Addendum: Basics of Gauss's law.

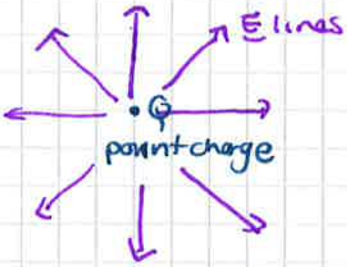
$$\underline{E} = \frac{1}{4\pi\varepsilon_0}\frac{q\hat{r}}{r^2} \quad \text{Coulomb's law}$$

Are they equivalent?

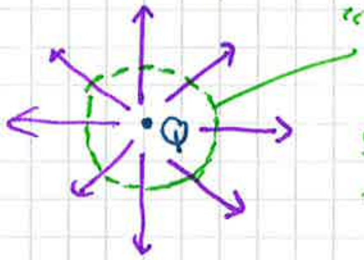
$$\Phi_E = \oint_S \underline{E}\cdot d\underline{a} = \frac{Q_{enc}}{\varepsilon_0} \quad \text{Gauss's law}$$

If charges are not moving (statics)
YES! as soon as charges move no.

what is $\underline{E}$ and what is Φ_E?
What is the difference between them?



what is Φ_E?



"Gaussian surface"

Φ_E is the flux of $\underline{E}$ through that surface

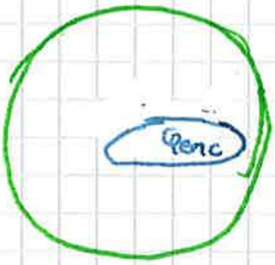
$$\Phi_E = \int_S \underline{E}\cdot d\underline{a}$$

We can use the statement of Gauss's law to calculate $\underline{E}$ when we can find a surface where $\underline{E}$ is constant and $\underline{E}$ is 'aligned' with $d\underline{A}$ i.e they are parallel (or $\frac{\underline{E}\cdot d\underline{A}}{|E||dA|} = 1$).

this ensures we can take $\underline{E}$ out of the $\int$

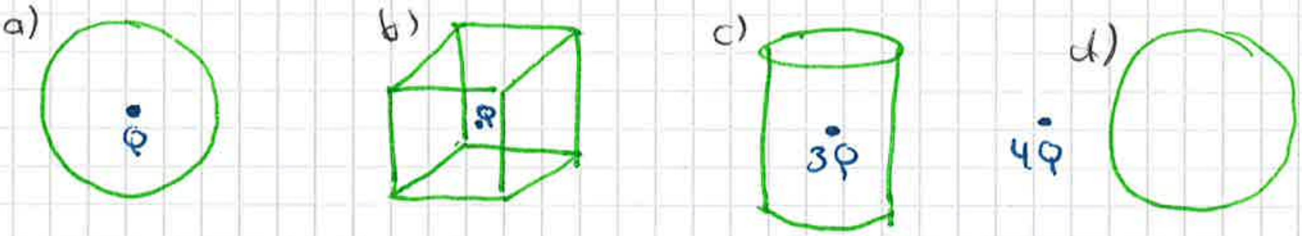
For example in this case, can we use Gauss's law to calculate $\underline{E}$?

Is $\underline{E} = \frac{Q_{enc}}{4\pi r^2 \varepsilon_0}$?



NO because $\underline{E}$ is not aligned with $d\underline{A}$ & $\underline{E}$ is not constant along the Gaussian surface.

To further understand the concept of flux of $\underline{E}$ i.e Φ_E. Can you rank which surfaces have the least & greatest flux of $\underline{E}$?



c has the largest flux
a & b have the same flux
d has zero flux.

Lecture 6. Conductors

Last time:

* Poisson's equation:

$$\nabla^2 \varphi = -\frac{1}{\varepsilon_0}\rho$$ can be derived from Gauss's law & our definition of potential

* Laplace's equation

$$\nabla^2 \varphi = 0$$ in regions where there is no charge, Poisson's reduces to Laplace's equation.

Fundamental equation for lot's of branches of physics → steady state heat conduction
→ irrotational flows
→ electric potential

properties of the solution of Laplace's equation:

- Value of φ at (x,y,z) is equal to the average value of φ around this point.
- φ has no local min or max. Extrema occur at the boundary

* First uniqueness theorem: The solution to Laplace's equation in some volume V is UNIQUELY determined if φ is specified on the boundary surfaces.
(applies to both Laplace & Poisson's equation)

* Stokes theorem

$$\int_S (\nabla \times \underline{v})\cdot d\underline{a} = \oint_P \underline{v}\cdot d\underline{l} \Rightarrow \nabla \times \underline{E} = 0$$

Today: Conductors

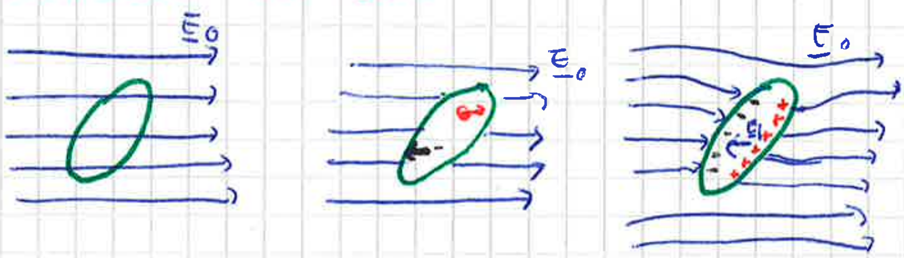
Conductor → Material w/ free $\bar{e}$ → Metals e.g. Au, Ag, Cu excellent conductors
→ Ionic solutions NaCl in H_2O OK conductors

Insulator → Material w/o free $\bar{e}$ → organic materials eg rubber, plastic
→ Inorganic materials e.g. quartz, glass

Demo 418 Are you a conductor?

Basic properties of conductors

(i) $\underline{E}$ inside a conductor is zero



- Initially $\underline{E}_0$ will drive ⊕ charges to the right and ⊖ to the left (Remember in conductors by def. charges are free to move)
In reality $\bar{e}$ move but this leaves an excess ⊕ on one side.

- When charges arrive @ the end, they 'pile up' creating a field of their own $\underline{E}_1$ in the opposite dir than $\underline{E}_0$
- Charge will continue to flow until $\underline{E}_1$ cancels $\underline{E}_0$ <u>inside</u> the conductor
- <u>Outside</u> the conductor $\underline{E}_1$ will NOT cancel $\underline{E}_0$.

(ii) $\rho = 0$ inside a conductor / φ inside a conductor is constant

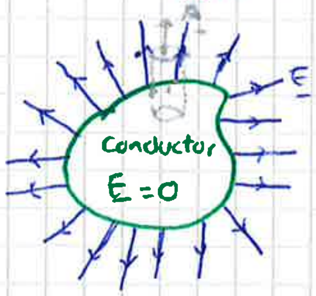
- If we have $\underline{E} = 0$ inside the conductor, by Gauss's law $\nabla \cdot \underline{E} = \frac{\rho}{\varepsilon_0} = 0 \Rightarrow \rho = 0$.
- This doesn't mean there is no charge inside, rather there is the same amount of ⊕ & ⊖ so they cancel out. The precise statement is then:

  The net charge density inside a conductor is zero.
- Since $\underline{E} = 0$ inside a conductor, it follows that $\varphi(\underline{a}) - \varphi(\underline{b}) = \int_a^b \underline{E} \cdot d\underline{s} = 0 \Rightarrow \varphi(a) = \varphi(b)$

(iii) Any net charge resides on the surface

it's the only place left. It is NOT uniformly distributed

(iv) $\underline{E}$ is ⊥ to the surface, just outside the conductor



- If there was a tangential component of $\underline{E}$ @ the surface ⇒ charge would move & everything would rearrange until $\underline{E}$ ⊥ surface of conductor
- To calculate the field strength just outside the surface consider the gaussian pillbox depicted

$$\Phi_E = \Phi_{top} + \cancelto{0}{\Phi_{bottom}} = \Phi_{top} = E\int da = \frac{Q_{enc}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

b/c $\underline{E} = 0$ inside a conductor

$$\Rightarrow \boxed{E_n = \frac{\sigma}{\varepsilon_0}}$$

For any shape conductor so $\underline{E} = \frac{\sigma}{\varepsilon_0}\hat{n}$

(v) A conductor is an equipotential

- $\underline{E}$ is ⊥ to the surface of the conductor ⇒ surface of the conductor is an equipotential.
- Formally, for any two points $\underline{a}, \underline{b}$ on the surface of the conductor

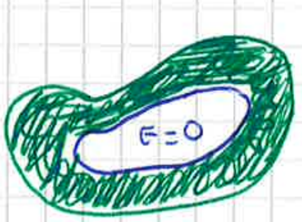
$$\varphi(\underline{a}) - \varphi(\underline{b}) = -\int_a^b \underline{E} \cdot d\underline{s} = 0 \qquad \Rightarrow \varphi(\underline{a}) = \varphi(\underline{b})$$

Demo 414

What are the implications of these statements?

Corollary 1. Shielding of external $\underline{E}$

In a hollow region inside a conductor, φ = const and $\underline{E} = 0$ if there are no charges in the cavity.



For nicer figure see Purcell fig 3.8

- Potential inside the hollow region must satisfy Laplace's eq. $\nabla^2\varphi = 0$
- The boundary of the hollow region is a conductor ⇒ it's an equipotential ⇒ $\varphi = \varphi_0$ constant
- A solution for φ that fulfills this BC is $\varphi = \varphi_0$ → This is THE solution ~~applying the~~ (guaranteed by the first uniqueness theorem)

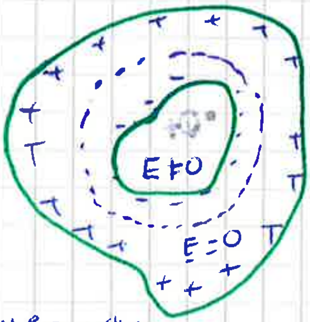
∴ φ = const in the hollow region
⇒ $\underline{E} = 0$

This is how a Faraday cage works!

Demo 525

Corollary 2. Induced charges

A charge +Q in the cavity of a conductor will induce a charge +Q on the outside of it.



(see fig 2.45 Griffiths)

- Total charge induced on the cavity wall is equal & opposite to the charge inside.
  Take the Gaussian surface shown (all inside the conductor)

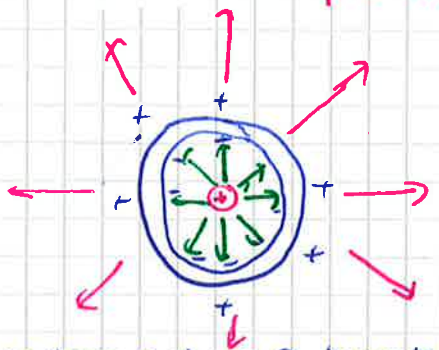
$$\oint \underline{E} \cdot d\underline{a} = 0 = Q_{enc}$$

$\uparrow$ b/c inside a conductor $\underline{E} = 0$

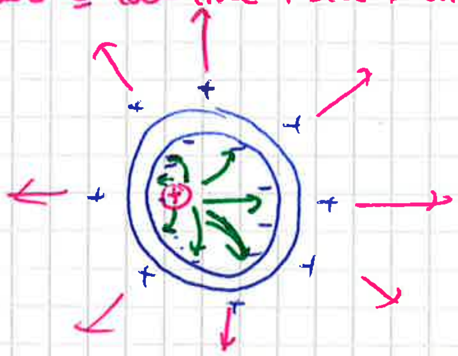
and $q_{enc} = +Q + q_{ind} \Rightarrow q_{ind} = -Q$

- If the conductor is electrically neutral $\Rightarrow$ there must be a charge +Q on its outer surface.

Example 1. What does $\underline{E}$ look like inside + outside of the conductor?



encloses a charge @ the center



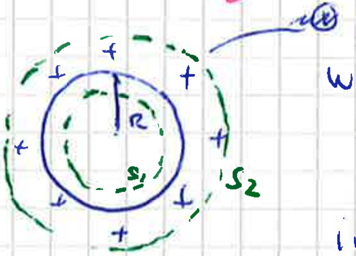
- encloses a charge off-center

for both cases $\underline{E} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{r^2}\hat{r}$ outside

why? $\underline{E}$ has to be $\perp$ to the surface of the conductor everywhere
$\Rightarrow$ it's radial

q is the induced charge due to the presence of q inside the cavity.

Example 2. Charge Q on a conductor sphere (solid)



What is $\underline{E}$?

i) For $r < R$ consider the Gaussian surface S_1

$$\oint_{S_1} \underline{E} \cdot d\underline{A} = 0 \Rightarrow \underline{E} = 0 \text{ everywhere}$$

by "definition" of a conductor, charge goes to the surface

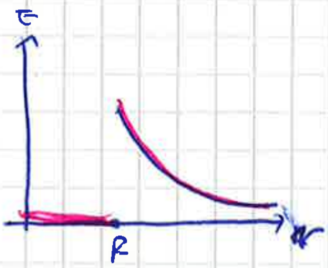
(this is ≠ from a non-conducting solid sphere. Remember lecture 3?)

ii) For $r > R$ consider the surface S_2

$$\oint_{S_2} \underline{E} \cdot d\underline{A} = \frac{Q}{\varepsilon_0}$$

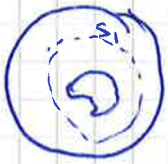
$$\Rightarrow \underline{E} \int_{S_2} dA = \frac{Q}{\varepsilon_0} \Rightarrow E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

For a conductor $\underline{E} \parallel d\underline{A} \parallel \hat{r}$



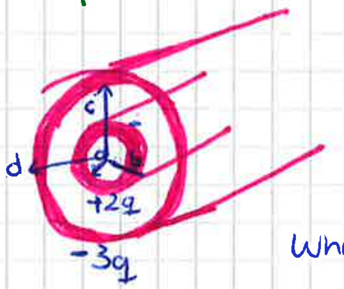
discontinuity @ surface of conductor

Example 3. What happens if The conductor has a hole inside?



Nothing E is identical to the case of a solid conductor.

Example 4. What is the charge on the surface of the following conductor?



A long hollow cylindrical conductor

Inside the hollow cylinder there is a charge +2q
outside the hollow cylinder there is a charge −3q

What is the charge on each surface of the conductor?

For $r \leq a$: $Q = 0$

For $r = b$: $Q = +2q$ ← info from the problem

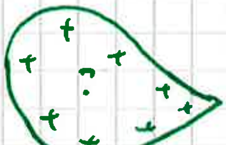
For $r = c$ $Q = -2q$

This is because E inside the conductor must be zero

$$\Rightarrow Q_{enc} = 0 = +2q + \boxed{q_{at\,c}} \rightarrow \text{must be } -2q$$

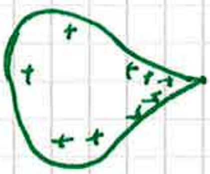
then I need $-q$ at surface d such that the total charge outside the largest cylinder is $-3q$.

Example 5. How is charge distributed on a conductor?



if we deposit a charge Q on a teardrop shaped conductor. How will it distribute itself over the surface?

experimentally:



$\sigma_{tip} >> \sigma_{flat}$

Important consequence: $E = \frac{\sigma}{\epsilon_0}$

$\Rightarrow E_{tip} >> E_{flat}$

But the condition that φ is constant over the entire surface still holds true.
(it's a requirement of conductors)

why?

Demo 484

we can also provide some intuition on this by considering the next problem:

Consider 2 conducting spheres connected by a wire:
conductive

$r_1 >> r_2$

Deposit some charge on them

→ Charge will redistribute until $\varphi = const$. Let's assume the charges on each sphere at eq. are q_2 & q_1. Neglecting the effect of the wire, the equipotential condition for conductors implies:

$$\varphi_1 = \varphi_2$$

$$\Rightarrow \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_1} = \frac{1}{4\pi\varepsilon_0}\frac{q_2}{r_2} \Rightarrow \frac{q_1}{r_1} = \frac{q_2}{r_2} \quad *$$

Assuming that the spheres are far apart so σ is uniform on the surface of both spheres, we can write E as:

$$E_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_1^2} = \frac{\sigma_1}{\varepsilon_0} \quad ; \quad E_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_2}{r_2^2} = \frac{\sigma_2}{\varepsilon_0} \quad **$$

Combining $*$ and $**$ we have:

$$\frac{E_1}{E_2} = \frac{\sigma_1}{\sigma_2} = \frac{r_2}{r_1}$$

we can see that $\sigma_i \propto \frac{1}{r_i} \Rightarrow$ if r_i is large $\Rightarrow \sigma_i$ small
if r_i small $\Rightarrow \sigma_i$ large

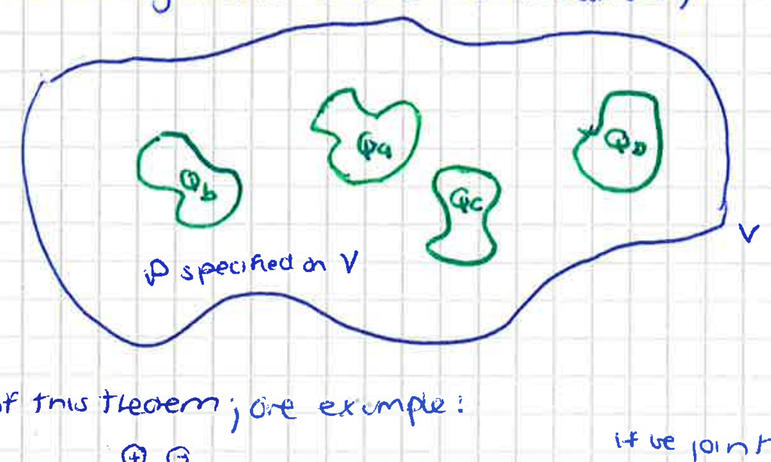
∴ Regions w/ smallest radius of curvature have the greatest σ.
So $\underline{E}$ on the surface of a conductor is the greatest at the sharpest points.

This principle is used in designing lightning rods.

Application of the uniqueness theorem and a second uniqueness theorem for conductors

Second uniqueness theorem:

In a volume V surrounded by conductors & containing a specified charge density ρ, the electric field is uniquely determined if the total charge on each conductor is given. (The region as a whole can be bounded by another conductor or unbounded)



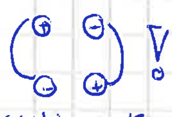
Proof: Griffiths section 3.1.6 for fun.

Implications of this theorem; one example:

4 conductors ~~consisting of~~ with charges $\pm Q$ ~~formed by~~

is this a possible charge configuration?

if we join them by a wire:



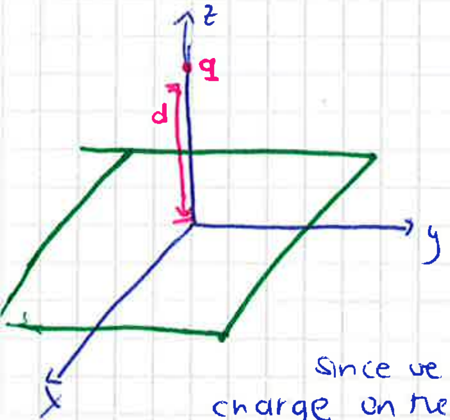
impossible config

after joining w/ wire we now have 2 conductors w/ total charge 0.

One possibility to distrib Q is as shown ∃ is the ONLY

Applications of the uniqueness Theorems. Method of images

The classic image problem:



Suppose we have a point charge q above a grounded infinite conducting plane.

grounded means $\varphi = 0$ at the conductor's surface

What is the potential in the region above the plane?

We're tempted to say $\frac{1}{4\pi\varepsilon_0}\frac{q}{r}$ but this is WRONG, we need to consider the induced charge.

Since we have a conductor q will induce a certain amount of negative charge on the nearby surface of the conductor

$\Rightarrow \varphi_{tot} = \varphi_q + \varphi_{q_{ind}}$

How can we calculate φ_{tot} if we don't know q_{ind}?

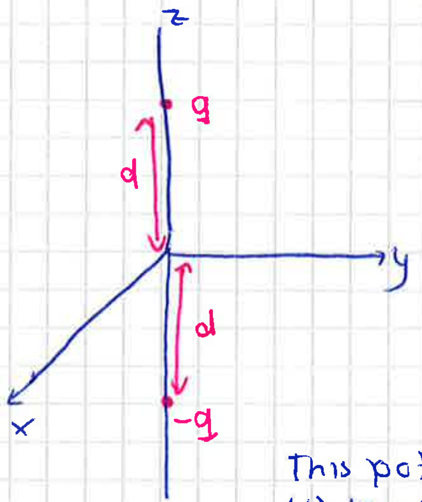
Mathematically our problem is to solve Poisson's eq in the region $z>0$ with a single point charge q at $(0,0,d)$, subject to the BC:

(1) $\varphi = 0$ at $z=0$ (i.e the conductor is grounded)
(2) $\varphi \to 0$ away from q, i.e for points $(x^2+y^2+z^2)^{1/2} >> d$

By the 1st uniqueness theorem, if φ satisfies Poisson's eq & the BC $\Rightarrow \varphi$ is a unique solution to the problem.

$\Rightarrow$ We can substitute our problem for one that is much easier to solve. As long as we satisfy the BC the solution that we find will be the correct one!

We will substitute our problem by the following configuration of charges



We have 2 point charges: $+q$ at $(0,0,d)$ and $-q$ at $(0,0,-d)$

we got rid of the conducting plane

For this charge configuration we can easily write the potential:

$$\varphi(x,y,z) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{[x^2+y^2+(z-d)^2]^{1/2}} - \frac{q}{[x^2+y^2+(z+d)^2]^{1/2}}\right]$$

by superposition we can just add the potential of 2 charges

denominator is just the distance of (x,y,z) to the charge $+q$

distance of a pt (x,y,z) to the charge $-q$

This potential satisfies the BC of the original problem, namely:

(1) $\varphi = 0$ at $z=0$
(2) $\varphi \to 0$ for $(x^2+y^2+z^2)^{1/2} >> d$

And the only charge in the region $z>0$ is q.
So we have all the conditions of the original problem.

$\therefore$ This potential is the solution to our original problem
the uniqueness theorem guarantees this.

So now that we have the potential, we can actually calculate the induced surface charge on the conductor:

We know that at the surface of the conductor

$$\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{n} \qquad \text{rewriting this in terms of } \varphi:$$

$$\sigma = -\varepsilon_0 \frac{\partial \varphi}{\partial n}$$ where $\frac{\partial \varphi}{\partial n}$ is the normal derivative at the surface, in this case.

$$\sigma = -\varepsilon_0 \frac{\partial \varphi}{\partial z}\Big|_{z=0}$$

Now if we calculate $\frac{\partial \varphi}{\partial z}$ for the potential we found:

$$\frac{\partial \varphi}{\partial z} = \frac{1}{4\pi\varepsilon_0}\left\{ \frac{-q(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} + \frac{q(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}} \right\}$$

$$\Rightarrow \boxed{\sigma(x,y) = \frac{-qd}{2\pi(x^2+y^2+d^2)^{3/2}}}$$

induced surface charge is negative ✓
it's greatest at $x=y=0$

Now if we compute the total induced charge:

$$Q = \int \sigma \, da$$

for convenience we compute this $\int$ in polar coordinates (r, φ) with $r^2 = x^2+y^2$ & $da = r\,dr\,d\varphi$

so $$\sigma(r) = \frac{-qd}{2\pi(r^2+d^2)^{3/2}}$$

Now computing the $\int$:

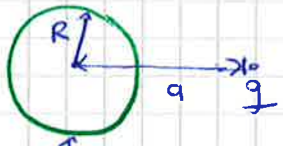
$$Q = \int_0^{2\pi}\int_0^{\infty} \frac{-qd}{2\pi(r^2+d^2)^{3/2}}\, r\,dr\,d\varphi = \frac{qd}{(r^2+d^2)^{1/2}}\Bigg|_0^{\infty} = -q$$ which makes sense.

Method of images

The method we just described can be used for any stationary charge distribution near a grounded conducting plane.

A "mirror" image" charge distribution can be introduced.
- Remember that it should have the opposite sign so φ at the ~~conductor~~ place where the conductor would be is zero.

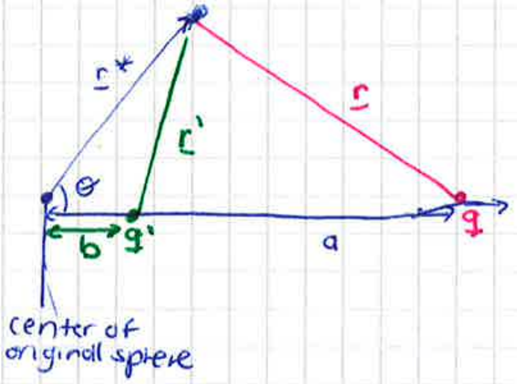
Another example:
A point charge q is situated at a distance a from the center of a grounded conducting sphere of radius $R<a$. Find the potential outside the sphere.



$\varphi = 0$
on the surface of the conductor

- I need to find a charge distrib that satisfies $\varphi=0$ on the surface of the sphere
- I need to add the q inside the sphere (so I don't change anything outside)
region of interest

we can substitute our problem by the following distrib of charges:



we put a mirror charge q' a distance b from the would be center of the sphere. we define b such that:

$$\frac{q}{b} = \frac{a^2}{R^2}$$

$$\Rightarrow b = \frac{R^2}{a}$$ is a point just to the right of the center of the sphere.

the point charge $q' = -\frac{R}{a} q$

So the potential for the 2 pt charge at a point r^* is:

$$\varphi(\underline{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} + \frac{q'}{r'}\right)$$ where $\underline{r}$ & $\underline{r}'$ are indicated in the diagram.

~~You can check~~ Using the law of cosines this potential can be re-written as:

$$\varphi(r,\theta) = \frac{1}{4\pi\varepsilon_0}\left\{\frac{q}{\left[r^2-a^2-2ra\cos\theta\right]^{1/2}} - \frac{q}{\left[R^2+(ra/R^2)-2ra\cos\theta\right]^{1/2}}\right\}$$

where θ, r are the usual spherical polar coord. w/ the z axis along the line through q.

Here you can see that $\varphi = 0$ for $r = R$ so it's zero on the surface of the sphere.