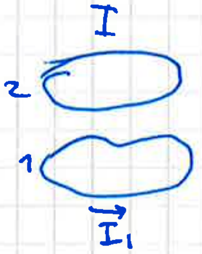
Lecture 20. Alternating current circuits.

Last time!

* Mutual inductance M :

M ≡ proportionality constant between the magnetic flux induced on a loop due to a current on another nearby loop



$\Phi_2 = M I_1$

and if the currents is in loop 2:

$\Phi_1 = M I_2$

$[M] \equiv$ Henry

*Self inductance L

$\Phi = L I$ intrinsically positive quantity that depends

* Magnetic energy stored in an inductor:

$$\boxed{U_B = \frac{1}{2} L I^2}$$

* Kirchhoff's loop rule for inductors:

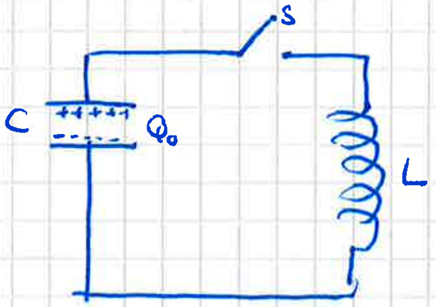
the "potential change" due to an inductor is

$-L\left(\frac{dI}{dt}\right)$ if L is traversed in the direction of I

$L\left(\frac{dI}{dt}\right)$ if L is traversed in the direction opposite to I

Today: LC circuits and circuits with an AC source.

Oscillations in LC circuits



Consider a circuit where the capacitor (C) has charge Q_0 at $t<0$.

At $t=0$ we close the switch → capacitor begins to discharge. current generated from the discharging process generates magnetic energy. stored in L.

In the absence of a resistance, the energy is transformed back & forth between electric energy in the capacitor & magnetic energy in the inductor. This phenomenon is called electromagnetic oscillation.

The total energy in the LC circuit at $t \geq 0$ (after closing the switch):

$$U = U_C + U_L = \frac{1}{2}\frac{Q^2}{C} + \frac{1}{2} L I^2$$

And since there is no resistance the energy remains constant: so!

$$\frac{dU}{dt} = 0$$

$$\Rightarrow \frac{dU}{dt} = \frac{d}{dt}\left(\frac{1}{2}\frac{Q^2}{C} + \frac{1}{2} L I^2\right) = \frac{Q}{C}\frac{dQ}{dt} + L I \frac{dI}{dt} = 0$$

$$\Rightarrow \frac{Q}{C} + L\frac{d^2Q}{dt^2} = 0 \quad (**)$$

where we have used that $I = -\frac{dQ}{dt}$ and $\frac{dI}{dt} = -\frac{d^2Q}{dt^2}$ (*)

We can obtain the same equation if we apply the modified Kirchhoff's rule:

$$\frac{Q}{C} - L\frac{dI}{dt} = 0 \qquad \left[\text{you can check both eqs are equivalent taking into account the definitions } * \right]$$

You can check that the general solution for equation ($\circledast$) is:

$$Q(t) = Q_0 \cos(\omega_0 t + \phi)$$

where $Q_0 \equiv$ charge amplitude

$\phi \equiv$ phase

$\omega_0 \equiv$ angular frequency $= \frac{1}{\sqrt{LC}}$

~~Frobal that~~ Having this expression we can now calculate the current on the inductor:

$$\underbrace{I(t) = -\frac{dQ}{dt}}_{\text{definition of current}} = \omega_0 Q_0 \sin(\omega_0 t + \phi) = I_0 \sin(\omega_0 t + \phi)$$

where we have ~~used~~ defined: $I_0 = \omega_0 Q_0$

From the initial conditions $Q(t=0) = Q_0$ and $I(t=0) = 0$, we can determine ϕ:

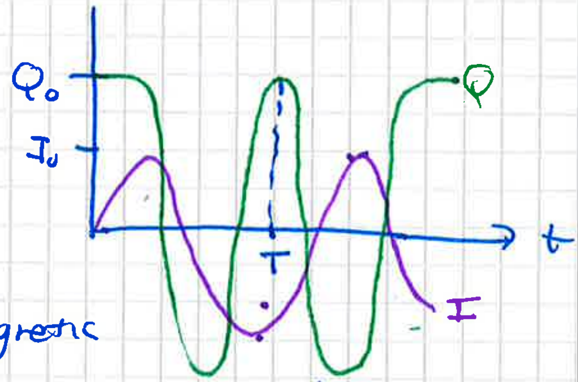
$$I(0) = I_0 \sin(\phi) = 0 \Rightarrow \phi = 0$$

So our expressions for Q and I become:

$$Q(t) = Q_0 \cos(\omega_0 t)$$

$$I(t) = I_0 \sin(\omega_0 t)$$

so this looks like:



oscillating or alternating current

With these expressions we can calculate the electric & magnetic energy at any instant in time:

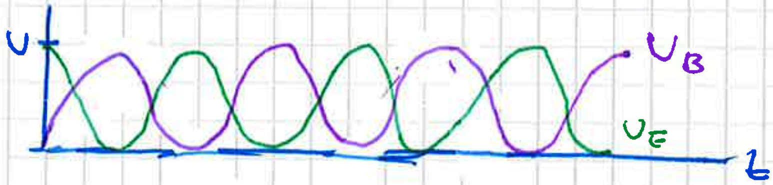
$$U_E = \frac{Q^2(t)}{2C} = \left(\frac{Q_0^2}{2C}\right)\cos^2(\omega t)$$

$$U_B = \frac{1}{2} L I^2(t) = \frac{L I_0^2}{2}\sin^2(\omega t) = \frac{L(-\omega_0 Q_0)^2 \sin^2(\omega_0 t)}{2} = \left(\frac{Q_0^2}{2C}\right)\sin^2(\omega_0 t)$$

You can verify that the total energy $U = U_E + U_B$ remains constant:

$$U = U_E + U_B = \left(\frac{Q_0^2}{2C}\right)\cos^2(\omega_0 t) + \left(\frac{Q_0^2}{2C}\right)\sin^2(\omega_0 t) = \frac{Q_0^2}{2C}$$

which looks like:



Does this remind you of something?

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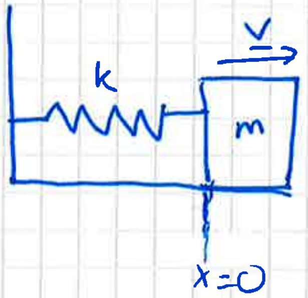


If a mass m attached to a spring w/ spring constant k is displaced from eq by x and is moving at speed V. The total energy of the system is:

$$U = K + U_{ps} = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

kinetic energy of mass — pot energy of spring

In the absence of friction U is conserved

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2 + \frac{1}{2}kx^2\right) = mv\frac{dv}{dt} + kx\frac{dx}{dt} = 0 \quad \left(\text{Using } v = \frac{dx}{dt}\ ;\ \frac{dv}{dt} = \frac{d^2x}{dt^2}\right)$$

$$\Rightarrow m\frac{d^2x}{dt^2} + kx = 0$$

Harmonic oscillator which we know has a general solution $x(t) = x_0\cos(\omega_0 t + \phi)$

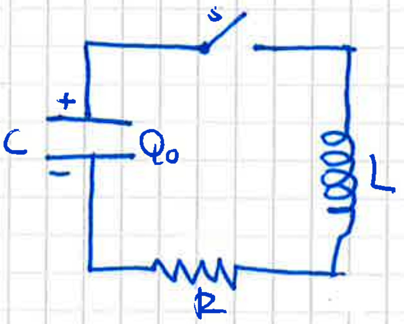
where $\omega_0 = \sqrt{\frac{k}{m}}$

$x_0 \equiv$ amplitude of oscillation

$\omega_0 \equiv$ angular frequency.

The RLC series circuit

we can now put our 3 electrical elements together:



We have a capacitor (C) with some stored charge Q_0.

As we close the switch at $t=0$, current begins to flow. In this case energy will be dissipated due to the resistance. The rate at which energy is dissipated is:

$$\frac{dU}{dt} = -I^2R \quad (*)$$

In Lecture 12 we saw that the rate of energy loss through a resistor is:

$\frac{\Delta U}{\Delta t} = I\Delta V$ and by ohms law $V = RI$

$$\Rightarrow \frac{dU}{dt} = RI^2$$

We add a minus sign since the total energy is decreasing.

Subs the expression for $\frac{dU}{dt}$ from eq (*):

$$\frac{Q}{C}\frac{dQ}{dt} + LI\frac{dI}{dt} = -I^2R$$

again, since the current is due to a decrease in charge in the capacitor plates:

$I = -\frac{dQ}{dt}$, Dividing both sides by I:

$$\Rightarrow \boxed{L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0} \Leftrightarrow \boxed{L\frac{d^2V}{dt^2} + R\frac{dV}{dt} + \frac{V}{C} = 0}$$

since $Q = CV$ — eq in Purcell

~~For a small R (underdamped case) we can verify that the solution to the above equation is: $Q(t) = Q_0 e^{-\gamma t}\cos(\omega' t + \phi)$ where $\gamma = \frac{R}{2L}$ & $\omega' = \sqrt{\omega_0^2 - \gamma^2}$ angular~~

For a small R ~~we can see~~ (underdamped case) we can verify that the solution to the above equation is:

$$Q(t) = Q_0 \, e^{-\gamma t} \cos(\omega' t + \phi)$$

where $\gamma = \dfrac{R}{2L}$ is the damping factor and $\omega' = \sqrt{\omega_0^2 - \gamma^2}$ is the angular freq of damped oscillations.

- The constants Q_0 and ϕ are real quantities to be determined from the initial conditions. In the limit where $R \to 0$ we recover the first case we looked at with natural angular frequency $\omega_0 = \dfrac{1}{\sqrt{LC}}$

Demo. 463 Damped LC circuit

The mechanical analogue of the series RLC circuit is of course the damped harmonic oscillator:

$$m\frac{d^2x}{dt^2} + \underbrace{b\frac{dx}{dt}}_{\substack{\text{Non conservative} \\ \text{dissipative force.}}} + kx = 0$$

b = damping coefficient.

AC circuits

The circuit we just discussed has no ext energy source so the activity will die out.

Now we will consider an alternating current circuit, where at steady state, a current and voltage oscillate sinusoidally without change in amplitude. In this case we have an oscillating emf driving the system:

The frequency f on an alternating current is expressed in cycles/second or Hertz H_z. The angular frequency $\omega = 2\pi f$ is the quantity we see in our equations, always assumed to be in radians/second. ~~Note ...~~

Let's consider now the voltage source

$V(t) = V_0 \sin \omega t$ where $V_0 \equiv$ amplitude (max. value)

Note that V varies from $-V$ to V since $\sin\theta \in [0,1]$ $\forall \theta$.

If T is the period of the sine function
$\Rightarrow V(t) = V(t+T)$

And recall that the frequency is defined as:

$$f = \frac{1}{T} \quad ; \quad [f] = \text{Hertz} = \frac{1}{s}$$

and the angular frequency is ~~...~~ related to the frequency by: $\omega = 2\pi f$

Once we connect a voltage source to an RLC circuit, energy is provided to compensate the dissipation in the resistor $\to$ the oscillation will no longer damp out.

After an initial "transient time" an AC current will flow in the circuit in response to the driving voltage.

The current through the circuit will be given by:

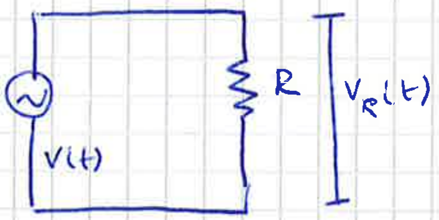
$$I(t) = I_0 \sin(\omega t - \phi) \quad *$$

You can see that the current oscillates w/ the same frequency as the voltage source with an amplitude I_0 and a phase ϕ that depends on the driving frequency.

Simple AC circuits

Before examining the driven RLC circuit (i.e. an RLC circuit with an AC source) we will look at one element at the time.

Purely resistive load



Consider a purely resistive circuit: A resistor connected to an AC generator $V(t) = V_0 \sin(\omega t)$

Applying Kirchhoff's loop rule gives:

$$V(t) - V_R(t) = V(t) - \underbrace{I_R(t) R}_{\text{instantaneous voltage drop across the resistor.}} = 0$$

The voltage drop across the resistor is:

$$\underset{\text{ohm's law}}{V_R(t) = I_R(t) R} \Rightarrow I_R(t) = \frac{V_R(t)}{R} = \frac{V_0 \sin(\omega t)}{R} = \frac{V_{R_0} \sin(\omega t)}{R} = I_{R_0} \sin(\omega t) \quad **$$

we know this is the form of (~)

where we have defined:

$$V_{R_0} = V_0 \quad \text{and} \quad I_{R_0} = \frac{V_{R_0}}{R} \text{ is the maximum current}$$

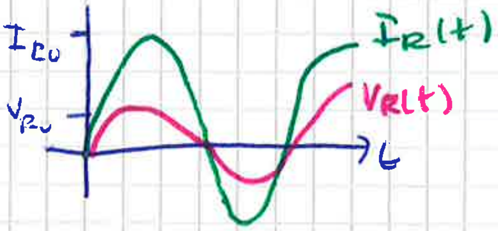
From equations * and ** we have:

$V_R(t) = V_{R_0}\sin(\omega t)$

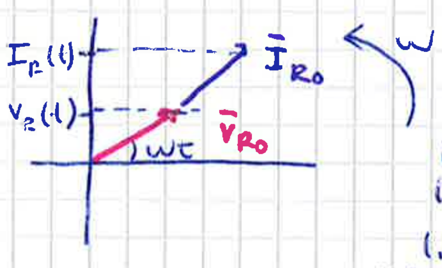
and

$$I_R(t) = I_{R_0} \sin(\omega t) = I_0 \sin(\omega t - \phi) \Rightarrow \phi = 0$$

V_R and I_R are in phase, i.e. they reach their maximum or minimum values at the same time.



We can represent this using a phasor diagram:



$\bar{V}_{R_0}$ and $\bar{I}_{R_0}$ are phasors:
A phasor is a rotating vector w/ the following properties:
(i) length → amplitude
(ii) angular speed ω → angular freq of current
(iii) Projection along vertical axis is the value of alternating quantity qty at t

From the last diagram we can see that the phasor $\bar{V}_{R_0}$ has a constant magnitude V_{R_0}. Its projection along the vertical axis is $V_{R_0}\sin\omega t$, which is equal to $V_R(t)$, the voltage drop across the resistor at time t. A similar interpretation applies to $\bar{I}_{R_0}$. From the phasor diagram we can see that $I_R(t)$ & $V_R(t)$ are in phase.

The average value of the current over one period is:

$$\langle I_R(t)\rangle = \frac{1}{T}\int_0^T I_R(t)\,dt = \frac{1}{T}\int_0^T I_{R_0}\sin\omega t\,dt = \frac{I_{R_0}}{T}\int_0^T \underbrace{\sin\frac{2\pi t}{T}\,dt}_{0} = 0$$

But the average square of the current is non-vanishing:

$$\langle I_R^2(t)\rangle = \frac{1}{T}\int_0^T I_R^2(t)\,dt = \frac{1}{T}\int_0^T I_{R_0}^2\sin^2\omega t\,dt = I_{R_0}^2\,\frac{1}{T}\int_0^T \sin^2\left(\frac{2\pi t}{T}\right)dt = \frac{1}{2}I_{R_0}^2$$

When discussing AC circuit it's convenient to define:

$$I_{rms} \equiv \left(\langle I_R^2(t)\rangle\right)^{1/2} \quad \text{root mean square (rms) current}$$

$$= \frac{I_{R_0}}{\sqrt{2}}$$

Similarly:

$$V_{rms} \equiv \left(\langle V_R^2(t)\rangle\right)^{1/2} \quad \text{root mean square voltage}$$

$$= \frac{V_{R_0}}{\sqrt{2}}$$

In a typical household $V_{rms} = 230\,V$ at $f = 50\,Hz$ and in the US $V_{rms} = 120\,V$ at $f = 60\,Hz$

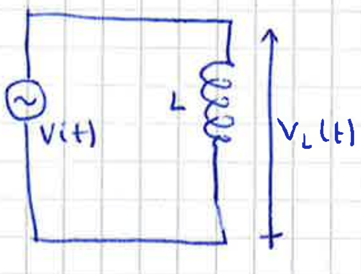
The power dissipated in the resistor is:

$$P_R(t) = I_R(t)\,V_R(t) = I_R^2(t)\,R$$ and its average over one period is:

$$\langle P_R(t)\rangle = \langle I_R^2(t)R\rangle = \frac{1}{2}I_{R_0}^2 R = I_{rms}^2 R = I_{rms}V_{rms} = \frac{V_{rms}^2}{R}$$

Purely inductive load

Consider now a purely inductive ~~load~~ circuit w/ an inductor connected to an AC generator



$V(t) = V_0\sin\omega t$

Applying the modified Kirchhoff's rule for inductors:

$$V(t) - V_L(t) = V(t) - L\frac{dI_L}{dt} = 0$$

$$\Rightarrow \frac{dI_L}{dt} = \frac{V(t)}{L} = \frac{V_{L_0}\sin\omega t}{L} \quad \text{where } V_{L_0} = V_0$$

Integrating the eq we find:

$$I_L(t) = \int dI_L = \frac{V_{L_0}}{L}\int \sin\omega t = -\left(\frac{V_{L_0}}{\omega L}\right)\cos\omega t = \left(\frac{V_{L_0}}{\omega L}\right)\sin\left(\omega t - \frac{\pi}{2}\right)$$

since $-\cos\omega t = \sin\left(\omega t - \frac{\pi}{2}\right)$

Now we had found that the current ~~generally~~ should have the form:
$I(t) = I_0\sin(\omega t - \varphi)$, comparing this with the expression we obtained we see that:

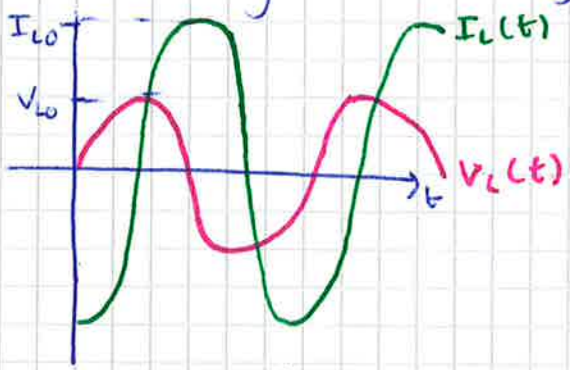
$$I_{L0} = \frac{V_{L0}}{\omega L} = \frac{V_{L0}}{X_L}$$ where $X_L \equiv \omega L$ inductive reactance

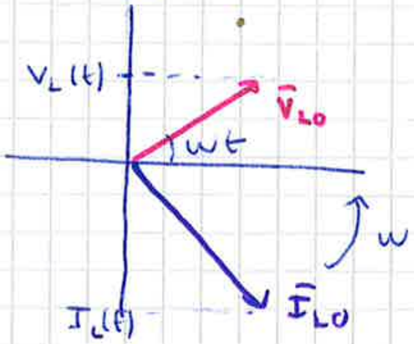
and $\phi = \frac{\pi}{2}$

$[X_L] = $ Ohms, just like resistance.

However unlike resistance, it depends linearly on the angular freq. ω.
$\Rightarrow$ resistence to current flow increases w/ frequency.

The corresponding current & voltage plots & phasor diagram are:



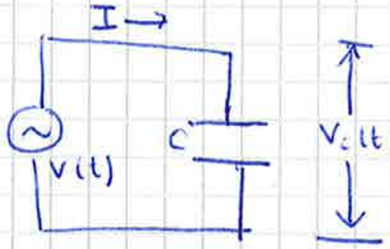


$I_L(t)$ is out of phase with $V_L(t)$ by $\phi = \pi/2$, i.e. it reaches its max value after $V_L(t)$ does by one quarter of a cycle.

$\boxed{\text{The current lags voltage by } \pi/2 \text{ in a purely inductive circuit}}$

Purely capacitive load

We now consider the case where we only have a capacitor:



$V(t) = V_0 \sin \omega t$

Again using Kirchhoff's law:

$$V(t) - V_c(t) = V(t) - \frac{Q(t)}{C} = 0$$

$$\Rightarrow Q(t) = CV(t) = CV_c(t) = CV_{c0} \sin \omega t \quad (*)$$

where $V_{c0} = V_0$

with this we can get the current.

$$I_c(t) = \frac{dQ}{dt} = \omega C V_{c0} \cos \omega t = \omega C V_{c0} \sin\left(\omega t + \frac{\pi}{2}\right)$$

differentiating eq (*)

since $\cos \omega t = \sin\left(\omega t + \frac{\pi}{2}\right)$

Again comparing this equation to our expression for current:

$I(t) = I_0 \sin(\omega t - \phi)$ we can see that

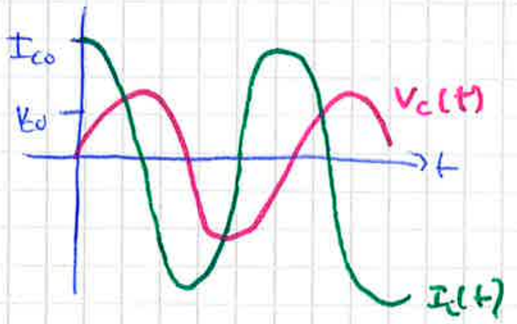
$$I_{c0} = \omega C V_{c0} = \frac{V_{c0}}{X_c}$$ where $X_c \equiv$ capacitance reactance $= \frac{1}{\omega C}$

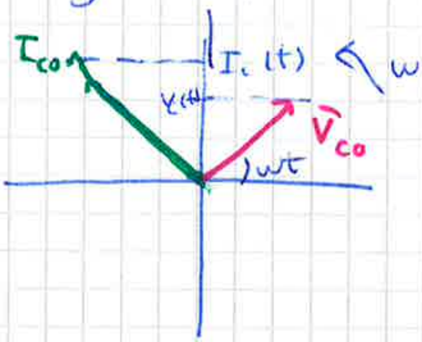
X_c inv prop to ωC
as $\omega \to 0$ X_c diverges

and $\phi = -\frac{\pi}{2}$

$[X_c] = $ Ohms represents effective resistance for a purely capacitive circuit.

The current & voltage plots as well as the phasor diagram are:

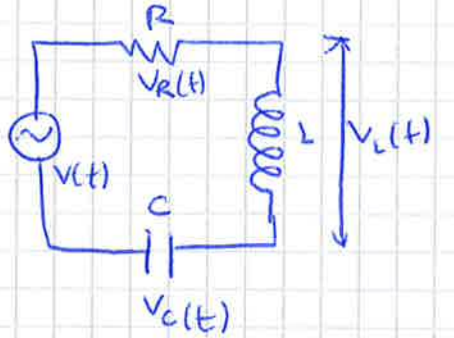




At $t=0$, voltage across the capacitor is zero while the current in the circuit is maximum. In fact, $I_c(t)$ reaches its maximum before $V_c(t)$ by one quarter cycle ($\phi = \frac{\pi}{2}$) so:

The current leads the voltage by $\pi/2$ in a capacitive circuit

The RLC series circuit



Applying Kirchhoff's loop rule we obtain:

$$V(t) - V_R(t) - V_L(t) - V_C(t) = 0$$

$$= V(t) - IR - L\frac{dI}{dt} - \frac{Q}{C} = 0$$

$$\Rightarrow L\frac{dI}{dt} + IR + \frac{Q}{C} = V_0 \sin \omega t.$$

Assuming that the capacitor is initially uncharged so that $I = \frac{dQ}{dt}$ is proportional to the increase of charge in the capacitor, we can rewrite the last equation as:

$$\boxed{L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = V_0 \sin \omega t}$$

One possible solution to this equation is:

$$Q(t) = Q_0 \cos(\omega t - \phi)$$

Where the amplitude & the phase are:

$$Q_0 = \frac{V_0/L}{\sqrt{(R\omega/L)^2 + (\omega^2 - 1/LC)^2}} = \frac{V_0}{\omega\sqrt{R^2 + (\omega L - 1/\omega C)^2}} = \frac{V_0}{\omega\sqrt{R^2 + (X_L - X_C)^2}}$$

and

$$\tan\phi = \frac{1}{R}\left(\omega L - \frac{1}{\omega C}\right) = \frac{X_L - X_C}{R}$$

The corresponding current is then:

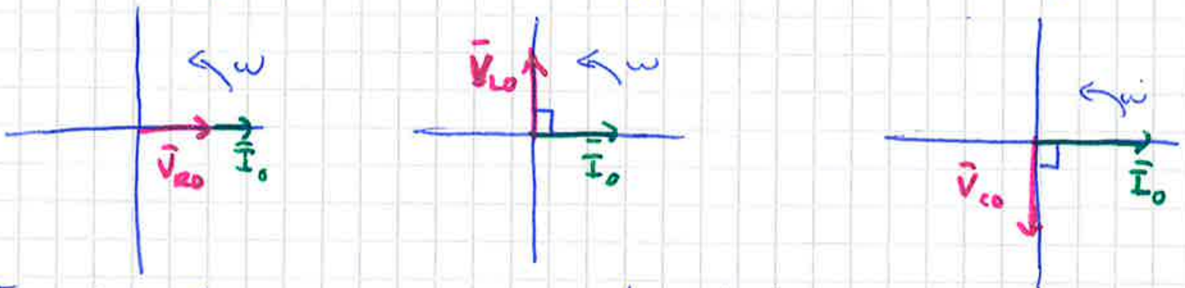
$$I(t) = \frac{dQ}{dt} = I_0 \sin(\omega t - \phi)$$

with an amplitude

$$I_0 = -Q_0\omega = \frac{-V_0}{\sqrt{R^2 + (X_L - X_C)^2}}$$

The current has the same amplitude & phase at all points in the RLC circuit.
The instantaneous voltage ~~drop~~ across each circuit element has a different amplitude & phase relation w/ the current.

From our analysis of all the elements by themselves we know the phasor diagrams are:



From these diagrams we can obtain the instantaneous voltages:

$$V_R(t) = I_0 R \sin\omega t = V_{R0} \sin\omega t$$

$$V_L(t) = I_0 X_L \sin\left(\omega t + \frac{\pi}{2}\right) = V_{L0} \cos\omega t$$

$$V_C(t) = I_0 X_C \sin\left(\omega t - \frac{\pi}{2}\right) = -V_{C0} \cos\omega t$$

where

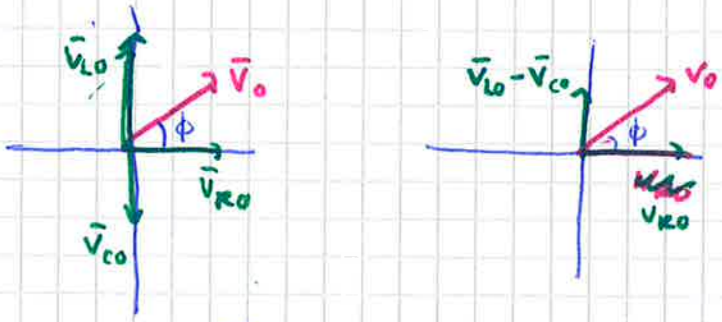
$$V_{R0} = I_0 R \quad , \quad V_{L0} = I_0 X_L \quad , \quad V_{C0} = I_0 X_C$$

are the amplitudes of the voltages across the circuit elements. The sum of all 3 is equal to the instantaneous voltage supplied by the AC source:

$$V(t) = V_R(t) + V_L(t) + V_C(t)$$

Or in phasor representation:

$$\bar{V}_0 = \bar{V}_{R0} + \bar{V}_{L0} + \bar{V}_{C0}$$



From these diagrams we can see what is the relationship between different voltage amplitudes:

$$V_0 = |\bar{V}_0| = |\bar{V}_{R0} + \bar{V}_{L0} + \bar{V}_{C0}| = \left[V_{R0}^2 + (V_{L0} - V_{C0})^2\right]^{1/2}$$

$$= \left[(I_0 R)^2 + (I_0 X_L - I_0 X_C)^2\right]^{1/2} = I_0 \left[R^2 + (X_L - X_C)^2\right]^{1/2}$$

Since the voltages are NOT in phase, the max amplitude of an AC voltage source V_0 is NOT equal to the sum of the max amplitudes of each element:

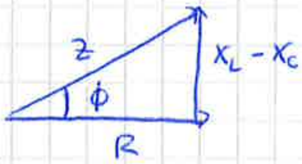
$$V_0 \neq V_{R0} + V_{L0} + V_{C0}$$

Impedance

We can see from the last equation that the effective resistence of an RLC circuit in series is:

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \quad \text{impedance} \quad ; \quad [Z] \equiv \text{Ohms}$$

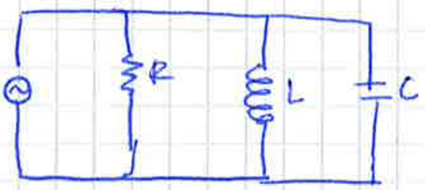
The relationship between Z, X_L and X_C can be represented by:



In terms of Z we can write the current as:

$$I(t) = \frac{V_0}{Z}\sin(\omega t - \phi)$$

Parallel RLC circuit



Here since we have the elements in ∥. The instantaneous voltage across R, L, C is the same.
Each voltage is in phase with the current through the resistor.
However, currents through each element are ≠.

The current in the resistor is

$$I_R(t) = \frac{V(t)}{R} = \frac{V_0}{R}\sin\omega t = I_{R0}\sin\omega t \quad \text{where } I_{R0} = \frac{V_0}{R}.$$

To obtain the current across the inductor we need to calculate first the voltage across it:

$$V_L(t) = V(t) = V_0 \sin\omega t = L\frac{dI_L}{dt}$$

$$\Rightarrow I_L(t) = \int_0^t \frac{V_0}{L}\sin\omega t'\,dt' = -\frac{V_0}{\omega L}\cos\omega t = \frac{V_0}{X_L}\sin\left(\omega t - \frac{\pi}{2}\right) = I_{L0}\sin\left(\omega t - \frac{\pi}{2}\right)$$

$$\text{where } I_{L0} = \frac{V_0}{X_L} \text{ and } X_L = \omega L$$

~~Tolda~~ Similarly, the voltage across the capacitor is:

$$V_C(t) = V_0 \sin\omega t = \frac{Q(t)}{C}$$

$$\Rightarrow I_C(t) = \frac{dQ}{dt} = \omega C V_0 \cos\omega t = \frac{V_0}{X_C}\sin\left(\omega t + \frac{\pi}{2}\right) = I_{C0}\sin\left(\omega t + \frac{\pi}{2}\right)$$

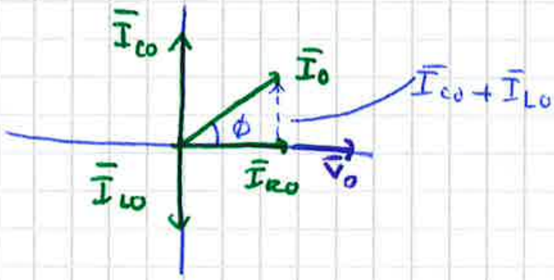
$$\text{where } I_{C0} = \frac{V_0}{X_C} \text{ and } X_C = \frac{1}{\omega C}$$

Using Kirchhoff's junction rule, the total current is the ~~Σ of~~ sum of all currents:

$$I(t) = I_R(t) + I_L(t) + I_C(t)$$

$$= I_{R0}\sin\omega t + I_{L0}\sin\left(\omega t - \frac{\pi}{2}\right) + I_{C0}\sin\left(\omega t + \frac{\pi}{2}\right)$$

We can represent the current in a phasor diagram:



From the phasor diagram we see:

$$\bar{I}_0 = \bar{I}_{R0} + \bar{I}_{L0} + \bar{I}_{C0}$$

and the maximum amplitude of the total current I_0 can be obtained as:

$$I_0 = |I_0| = |\bar{I}_{R0} + \bar{I}_{L0} + \bar{I}_{C0}| = \sqrt{I_{R0}^2 + (I_{C0} - I_{L0})^2}$$

$$= V_0 \sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2} = V_0 \sqrt{\frac{1}{R^2} + \left(\frac{1}{X_C} - \frac{1}{X_L}\right)^2}$$

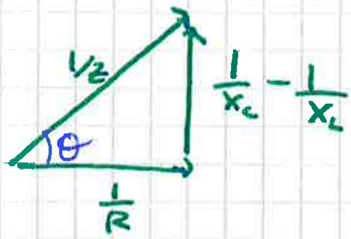
Note however, since $I_R(t)$, $I_L(t)$ and $I_C(t)$ are not in phase, I_0 is not equal to the sum of the max amplitudes of the currents:

$$I_0 \neq I_{R0} + I_{L0} + I_{C0}$$

With $I_0 = \frac{V_0}{Z}$, the inverse impedance of the circuit is:

$$\frac{1}{Z} = \sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2} = \sqrt{\frac{1}{R^2} + \left(\frac{1}{X_C} - \frac{1}{X_L}\right)^2}$$

The relationship between Z, R, X_L and X_C for a ll RLC circuit is:



we can obtain the phase as:

$$\tan\phi = \left(\frac{I_{C0} - I_{L0}}{I_{R0}}\right) = \frac{\frac{V_0}{X_C} - \frac{V_0}{X_L}}{\frac{V_0}{R}} = R\left(\frac{1}{X_C} - \frac{1}{X_L}\right) = R\left(\omega C - \frac{1}{\omega L}\right)$$