε_2
Resistances R_1, R_2, R_3 are all given

What is the current through each resistor?

How do we solve this problem?

1) Unknown quantities are I_1, I_2, I_3 associated w/ each resistors.
   we have 3 unknowns ⇒ we need 3 equations

2) It helps to consider specific junctions in the circuit & assign the directions of the currents:



3) Applying Rule #1 to junction b yields:

$$I_1 + I_2 = I_3$$ since I_2 and I_1 are leaving the junction while I_3 is entering the junction.

If we apply the rule to point c we get the same equation.

4) We need to obtain 2 more equations by using the loop rule. Rule #1, which says that the net potential difference across all elements in a closed loop is zero.

Going across befc in a clockwise direction; the voltages are:

$$-I_2 R_2 - \varepsilon_1 + I_1 R_1 - \varepsilon_2 = 0$$

Similarly, traversing the second loop abcd clockwise yields:

$$\varepsilon_2 - I_1 R_1 - I_3 R_3 = 0$$

Note that we could've also considered the entire loop abefcda

$$-I_2 R_2 - \varepsilon_1 - I_3 R_3 = 0$$

but this is NOT linearly independent of the other loop equations since it's the sum of them.

5) ok, so the equations we need to solve are:

$$I_3 = I_1 + I_2$$

$$-I_2 R_2 - \varepsilon_1 + I_1 R_1 - \varepsilon_2 = 0$$

$$\varepsilon_2 - I_1 R_1 - I_3 R_3 = 0$$

and after some tedious but straightforward algebra we get:

$$I_1 = \frac{\varepsilon_1 R_3 + \varepsilon_2 R_3 + \varepsilon_2 R_2}{R_1 R_2 + R_1 R_3 + R_2 R_3}$$

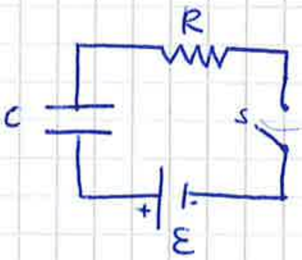
$$I_2 = -\frac{\varepsilon_1 R_1 + \varepsilon_1 R_3 + \varepsilon_2 R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3}$$

Note that I_2 is negative, this just indicates that the direction of I_2 is opposite of what we assumed initially.

$$I_3 = \frac{\varepsilon_2 R_2 - \varepsilon_1 R_1}{R_1 R_2 + R_1 R_3 + R_2 R_3}$$

RC circuits

Charging a capacitor



S ≡ switch

ε ≡ emf source

C ≡ capacitor

R ≡ resistor

t ≡ time

At $t<0$ the switch is open so there is no voltage across the capacitor

At $t=0$ ~~current begins to flow~~ the switch is closed & current begins to flow according to:

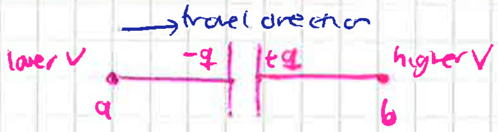
$$I_0 = \frac{\varepsilon}{R}$$

At this instant the potential difference from the battery terminals is the same as that across of the resistor. This initiates charging of the capacitor. As the capacitor charges the voltage across the capacitor increases in time:

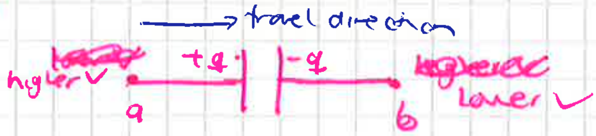
$$V_c(t) = \frac{q(t)}{C}$$

How much charge is stored on the plates?

Kirchoff's rule for capacitors:

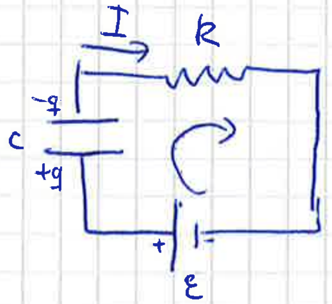


$$\Delta V = V_b - V_a = \frac{q}{C}$$



$$\Delta V = V_b - V_a = -\frac{q}{C}$$

Using this rule and traversing the circuit loop clockwise we obtain:



$$\varepsilon + \underbrace{V_c(t)}_{\text{voltage across the capacitor}} - \underbrace{I(t)R}_{\text{voltage across resistor}} = 0 \quad (*)$$

sum of voltages in a closed loop is zero

$$\Rightarrow \varepsilon - \frac{dq}{dt}R - \frac{q}{C} = 0 \quad (**)$$

I, must be the same in all parts of the series circuit, the current across the resistance R is equal to the rate of increase of charge on the capacitor plates.

Over time current flow in the circuit will decrease because the charge present on the capacitor ~~will~~ makes it harder to put more charge on the capacitor. Once the charge on the capacitor reaches its max value Q, the current in the circuit will drop to zero. This becomes clear if we re-write eq (*) as:

$$I(t)R = \varepsilon - V_c(t)$$ once V_c reaches ε current is zero

Now, equation (**) gives us a relation between the rate of change of charge to the charge on the capacitor:

$$\frac{dq}{dt} = \frac{1}{R}\left(\varepsilon - \frac{q}{C}\right)$$

$$\Rightarrow \frac{dq}{\left(\varepsilon - \frac{q}{C}\right)} = \frac{1}{R}dt \quad \Rightarrow \frac{dq}{q - C\varepsilon} = -\frac{1}{RC}dt$$

and integrating both sides we get:

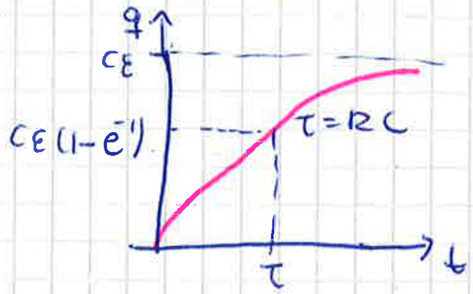
$$\int_0^q \frac{dq}{q - C\varepsilon} = -\frac{1}{RC}\int_0^t dt$$

$$\Rightarrow \ln\left(\frac{q - C\varepsilon}{-C\varepsilon}\right) = -\frac{t}{RC}$$

$$\Rightarrow \boxed{q(t) = C\varepsilon\left(1 - e^{-t/RC}\right) = Q\left(1 - e^{-t/RC}\right)}$$

where $Q = C\varepsilon$ is the maximum of charge stored on the plate

So if we plot $q(t)$ we have:



Once we know the charge on the capacitor we can also determine the voltage across the capacitor:

$$V_c(t) = \frac{q(t)}{C} = \mathcal{E}\left(1 - e^{-t/RC}\right)$$

, this has the same form as the figure above for q.

Now, the current that flows in the circuit is:

$$I(t) = \underbrace{\frac{dq}{dt}}_{\text{derivative in time of the charge}}$$

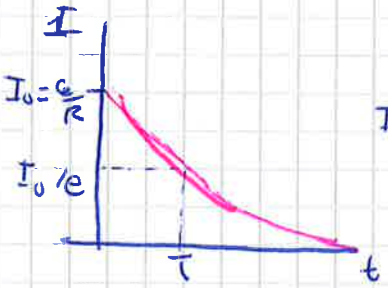
Remembering the expression for q:

$$q(t) = C\mathcal{E}\left(1 - e^{-t/RC}\right)$$

$$I = \frac{dq}{dt} = \frac{C\mathcal{E}}{RC}\left(e^{-t/RC}\right) = \frac{\mathcal{E}}{R}e^{-t/RC} = I_0 e^{-t/RC}$$

where I_0 is the initial current that flows in the circuit when the switch was closed at $t=0$.

so the current as a function of time is:



$$I(t) = I_0 e^{-t/\tau}$$

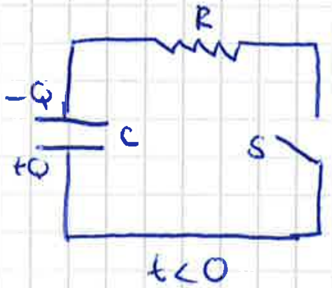
current decreases exponentially w/ the time constant $\tau = RC$

Discharging a capacitor

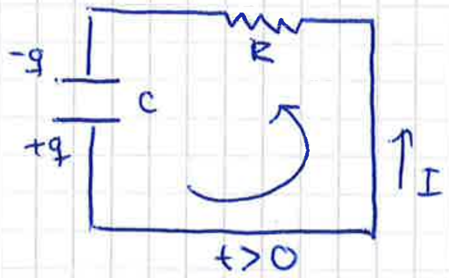
- Now we have a different situation where the capacitor was first charged to some value Q.
- At $t<0$ the switch is open and the potential difference across the capacitor is:

$$V_0 = \frac{Q}{C}$$

- Across the resistor the potential difference is zero because $I=0$



Now what happens at $t=0$ when we close the switch?



At $t>0$ the capacitor begins to discharge and there is a current

As current flows, the voltage across the capacitor decreases

Applying Kirchhoff's loop rule by traversing the loop counterclockwise we get:

$$\frac{q}{C} - IR = 0 \quad (*)$$

The current that flows away is proportional to the charge on the plate

$$I = -\frac{dq}{dt}$$ so subst this in eq $(*)$ we get:

this is negative b/c charge on the positive plate is decreasing as more positive charges leave the positive plate

$$\frac{q}{C} + R\frac{dq}{dt} = 0$$

we can solve this eq as we did for the charging capacitor:

$$\frac{dq}{q} = -\frac{1}{RC}dt \Rightarrow \int_Q^q \frac{dq}{q} = -\frac{1}{RC}\int_0^t dt$$

$$\Rightarrow \ln\left(\frac{q}{Q}\right) = -\frac{t}{RC}$$

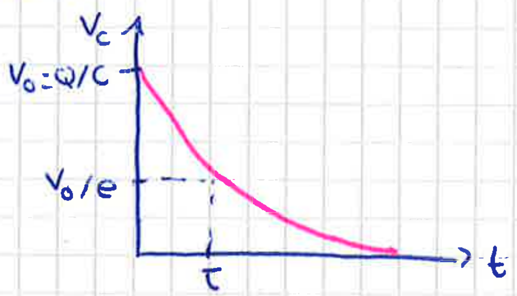
$$\Rightarrow \boxed{q(t) = Qe^{-t/RC}}$$

Now that we have the charge, we can also determine the voltage across the capacitor:

$$V_c(t) = \frac{q(t)}{C} = \left(\frac{Q}{C}\right)e^{-t/RC}$$

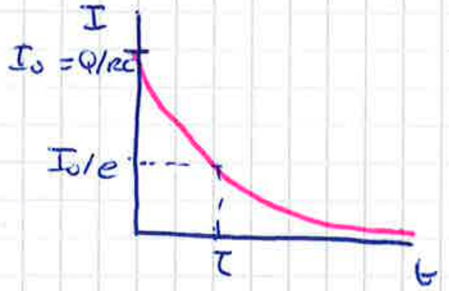
subs $q(t)$ we just found

This looks like:

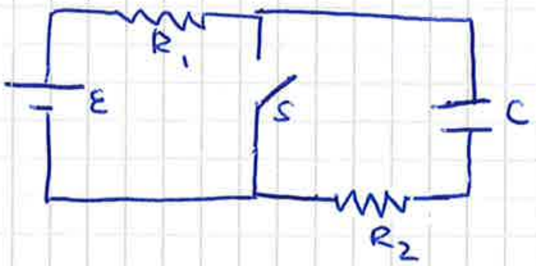


From the expression for the charge we can also calculate the current:

$$I = -\frac{dq}{dt} = \left(\frac{Q}{RC}\right)e^{-t/RC}$$ which has a similar form as V:



EXAMPLE: RC circuit.



a) What is the time constant before the switch is closed?
b) What is the time constant after the switch is closed?
c) Find the current through the switch as a function of time after the switch is closed.

a) Before the switch is closed, the two resistors R_1 and R_2 are in series with the capacitor, so the time constant is:

$$\tau = R_{eq} C = (R_1 + R_2) C$$

remember that the charge stored in the capacitor depends on the time constant as.

$$q(t) = C\varepsilon \left(1 - e^{-t/\tau}\right)$$

b) After the switch is closed the closed loop on the right becomes a decaying RC circuit, with a time constant $\tau^* = R_2 C$. Charge begins to decay according to:

$$q^*(t) = C\varepsilon\, e^{-t/\tau^*}$$

c) The current passing through the switch consists of two sources:

The steady current I_1 from the left of the circuit and the decaying I_2 from the RC circuit. These currents are:

$$I_1 = \frac{\varepsilon}{R_1}$$

$$I_2(t) = \frac{dq^*}{dt} = -\left(\frac{C\varepsilon}{\tau^*}\right) e^{-t/\tau^*} = -\left(\frac{\varepsilon}{R_2}\right) e^{-t/R_2 C}$$

the negative sign in I_2 indicates that the direction of flow is opposite ~~to that~~ of the charging process.

Since both I_1 and I_2 move downward across the switch, the total current is:

$$I(t) = I_1 + I_2(t) = \frac{\varepsilon}{R_1} + \left(\frac{\varepsilon}{R_2}\right) e^{-t/R_2 C}$$