Lecture 3. The electric field, Gauss's law & its applications

Last time:

① Coulomb's law

$$\underline{F}_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{r}$$ Force felt by charge q_2 due to q_1

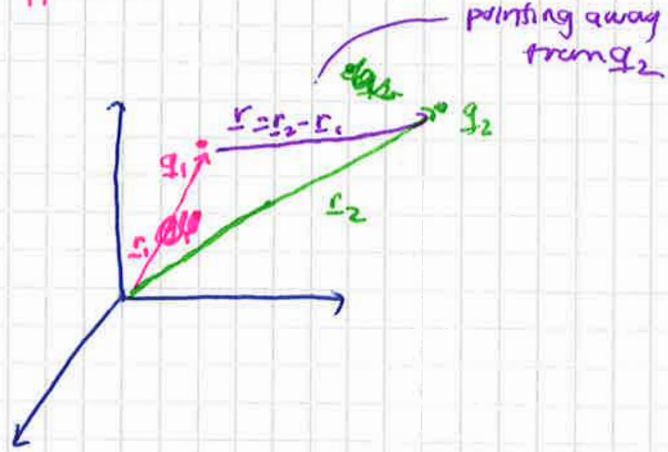
where $\underline{r} = \underline{r}_2 - \underline{r}_1$ ~~is the~~

vector w/ magnitude r

and direction $\hat{r}$

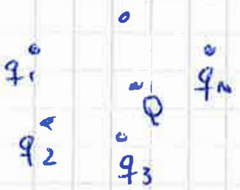
and $\underline{r}_2 \equiv$ position of q_2

$\underline{r}_1 \equiv$ position of q_1



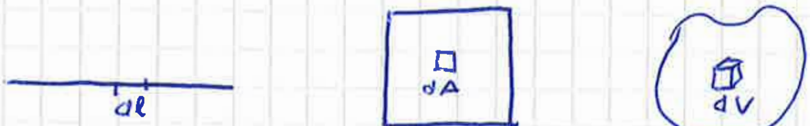
& it follows that $\underline{F}_2 = -\underline{F}_1$

② Principle of superposition



$$F_Q = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i Q}{r_i^2}\hat{r}_i$$

③ For continuous distributions of charge:

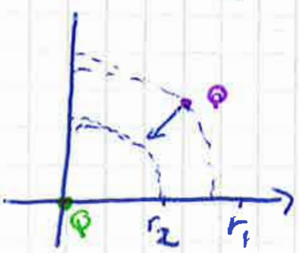


$$\lambda = \frac{dq}{dl} \qquad \sigma = \frac{dq}{dA} \qquad \rho = \frac{dq}{dV}$$

$$\underline{F}_q = \frac{q}{4\pi\varepsilon_0}\int_L \frac{\lambda\, dl\,\hat{r}}{r^2} = \frac{q}{4\pi\varepsilon_0}\int_A \frac{\sigma\, dA}{r^2}\hat{r} = \frac{q}{4\pi\varepsilon_0}\int_V \frac{\rho\, dV}{r^2}$$

F due to these distrib.

What is the potential energy of a system of charges?



How much work do I need to do to move Q from r_1 to r_2?

$$W = \int \underline{F}\cdot d\underline{s}$$

$\underline{F} = -\frac{Qq}{r^2}\hat{r}$ — minus b/c we need to work 'against' the $\underline{F}$

for our case $\underline{F} = \underline{F}_{coulomb}$

so if we assume a radial path:

$$W = -\int_{r_1}^{r_2}\frac{Qq\hat{r}}{r^2}\cdot d\hat{r} = \frac{Qq}{r_2} - \frac{Qq}{r_1}$$

now b/c $r_1 > r_2 \Rightarrow$ 2nd term is smaller so $W > 0$

$$\boxed{U = \frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1 \\ j\neq i}}^{N}\frac{1}{4\pi\varepsilon_0}\frac{q_i q_j}{r_{jk}}}$$

physically I don't need to inject energy to move the charge

in general: $W > 0$ for charges of same sign

$W < 0$ for charges $\neq$ sign

Now we call U total W to assemble a configuration of charges.

In general b/c of the superposition principle.

$$U = \frac{1}{2} \sum_{j=1}^{N} \sum_{\substack{k=1 \\ k \neq j}}^{N} \frac{1}{4\pi\varepsilon_0} \frac{q_j q_k}{r_{jk}}$$

Important corollaries:

① Work is path independent → If you draw any path, you can decompose a segment "ds" into components // to r
$\perp$ to r → 0 all $\perp$ components will be zero

this is similar for the gravitational $\underline{F}$ & in general true for a central $\underline{F}$

$$\underline{F}_c = F(r)\hat{r}$$

$$W_{AB} = \int_A^B F(r)\hat{r}\cdot d\underline{s}$$ in polar coord $d\underline{s} = d\underline{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + \boxed{r\sin\theta\, d\varphi\,\hat{\varphi}}$ for 3D

If we subs this in the integral we get:

$$W_{AB} = \int_{r_A}^{r_B} F(r)\,dr$$

② $\underline{F}$ is conservative

$$W_{11} = \oint \underline{F}\cdot d\underline{s} = 0$$ work done along a path that starts & ends @ the same point is zero

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The electric field

We always need to introduce a test charge "q" to ask w is the $\underline{F}$ acting on it due to the presence of another charge. To avoid this:

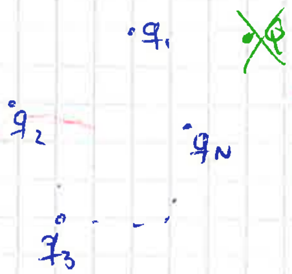
$$\underline{E} = \frac{\underline{F}}{q}$$ Mathematically

but what is it physically?

↳ $\underline{F}$ field?

remember the pictures I showed in the 1st lecture?

For a system of discrete charges

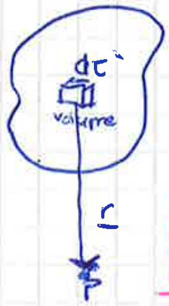


$$\boxed{\underline{E} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{r_i^2}\hat{r}_i}$$

- vector qty
- varies across space
- determined by the configuration of "source" charges.

For a cont. charge distrib:

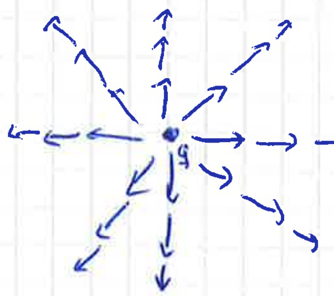
$$\underline{E}(\underline{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{1}{r^2}\hat{r}\,dq$$



If $dq = \rho\, d\tau'$

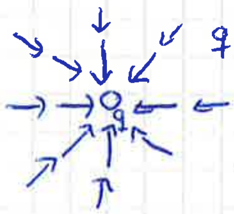
$$\boxed{\underline{E}(\underline{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\underline{r}')}{r^2}\hat{r}\,d\tau'}$$

How do we represent the electric field?



$q = +3$

positive charges are a source



$q = -1$

negative charges are sinks.

Remember $\underline{E}$ is a 3D entity, best we can do is look @ the plane where the charges are & indicate magnitude + direction of $\underline{E}$ w/ arrows

$\underline{E}$ is a vector field.

Now let's look @ the superposition of fields:

• +3

o −1

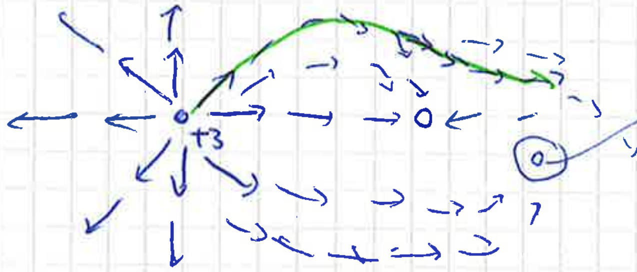
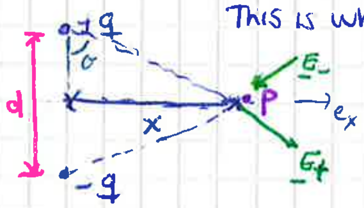


Fig 1.10 Purcell

we can connect the arrows + get the field lines.

Demo 412 the electric field

let's work through some examples:



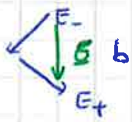
This is what we call an electric dipole.

d ~~separation~~ distance between +q + −q

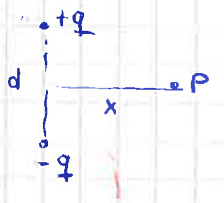
consider a pt P a distance x from the dipole axis

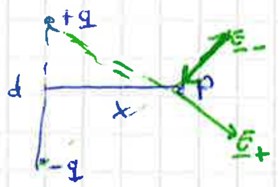
$$\underline{E}_P = \underbrace{\underline{E}_-}_{\text{due to } q_-} + \underbrace{\underline{E}_+}_{\text{due to } q_+}$$

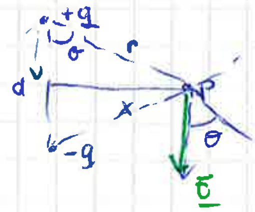
In which direction is the $\underline{E}_P$ pointing?



b/c of symmetry the horizontal components cancel.

① 

② 

③ 

$$|E| = 2E_+ \cos\theta$$

where E_+ is field of a pt charge

$$|E| = 2\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\right)\cos\theta$$

and from trig $r^2 = \left(\frac{d}{2}\right)^2 + x^2$ (*)

and $\cos\theta = \frac{d/2}{r}$

we rewrite this in terms of x + d using eq (*)

$$\cos\theta = \frac{d/2}{\left[(d/2)^2 + x^2\right]^{1/2}} \quad (**)$$

so subs (*) & (**) into the eq for $\underline{E}$:

$$E = \frac{q}{2\pi\varepsilon_0}\left(\frac{1}{\left(\frac{d}{2}\right)^2 + x^2}\right)\left(\frac{d/2}{\left[(d/2)^2 + x^2\right]^{1/2}}\right)$$

$$= \boxed{\frac{qd}{4\pi\varepsilon_0}\frac{1}{\left[x^2 + (d/2)^2\right]^{3/2}}}$$

Interesting limit case: if P is really far away ⇒ $x \gg d$

rewrite this term as

$$\left[x^2 + \left(\frac{d}{2}\right)^2\right]^{3/2} = x^3\left[1 + \left(\frac{d}{2x}\right)^2\right]^{3/2}$$

recognize this?

$(1+y)^n \approx 1 + ny$ if $y \ll 1$

so take $y = \left(\frac{d}{2x}\right)^2$

so then we have that the field is

$$E \simeq \frac{1}{4\pi\varepsilon_0}\frac{qd}{x^3} \propto \frac{1}{x^3}$$ cube of distance from dipole

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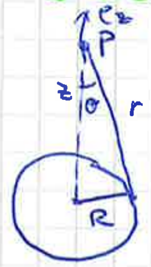
Now for a continuous charge distribution: (too much already!) ONLY DO THE DIPOLE



$$E = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda\, d\ell}{r^2}\cos\theta\,\hat{z}$$ this is just to set up the next problem.

$$= \frac{1}{4\pi\varepsilon_0}\frac{\lambda(2\pi R)z}{(R^2+z^2)^{3/2}}\hat{z} =$$

$=$

horizontal components cancel

Notice

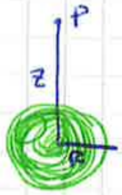
$r^2 = R^2 + z^2 \leftarrow$ const

$\cos\theta = \frac{z}{r} \leftarrow$ const

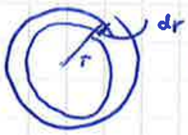
and

$\int d\ell = 2\pi r$

What happens if you now have a disk of charge?



we know the expression for a ring, we just integrate along the area:



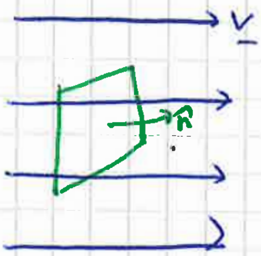
radius r & thickness dr

what is the total charge of each ring?

$\sigma\, 2\pi r\, dr = \lambda\, 2\pi r \Rightarrow \lambda = \sigma\, dr$

$$E_{ring} = \frac{1}{4\pi\varepsilon_0}\frac{(\sigma\, dr)2\pi r z}{(r^2+z^2)^{3/2}} \qquad E_{disk} = \frac{1}{4\pi\varepsilon_0}2\pi\sigma z\int_0^R \frac{r}{(r^2+z^2)^{3/2}}\,dr$$

~~Gauss's law & the concept of flux~~



what is the flux of $\underline{v}$ through the loop?

$\underline{v}(x, y, z)$

loop or surface S

loop area vector $\underline{A} \equiv A\hat{n}$

What is the flux of $\underline{v}$ through this surface?

$$\Phi_v = \underline{v} \cdot d\underline{A}$$

~~intuit~~

Intuitively this will depend on the orientation of $d\underline{A}$

$v \parallel d\underline{A}$ → everything will go thru max flux

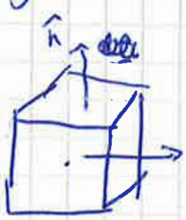
$v \perp d\underline{A}$ Φ is zero

mathematically remember the geom interp of the dot product:

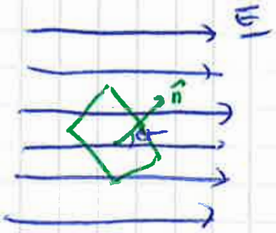
$\Rightarrow \Phi_v = |vA| \cos\theta$

where θ is the angle between v and A.

Notice that ~~the~~ $d\underline{A}$ is not uniquely defined, but we will consider it such that if a source of v is at the center of a surface, the flux through the surface is positive



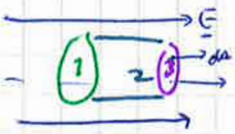
Now that we have an intuition for the idea of flux, let's go back & think about the flux of $\underline{E}$ through a surface:



$$\Phi_E = \int_S \underline{E} \cdot d\underline{A} = \underline{E} \cdot \underline{A} = EA\cos\theta$$

$\Phi_E \propto$ # of field lines going through the loop

what is the flux of $\underline{E}$ through a closed surface?



cylinder $\parallel$ to $\underline{E}$

$E = \Phi_1 + \Phi_2 + \Phi_3$

$\Phi_2 = 0$ because $\underline{E} \perp \hat{n}$

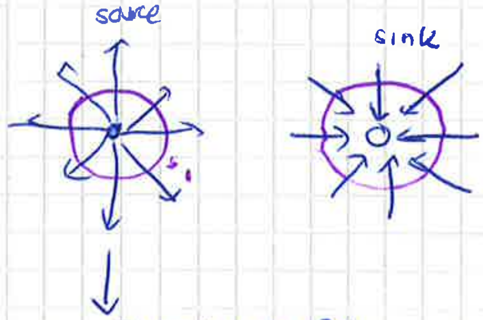
$|\Phi_1| = |\Phi_3|$ but they have the opposite sign

$\Rightarrow$ TOTAL FLUX IS ZERO

In general for any closed surface

lines going out ~~So the field lines~~ so $E = 0$

If the ~~[illegible]~~ surface has a charge ~~[illegible]~~ enclosed, $\underline{E}$ is NOT zero

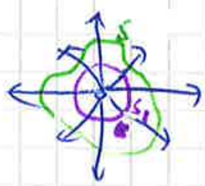


what is ~~[illegible]~~ the flux of $\underline{E}$ in this case?

For a point charge $\underline{E} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2}\hat{r}$
at a distance r:

$$\Phi = \int_{S_1} \underline{E} \cdot d\underline{A} = \int_{S_1} \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} dA = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \underbrace{\int_S dA}_{4\pi r^2} = \frac{1}{\varepsilon_0} Q$$

& For a generic surface?



$$\Phi_{S_1} = \Phi_S = \boxed{\oint_S \underline{E} \cdot d\underline{A} = \frac{1}{\varepsilon_0} Q_{enc}}$$ Gauss's law

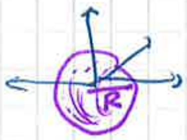
Relates $\underline{E}$ with the sources Q

Give me Q, I can find $\underline{E}$ → integral form
Give me $\underline{E}$ I can find Q → differential form

works great if the problem has symmetries.

Example ① what is $\underline{E}$ for a solid sphere of charge?

Painful integration



$\underline{E}$ due to a pt charge
$d\underline{E} = \frac{dq}{r^2}$
& integrate over the sphere.

Inside $r < R$

Outside $r > R$

$$\int_{r'=0}^{r'=r} d\underline{E} = \int_{r'=0}^{r'=r} \frac{dq}{r'} = \int_{r'=0}^{r'=r} \frac{\rho\, dV}{r'^2} = \int d\theta \int d\varphi \int_{r'=0}^{r'=r} \frac{\rho r'^2}{r'^2} \sin\theta\, d\theta\, d\varphi \ldots$$

OR we can use Gauss's law:

$$\Phi = \oint_S \underline{E} \cdot d\underline{A} = \frac{1}{\varepsilon_0} Q_{enc}$$

symmetries:
$\underline{E}$ is constant on the surface
$\underline{E} \parallel d\underline{A}$ ~~[illegible]~~

① For $r > R$ outside the sphere

$$\oint_S \underline{E} \cdot d\underline{A} = E \int dA = E\, 4\pi r^2 \overset{\text{Gauss's law}}{=} \frac{Q_{enc}}{\varepsilon_0}$$

$$\Rightarrow E = \frac{Q_{enc}}{\varepsilon_0 4\pi r^2}$$

② Inside the sphere $(r < R)$

$$\Phi = \oint_{S_2} \underline{E} \cdot d\underline{A} = E\, 4\pi r^2 = \frac{Q_{enc}}{\varepsilon_0}$$ — what is Q_{enc}?

$$Q_{enc} = \int \rho\, dV = \rho \frac{4}{3}\pi r^3$$

we have the same symmetries

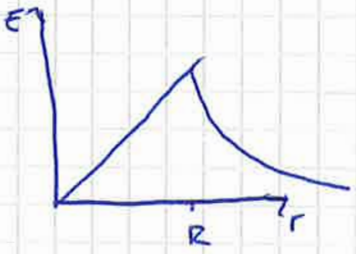
so then $E = \frac{1}{\varepsilon_0 4\pi r^2}\left(\rho \frac{4}{3}\pi r^3\right) = \frac{\rho r}{3\varepsilon_0}$

Are we done or are we missing something in this solutions?

$\underline{E}$ is a vector, we are missing the direction of E. Typically we would get this by symmetry. In this case $\underline{E}$ is radial

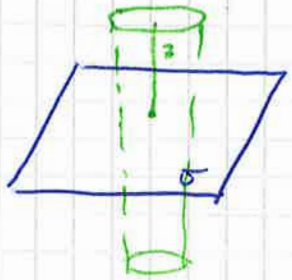
$$\underline{E} = \frac{Q}{4\pi r^2 \varepsilon_0}\hat{r} \quad \text{for } r > R$$

$$\underline{E} = \frac{\rho r}{3\varepsilon_0}\hat{r} \quad \text{for } r < R$$



One last example:

what is $\underline{E}$ at a distance z from a + charged infinite plane of surface charge density σ?



① what Gaussian surface would you choose?

we choose a cylinder

$$\Phi_{tot} = \Phi_{top} + \Phi_{bottom} + \cancel{\Phi_{sides}} \rightarrow 0 \text{ by symmetry}$$

$$\Phi_{top} = \Phi_{bottom}$$

so $\Phi_{tot} = \oint_{B} \underline{E} \cdot d\underline{A} = 2\int_{top} E\, dA = 2EA$ and $\sigma = \frac{Q}{A}$

$$= \frac{2EQ}{\sigma} = \frac{Q_{enc}}{\varepsilon_0}$$

Gauss's law

$$\Rightarrow E = \frac{\sigma}{2\varepsilon_0}\hat{z}$$