Lecture 24. Special relativity.

Last time:

* Poynting's theorem

$$\underbrace{\frac{dW}{dt}}_{\substack{\text{Work done}\\ \text{on charges}\\ \text{by EM force}}} = \underbrace{-\frac{d}{dt}\int \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau}_{\substack{\text{Decrease in energy}\\ \text{stored in the remaining}\\ \text{fields}}} \underbrace{- \boxed{\frac{1}{\mu_0}\oint_S (\underline{E}\times\underline{B})}^{\equiv S} \cdot d\underline{a}}_{\substack{\text{energy that has "flown}\\ \text{out" of the surface.}}} \iff \frac{dW}{dt} = -\frac{d}{dt}\int_V u\, d\tau - \oint_S \underline{S}\cdot d\underline{a}$$

* Poynting vector

$$\underline{S} \equiv \frac{1}{\mu_0}(\underline{E}\times\underline{B})$$

→ energy per unit time, per unit area transported by the fields.

→ points in the direction of EM energy flow i.e. direction of wave propagation

→ $|\underline{S}| = \frac{1}{\mu_0}|\underline{E}\times\underline{B}| = \frac{EB}{\mu_0} \equiv$ intensity

→ Units of S: $[S] = \frac{\text{Watt}}{m^2} = \frac{\text{power}}{\text{area}}$

* Radiation pressure

EM waves carry energy → carry momentum → exert radiation pressure on a surface
Force/unit area

$P = \frac{I}{c}$ (complete absorption) ; $P = \frac{2I}{c}$ (complete reflection) ; where $I \equiv$ intensity, $c \equiv$ speed of light.

Today: Special relativity

Special relativity naturally explains magnetic forces in terms of electric forces seen from a reference frame in motion. Theory formulated by Einstein in 1905. Based on 2 postulates:

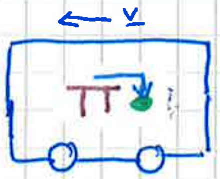
The postulates of special relativity:

1. The principle of relativity: The laws of physics are the same for all reference frames.
2. The universal speed of light: The speed of light (c) is the same in all reference frames.

We will be talking a lot about reference frames, so it's important to have this concept clear.

Inertial reference frame: System of coordinates in which the observer is non accelerating.
(inertial = non accelerating) The observer could be moving, but at a constant speed.

Examples of reference frames



We have a ball falling from a table
The table is in a train moving at constant velocity.

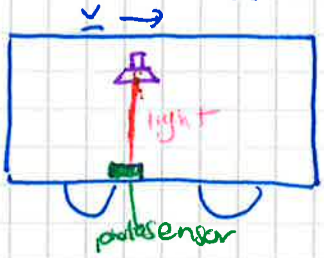
In which reference frames can we describe the motion of the ball falling?
It's useful to think in terms of observers (actual people observing the phenomenon) and their location.

How many reference frames can we have? Are all of them inertial?

We can identify 3 reference frames & 3 observers:
- Observer 1: sitting at a bench on the train station
- Observer 2: sitting on the table on the train
- Observer 3: A bug sitting on top of the falling ball

Who are the observers in an <u>inertial</u> reference frame?
ONLY observers 1 & 2 since the ball is falling w/ acceleration g.

Is time the same in all reference frames?
If we follow the consequences of the 2 postulates of special relativity we will see that time is NOT absolute. Consider the following situation:



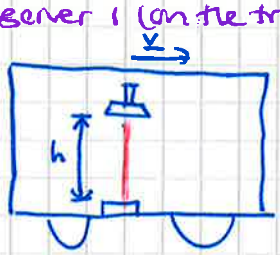
We have a train moving at velocity $\underline{v}$ // x-axis
We will look at the situation from the point of view of 2 observers:
- Observer 1: standing in the train
- Observer 2: at the train station (sitting on a bench)

Observer ① (on the train) flashes a pulse of light vertically to a photosensor mounted on the floor of the train.

Both observers measure the time it takes for the light to reach the photosensor (i.e time between when the light is emitted & when it reaches the sensor)

What is the time that each observer measures? We know that light travels at speed c.

Observer 1 (on the train)

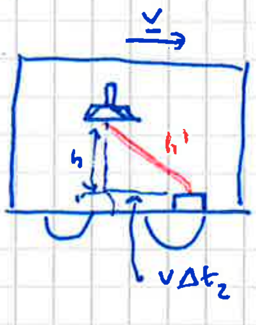


- Distance travelled by the light $= h$
- Velocity of light $= c$

$= \left(\text{Remember: } v = \frac{\text{distance}}{\text{time}} = c \right)$

$$\Rightarrow \underbrace{\Delta t_1}_{\text{time measured by observer 1}} = \frac{h}{c} \quad **$$

Observer 2 (at the train station)



- Distance traveled by light $= h' \underset{\uparrow \text{ pythagoras}}{=} \left[h^2 + (v\Delta t_2)^2\right]^{1/2}$ *

- Velocity of light $= c$

$$\Rightarrow \Delta t_2 = \frac{h'}{c}$$

We can write Δt_2 in terms of h by subs the expression for h' in eq *:

$$(\Delta t_2)^2 = \left(\frac{h'}{c}\right)^2 = \frac{h^2 + (v\Delta t_2)^2}{c^2} = \underbrace{\frac{h^2}{c^2}}_{= \Delta t_1^2 \text{ from eq **}} + \frac{v^2 \Delta t_2^2}{c^2} = \Delta t_1^2 + \frac{v^2}{c^2}\Delta t_2^2$$

$$\Rightarrow \Delta t_2^2 \left(1 - \frac{v^2}{c^2}\right) = \Delta t_1^2 \qquad \therefore \boxed{\Delta t_1 = \Delta t_2 \sqrt{1 - \frac{v^2}{c^2}}}$$

More generally if we denote the variables in the station reference frame with primes (i.e t' and l' for time and distance) and defining $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ we get:

$$\boxed{\Delta t' = \gamma \Delta t}$$ Time dilation

Let's think about this for a second:

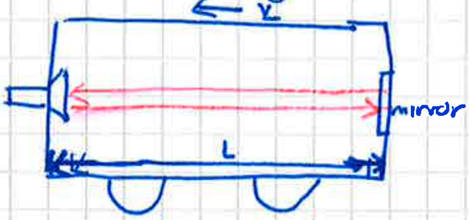
· We defined $\gamma \equiv \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\beta^2}} > 1$ (denominator is less than one $\Rightarrow \gamma > 1$)

· If $\gamma > 1 \Rightarrow \Delta t' = \gamma \Delta t > \Delta t$ always.

remember $\Delta t'$ time measured at the station
Δt time measured on the train

$\therefore$ $\boxed{\text{clocks in motion run slower (time dilation)}}$

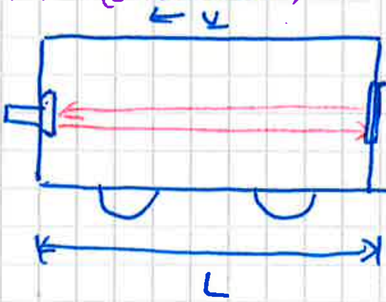
Is length the same in all reference frames?
Let's consider the following situation:



· We have a train moving at velocity $-\underline{v}$ // to x-axis.
· Observer ① (on the train) flashes a pulse of light horizontally from the left end of the train.
· Light is ~~absorbed~~ reflected by a mirror on the right end wall & detected by a photosensor on the left wall.

What is the length of the train measured by each observer?

Observer 1 (on the train)



If we know the time it takes for the light to travel the distance 2L and the speed of light we can know L.

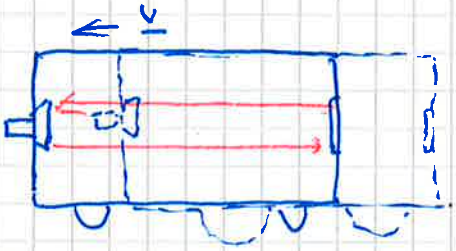
we added have 2L since light goes to the mirror & back to the sensor

This is equivalent to asking what is the time interval between emission & reception of light.

· The time between the 2 events is $\Delta t = \Delta t_{train}$ (time measured with a clock on the train)

· The length of the train will be: $L = \frac{c \Delta t}{2} \Leftrightarrow \Delta t = \frac{2L}{c}$

Observer 2 (at the train station)



We need to calculate separately the time the light ray takes to move from left to right (Δx_1)
And the time it takes to move from right to left (Δx_2)

Note that when light moves from L → R, it moves opposite to the train. (Δx_1)
when light moves from R → L, it moves in the same direction as the train. (Δx_2)

the distances traveled by the light Δx_1 & Δx_2 are different.

If we call $\Delta t_1'$ and $\Delta t_2'$ the times it takes to move Δx_1 & Δx_2 respectively:

$$\Delta t_1' = \frac{(L' - v\Delta t_1')}{c} \quad \text{shorter} \quad \left(\text{Remember } v = \frac{\text{distance}}{\text{time}} = c\right)$$

$$\Delta t_2' = \frac{(L' + v\Delta t_2')}{c} \quad \text{longer}$$

And solving for $\Delta t_1'$ and $\Delta t_2'$ from these eq we get. (You have 2 algebraic eq with 2 unknowns so just solve for $\Delta t_1'$ & $\Delta t_2'$)

$$\Delta t_1' = \frac{L'}{c+v} \quad ; \quad \Delta t_2' = \frac{L'}{c-v}$$

The total time as measured in the station reference frame is the sum of $\Delta t_1'$ & $\Delta t_2'$:

$$\Delta t' = \Delta t_1' + \Delta t_2' = \frac{L'}{c-v} + \frac{L'}{c+v} = \frac{L' \, 2c}{c^2 - v^2} = \frac{L' \, 2c}{c^2\left(1 - \frac{v^2}{c^2}\right)} = \frac{2L'\gamma^2}{c}$$

$$\therefore \Delta t' = \frac{2L'\gamma^2}{c}$$

and if we take into account time dilation $(\Delta t' = \gamma \Delta t)$ we have that:

$$\frac{2L'\gamma^2}{c} = \Delta t' = \gamma \Delta t = \gamma \frac{2L}{c} \Rightarrow \boxed{L' = \frac{L}{\gamma}}$$

using the result we derived for observer 1

length contraction

Moving objects appear contracted or shortened.

EXAMPLE: Cosmic ray muons.

Cosmic rays are energetic particles coming from somewhere in the universe. When they hit the atmosphere they produce a shower of particles, among those particles are muons. These particles however can penetrate the atmosphere & have a long lifetime (2.2 μs).

If the atmosphere is ~20 Km thick, and the velocity of muons 99.99% of c,

Can muons produced in the atmosphere reach the ground?

If we don't consider special relativity:

$$\Delta l = 0.99c\,\Delta t = (0.99c)(2.2\mu s) = 0.6 \text{ km} < 20 \text{Km}$$

NO, they can't reach the ground

but we can detect them!

We need to take the relativistic approach.

~~Our~~ From our perspective:

$\Delta t = 2.2 \mu s$ in the muon's reference frame

In our reference frame:

$$\Delta t' = \frac{\Delta t}{\gamma} = 71.22 \mu s = 156 \mu s$$

So then $\Delta l = (0.99c)(\Delta t') = 42 \text{Km}$ ✓

From the muon's perspective:

$$\Delta l = \frac{\Delta l'}{\gamma} = \frac{20 \text{Km}}{71} \sim 0.3 \text{ Km}$$, which can be traveled with $\Delta t = 2.2 \mu s$ ✓

One final comment on length contraction:

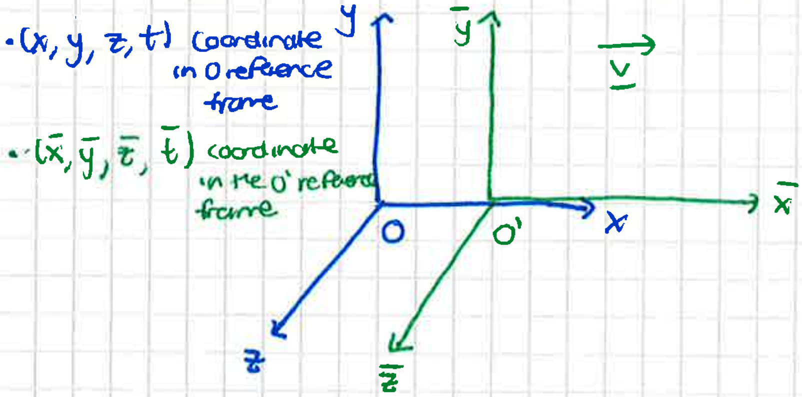
Dimensions perpendicular to the velocity of a moving object are NOT contracted.

(for an intuitive explanation read Griffiths p.517-518)

Lorentz transformations

Consider 2 inertial reference frames O and $\bar{O}$

$\bar{O}$ is moving with respect to O at velocity $\underline{v}$ parallel to the x-axis.



Lorentz transformation ≡ linear transformation that relates the coordinates in two reference frames.

* To go from the reference frame at rest O, to the reference frame in motion $\bar{O}$:

$$\bar{x} = \gamma\,(x - vt)$$

$$\bar{t} = \gamma\left(t - \frac{v}{c^2}x\right)$$

* To go from the reference frame in motion $\bar{O}$ to the reference frame at rest O, just change the sign of the velocity:

$$x = \gamma\,(\bar{x} + v\bar{t})$$

$$t = \gamma\left(\bar{t} + \frac{v}{c^2}\bar{x}\right)$$

(Note: If you are interested in the details of the derivation see slides in moodle)

* The other coordinates (y & z) are <u>not affected</u>.

* Consequence of Lorentz transformations → transformation of the velocity.

· An observer in motion $\bar{O}$ throws a ball with velocity $\bar{u}_x$ // $+x$ axis. What is the velocity of the ball measured by O?

$$u_x = \frac{dx}{dt} = \frac{d\,[\gamma(\bar{x} + v\bar{t})]}{d\,[\gamma(\bar{t} + \frac{v}{c^2}\bar{x})]} = \frac{d\bar{x} + v\,d\bar{t}}{d\bar{t} + \frac{v}{c^2}d\bar{x}} = \frac{d\bar{x}/d\bar{t} + v}{1 + \frac{v}{c^2}\frac{d\bar{x}}{d\bar{t}}}$$

$$= \frac{\bar{u}_x + v}{1 + \frac{v\bar{u}_x}{c^2}}$$

$$\therefore \quad u_x = \frac{\bar{u}_x + v}{1 + \frac{v\bar{u}_x}{c^2}} \quad \text{and} \quad \bar{u}_x = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}$$

Now if the velocity of the moving object is not // to the x-axis (direction of velocity of moving frame):

How do we sum velocity not // to the relative motion of the two reference frames?

· Let's assume the observer in motion $\bar{O}$ throws a ball with velocity $\bar{u}_y$ $\perp$ to $\underline{v}$.

What is the velocity of the ball u_y measured by O?

$$u_y = \frac{dy}{dt} = \frac{d\bar{y}}{d\left[\gamma\left(\bar{t} + \frac{v}{c^2}\bar{x}\right)\right]} = \frac{d\bar{y}}{\gamma\left(d\bar{t} + \frac{v}{c^2}d\bar{x}\right)} = \frac{d\bar{y}/d\bar{t}}{\gamma\left(d\bar{t} + \frac{v}{c^2}d\bar{x}\right)/d\bar{t}}$$

$$= \frac{\bar{u}_y}{\gamma\left(1 + \frac{v\bar{u}_x}{c^2}\right)}$$

$$\therefore\ u_y = \frac{\bar{u}_y}{\gamma\left(1 + \frac{v\bar{u}_x}{c^2}\right)} \quad \text{and} \quad \bar{u}_y = \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}$$

* More consequences of Lorentz transformations → momentum and energy.

$$\bar{E} = \gamma\left(E - \beta c p_x\right)$$
$$\bar{p}_x = \gamma\left(p_x - \frac{\beta E}{c}\right)$$
$$\bar{p}_y = p_y$$
$$\bar{p}_z = p_z$$

where ~~[illegible]~~

$$\gamma = \frac{1}{(1-\beta^2)} \quad \text{and} \quad \beta^2 = \frac{v^2}{c^2}$$

* More consequences of Lorentz transformations → Forces p...

Forces parallel to $\underline{v}$:

$$\bar{F}_x = F_x$$

Forces perpendicular to $\underline{v}$:

$$\bar{F}_y = \frac{F_y}{\gamma}$$

Now that we have Lorentz transformations we are ready to think about how relativity applies in the context of electromagnetism.

Interactions between a moving charge & other moving charges (Purcell 5.9)

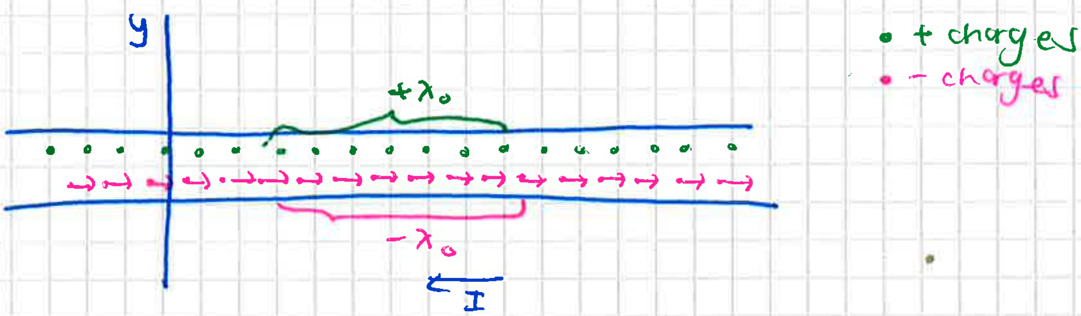
~~considering~~

If the postulates of special relativity are valid, electric charge is invariant and Coulomb's law holds → we get magnetism for free!

E & B are equivalent in different reference frames.

In the lab frame consider an electrically neutral wire carrying current I:

This is the situation we have:



- In the wire positively charged particles are at rest. Linear density of + charges is λ_+.
- The movement of $\bar{e}$ charges to the right creates a current. Linear density of − charges is $\lambda_- = -\lambda_+$ (i.e. it has the same magnitude as λ_+)

$\Leftrightarrow$ Any given length of wire at a given instant in time contains the same amount of + & $\bar{e}$ charges.

Now in the reference frame of the electrons, what is ~~their~~ their charge density?

Remember $\lambda = \frac{Q}{L}$ where L is the length. ~~[illegible]~~ Since charge is invariant we only need to consider what is the change in length with respect to the lab frame.

In the $\bar{e}$'s reference frame, the lab is moving at velocity $-v$

If the length of a line of $\bar{e}$'s measured in their own reference frame is L^*, this length will be contracted by γ in the lab frame $\Rightarrow$ $L^* = \gamma L_{Lab}$

Remember when we derived at the beginning of the lecture that if L' is the lenght measured in the moving reference frame, ~~[illegible]~~ and L is the length measured in the frame at rest:

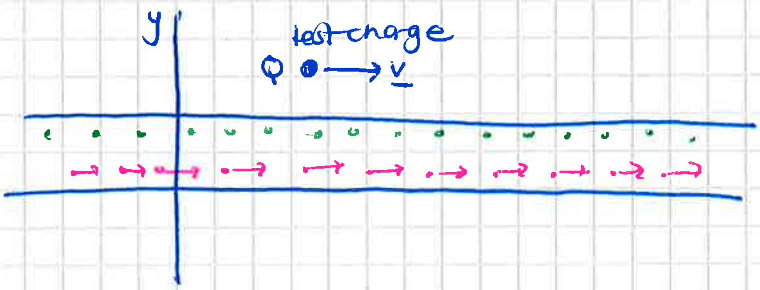
$L' = \frac{L}{\gamma}$ In this case the moving reference frame is the lab frame (so $L' = L_{lab}$ and $L = L^*$ is the length measured in the frame at rest, in this case the $\bar{e}$'s are at rest in their own reference frame)

$\therefore L_{lab}\gamma = L^*$

This is why thinking about observers is useful, the $\bar{e}$'s are the observers in this reference frame.

So we have: $\lambda_-^* = \frac{Q}{L^*} = \frac{Q}{\gamma L} = \frac{-\lambda_0}{\gamma}$ is the charge density in the ~~[illegible]~~ reference frame of the $\bar{e}$'s.

Now that we have spelled this out we can add a test charge to the problem?
What is the $\underline{F}$ experienced by a test charge ~~[illegible]~~ moving with velocity $\underline{v}$ near the wire?



What are the forces acting on the test charge Q? In which frame:

In the lab frame:

→ Wire is neutral ⇒ No electric field $\underline{E}$

→ There is a current ⇒ this will generate a magnetic field $\underline{B}$

We can calculate the magnetic field using ~~A~~ Ampère's law.

$$\int \underline{B} \cdot d\underline{l} = \mu_0 I_{enc} \Rightarrow B = \frac{\mu_0 I_{enc}}{2\pi r}$$

What is I_{enc}?

$$I = \frac{dQ}{dt} = \lambda_0 \frac{dx}{dt} = \lambda_0 u$$

u ← vel of electrons

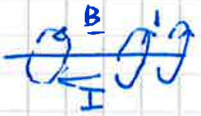
↑ since $\lambda = \frac{Q}{L} \Rightarrow Q = \lambda x$

Subs this into the eq. above: $B = \frac{\mu_0}{2\pi r} \lambda_0 u$

So the magnetic force acting on the test charge Q is: $F = |Q(\underline{v} \times \underline{B})| = QvB$

$$= Q \frac{\mu_0 \lambda_0 u v}{2\pi r}$$

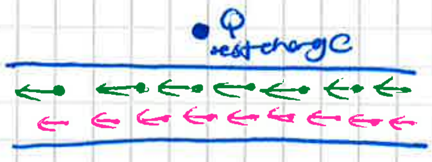
The direction of the F is upwards → repulsive force



Thumb direction of $\underline{v}$, index direction of $\underline{B}$
middle finger direction of $\underline{F}$

In the reference frame of the charge:

- By definition Q is at rest so $\underline{v} = 0$ ⇒ NO magnetic force.
- But in this case the particle "sees" both the negative & positive charges moving to the left.



The positive charges will be moving to the left at a velocity v

To obtain ~~we~~ the velocity of the ~~charges~~ negative charges we need to substract v (vel of the test charge) from ~~the vel of~~ $u = \frac{dx}{dt}$ ~~velocity moving~~ relativistically, i.e using Lorentz's transformations:

u ← vel of e's in lab frame

$$u' = \frac{u - v}{1 - \frac{vu}{c^2}}$$

Is there a force acting on Q? There must be since according to special relativity the physics has to be the same in any reference system

As you can guess by now there will be an electric force on the test charge, this means that the wire will no longer be neutral in this reference frame.

Let's see if this is actually the case.

What happens to the charges in the wire?

The charge density for the positive charges in the wire is:

$$\lambda'_+ = \frac{Q}{L'} = \gamma_v \lambda_0 \quad \text{where } \gamma_v = \frac{1}{1-\beta^2} = \frac{1}{1-\frac{v^2}{c^2}}$$

Now for the negative charges the charge density is:

$$\lambda'_- = \gamma_{u'} \lambda^*_- = \gamma_{u'} \left(\frac{-\lambda_0}{\gamma_u}\right)$$

here $\gamma_{u'} = \frac{1}{1-\frac{u'^2}{c^2}}$; $\gamma_u = \frac{1}{1-\frac{u^2}{c^2}}$

We need to calculate $\gamma_{u'}$, we can start by calculating $\frac{1}{\gamma_{u'}^2} = 1-\beta_{u'}^2$:

$$1-\beta_{u'}^2 = 1-\left(\frac{u'}{c}\right)^2 = 1-\frac{\left(\frac{u-v}{1-uv/c^2}\right)^2}{c^2} = 1-\left(\frac{\beta_u-\beta_v}{1-\beta_u\beta_v}\right)^2$$

$$= \frac{1-2\beta_u\beta_v+\beta_u^2\beta_v^2-(\beta_u-\beta_v)^2}{(1-\beta_u\beta_v)^2} = \frac{(1-\beta_v^2)(1-\beta_u^2)}{(1-\beta_u\beta_v)^2} = \frac{1}{\gamma_u^2\gamma_v^2(1-\beta_u\beta_v)^2}$$

$\Rightarrow \gamma_{u'} = \gamma_u\gamma_v(1-\beta_u\beta_v)$, so subs this in the eq for λ'_- :

$$\lambda'_- = \gamma_u\gamma_v(1-\beta_u\beta_v)\left(\frac{-\lambda_0}{\gamma_u}\right) = -\gamma_v(1-\beta_u\beta_v)(\lambda_0)$$

So the total charge density of the wire in Q's reference frame is:

$$\lambda'_{Net} = \lambda'_+ + \lambda'_- = \gamma_v\lambda_0 - \gamma_v(1-\beta_u\beta_v)\lambda_0 = +\gamma_v\beta_u\beta_v\lambda_0 = \gamma_v\frac{uv}{c^2}\lambda_0$$

∴ there is a net charge in the wire ⇒ there is an $\underline{E}'$ field.

So there will be a force F' that acts on the charge Q due to $\underline{E}$: To calculate F' we first calculate the electric field:

$$E' = \frac{\lambda'_{Net}}{2\pi\varepsilon_0 r} = \frac{\gamma_v uv\lambda_0}{2\pi\varepsilon_0 rc^2} = \frac{\mu_0\gamma_v uv\lambda_0}{2\pi r} \Rightarrow F' = QE' = Q\left(\frac{\mu_0\gamma_v uv\lambda_0}{2\pi r}\right)$$ Repulsive force.

With all these moving charge is there a magnetic field? Yes → but the test charge feels no force due to it since it's at rest.

So what did we find?

In the lab frame:

Repulsive magnetic force acting on charge Q:

$$F = Q\frac{\mu_0\lambda_0 uv}{2\pi r}$$

In the moving charge reference frame:

Repulsive electric force acting on Q:

$$F = \gamma_v Q\left(\frac{\mu_0\lambda_0 uv}{2\pi r}\right)$$

This is consistent with Lorentz's transformations. Forces perpendicular to $\underline{v}$ transform as:

$$F'_y = \frac{F_y}{\gamma}$$