lecture 14. Magnetostatics II. Biot-Savart & Ampère's laws.

Last time:

* Magnetic force on a charge Q:

$$\underline{F} = Q(\underline{v} \times \underline{B})$$

* Magnetic force on a current I:

$$\underline{F} = I\int_a^b d\underline{s} \times \underline{B}$$ where a, b are the ends of the wire, ds is a length element along the wire

* Lorentz force law:

$$\underline{F} = Q[\underline{E} + (\underline{v} \times \underline{B})]$$ force on a charge Q due to E & B

Today: Steady currents & the fields they produce

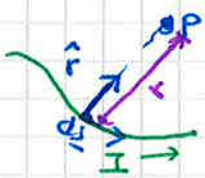
$$\boxed{\begin{array}{l}\text{Stationary charges} \rightarrow \text{constant } \underline{E} \text{ (electrostatics)}\\ \text{Steady currents} \rightarrow \text{constant } \underline{B} \text{ (magnetostatics)}\end{array}}$$

Saying stationary current is an oxymoron so instead we call it a steady current. But the point is that it doesn't change in time

$$\frac{\partial \underline{J}}{\partial t} = 0$$ where $\underline{J}$ is the current density.

Physical interpretation of steady current
→ charge is NOT piling up anywhere $\left(\frac{\partial \rho}{\partial t} = 0\right)$
→ Magnitude of current in the wire is the same everywhere.
And then → continuity eq. For steady currents: $\nabla \cdot \underline{J} = 0$

What is the magnetic field of a steady current?



r ≡ distance from the current source to point P
$\hat{r}$ unit vector in direction of r

We want to calculate the magnetic field (dB) at a point P due to an infinitesimal wire element ds:

The source of the magnetic field $d\underline{B}$ is the infinitesimal current segment $I d\underline{s}$.
The Biot-Savart law gives an expression for $d\underline{B}$ from the current source $I d\underline{s}$:

$$d\underline{B} = \frac{\mu_0}{4\pi}\frac{I\, d\underline{s} \times \hat{\underline{r}}}{r^2}$$ where $\mu_0 \equiv$ permeability of free space $= 4\pi \times 10^{-7}\ T \cdot m/A$

We choose those units for μ_0 so the units of $\underline{B}$ are $\frac{N}{A \cdot m} \equiv$ Tesla

(A common unit used for $\underline{B}$ is the gauss (cgs unit)

$1T = 10^4$ gauss $\qquad$ $\underline{B}$ of earth ~ 0.5 gauss

strong $\underline{B}$ in the lab $\sim 10\,000$ gauss)

Now to actually obtain the field we need to integrate $d\underline{B}$ over the wire:

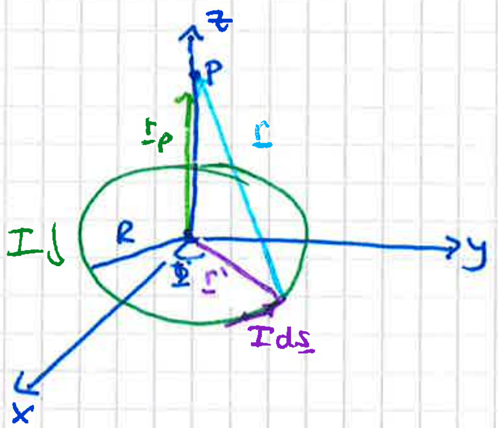
$$\underline{B} = \int_{wire} d\underline{B} = \frac{\mu_0 I}{4\pi} \int_{wire} \frac{d\underline{s} \times \hat{r}}{r^2}$$

where $\hat{r}$ is the unit vector in the direction of $\underline{r}$, which is the distance from the current source to the field point. $\underline{r} = r\hat{r}$

we have a x product so this is a vector integral, which means that the expression for $\underline{B}$ is really 3 integrals, one for each component of $\underline{B}$.

~~we have so~~

EXAMPLE: Magnetic field of a circular current loop



Find the magnetic field at a distance z above the center of a circular loop of radius R, which carries a steady current I.

We will use Biot-Savart's law:

$$\underline{B} = \frac{\mu_0 I}{4\pi} \int_{wire} \frac{d\underline{s} \times \hat{r}}{r^2}$$

General methodology for these problems:

1) source point: choose an appropriate coord system and write down an expression for the differential current element $Id\underline{s}$ and the vector $\underline{r}'$ describing the position of the source w/ respect to your coord system.

2) field point: It's the point where we are interested in calculating the field. The position of this point w/ respect to our coord system is given by:

$\underline{r}_p$, note $|\underline{r}_p|$ is the distance between the origin & the field pt.

3) Relative position vector: Vector between the source & the field pt.

$$\underline{r} = \underline{r}_p - \underline{r}'$$ ~~and the unit vector is~~ w/ corresponding unit vector:

$$\hat{r} = \frac{\underline{r}}{r} = \frac{\underline{r}_p - \underline{r}'}{|\underline{r}_p - \underline{r}'|}$$

4) we need to first establish what these vectors are because we are after calculating ~~the~~ the x product in the integral $d\underline{s} \times \hat{r}$. The resultant vector will give the direction of $\underline{B}$.

5) Once this is calculated, subs into Biot-Savart's law & simplify. Remember to use trig identity to simplify.

6) Integrate the expression to obtain $\underline{B}$. The geometry of the system will determine the integration limits. Change of variables using trig. might help getting nicer integrals.

OK, let's solve this now:

We can go back to our drawing & draw $\underline{r}_p$, $\underline{r}'$

1) First, what is the source pt $\underline{r}'$?

The differential current element $I d\underline{s}$ located at $\underline{r}' = R(\cos\Phi'\hat{\imath} + \sin\Phi'\hat{\jmath})$ can be written as:

$$I d\underline{s} = I\left(\frac{d\underline{r}'}{d\Phi'}\right) d\Phi' = I d\Phi' \frac{d}{d\Phi'}\left[R(\cos\Phi'\hat{\imath} + \sin\Phi'\hat{\jmath})\right]$$

$$= IR\,d\Phi'(-\sin\Phi'\hat{\imath} + \cos\Phi'\hat{\jmath})$$

2) Now, what is the field point?

$$\underline{r}_p = z\hat{k} \quad , \text{ a pt in the z axis}$$

3) What is the relative position vector?

$$\underline{r} = \underline{r}_p - \underline{r}' = -R\cos\Phi'\hat{\imath} - R\sin\Phi'\hat{\jmath} + z\hat{k}$$

and it's magnitude:

$$r = |\underline{r}| = \left[(-R\cos\Phi')^2 + (-R\sin\Phi')^2 + z^2\right]^{1/2} \underset{\text{because } \sin^2\Phi' + \cos^2\Phi' = 1}{=} \left[R^2 + z^2\right]^{1/2}$$

And we can re-write the corresponding unit vector

$$\hat{r} = \frac{\underline{r}}{r} = \frac{\underline{r}_p - \underline{r}'}{|\underline{r}_p - \underline{r}'|}$$

4) Calculate & simplify the cross product $d\underline{s} \times \hat{\underline{r}}$

$$d\underline{s} \times (\underline{r}_p - \underline{r}') = R\,d\Phi'(-\sin\Phi'\hat{\imath} + \cos\Phi'\hat{\jmath}) \times \left[-R\cos\Phi'\hat{\imath} - R\sin\Phi'\hat{\jmath} + z\hat{k}\right]$$

$$= R\,d\Phi'\left[z\cos\Phi'\hat{\imath} + z\sin\Phi'\hat{\jmath} + R\hat{k}\right]$$

5) Writing down $d\underline{B}$:

$$d\underline{B} = \frac{\mu_0 I}{4\pi}\frac{d\underline{s} \times \hat{r}}{r^2} = \frac{\mu_0 I}{4\pi}\frac{d\underline{s} \times \underline{r}}{r^3} = \frac{\mu_0 I}{4\pi}\frac{d\underline{s} \times (\underline{r}_p - \underline{r}')}{|\underline{r}_p - \underline{r}'|^3}$$

$$= \frac{\mu_0 IR}{4\pi}\frac{z\cos\Phi'\hat{\imath} + z\sin\Phi'\hat{\jmath} + R\hat{k}}{(R^2+z^2)^{3/2}}\,d\Phi'$$

6) Integrate:

$$\underline{B} = \frac{\mu_0 IR}{4\pi}\int_0^{2\pi}\frac{z\cos\Phi'\hat{\imath} + z\sin\Phi'\hat{\jmath} + R\hat{k}}{(R^2+z^2)^{3/2}}\,d\Phi'$$

As we said we have 3 integrals:

$$B_x = \frac{\mu_0 I R z}{4\pi (R^2+z^2)^{3/2}} \int_0^{2\pi} \cos\Phi' d\Phi' = \frac{\mu_0 I R z}{4\pi (R^2+z^2)^{3/2}} \sin\Phi' \Big|_0^{2\pi} = 0$$

$$B_y = \frac{\mu_0 I R z}{4\pi (R^2+z^2)^{3/2}} \int_0^{2\pi} \sin\Phi' d\Phi' = \frac{-\mu_0 I R z}{4\pi (R^2+z^2)^{3/2}} \cos\Phi' \Big|_0^{2\pi} = 0$$

$$B_z = \frac{\mu_0}{4\pi} \frac{I R^2}{(R^2+z^2)^{3/2}} \int_0^{2\pi} d\Phi' = \frac{\mu_0}{4\pi} \frac{2\pi I R^2}{(R^2+z^2)^{3/2}} = \frac{\mu_0 I R^2}{2(R^2+z^2)^{3/2}}$$

so $\underline{B}$ has only a component along the z-axis.

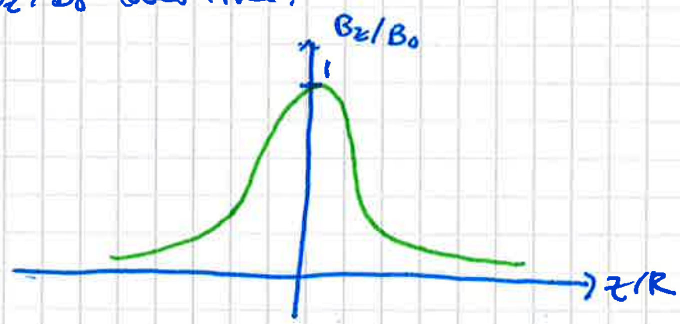
What does this field look like? Can we get some intuition?

$$B_z(z) = \frac{\mu_0 I R^2}{2(R^2+z^2)^{3/2}}$$ at $z=0$ the strenght of the field is:

$$B_z(0) \equiv B_0 = \frac{\mu_0 I R^2}{2(R^2)^{3/2}} = \frac{\mu_0 I}{2R}$$, using this and considering $z^* = \frac{z}{R}$ we can re-write B as:

$$B_z(z^*) = \frac{\mu_0 I R^2}{2(R^2+z^{*2}R^2)^{3/2}} = \frac{\mu_0 I R^2}{2[R^2(1+z^{*2})]^{3/2}} = \frac{\mu_0 I R^2}{2R^3(1+z^{*2})^{3/2}}$$
$$= \frac{\mu_0 I}{2R(1+z^{*2})^{3/2}}$$

so B_z/B_0 looks like:



Is there an easier way to do this? Yes! we can use the physicist's approach & look for symmetries as in the next page.

The physicist's approach:

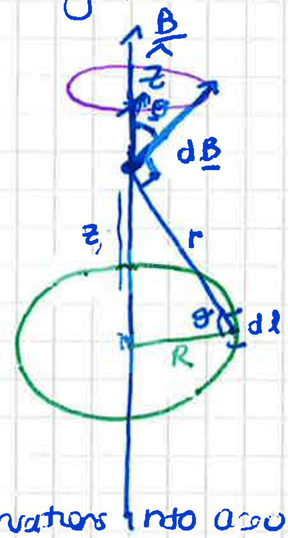
EXAMPLE: Find the magnetic field a distance z above the center of a circular loop of radius R, which carries a steady current I.

steady current → we can use Biot-Savart's law

$$B(\underline{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\underline{l} \times \hat{r}}{r^2}$$

(here Don't be confused by the change in notation, here $d\underline{l} = ds$ from before)

Always make a drawing first:



Observations:

* $\underline{B}$ should point in the direction $d\underline{l} \times \hat{\underline{r}}$, i.e. it will be ⊥ to the plane these vectors make.
* As we integrate around the circle, the vector $d\underline{B}$ will "sweep out" a cone.
* Because of symmetry the horizontal components of $d\underline{B}$ cancel, leaving only the projection with the z axis given by $\cos\theta$.

Taking these observations into account we can see:

$$B(z) = \frac{\mu_0}{4\pi} I \int \frac{d\underline{l} \times \hat{r}}{r^2} = \frac{\mu_0}{4\pi} I \int \frac{d\underline{l}}{r^2}\cos\theta = \frac{\mu_0}{4\pi} I \frac{\cos\theta}{r^2} \int_0^{2\pi} dl$$

$$= \frac{\mu_0}{4\pi} I \frac{\cos\theta}{r^2} 2\pi R = \frac{\mu_0 I R}{2r^2} \underbrace{\left(\frac{R}{r}\right)}_{\text{this is } \cos\theta \text{ by trigonometry}}$$

$$= \frac{\mu_0 I R^2}{2r^3} = \frac{\mu_0 I R^2}{2(R^2+z^2)^{3/2}} \blacksquare$$

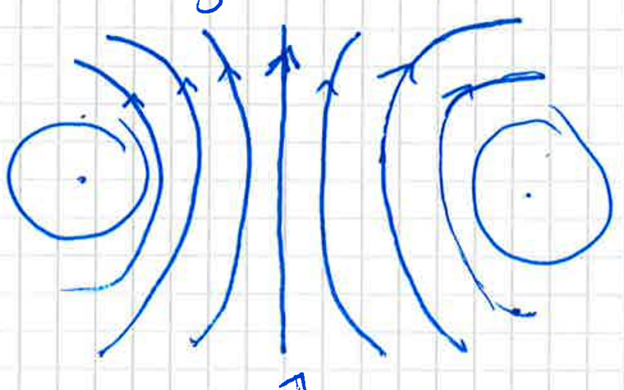
↑ because $r = (R^2+z^2)^{1/2}$ by pythagoras.

For fun let's look more closely at this expression. At the center of the ring, $z=0$

$$B_z(0) = \frac{\mu_0 I R^2}{2(R^2)^{3/2}} = \frac{\mu_0 I R^2}{2R^3} = \frac{\mu_0 I}{2R}$$

Note that $\underline{B}$ points upward at any point along the z-axis.

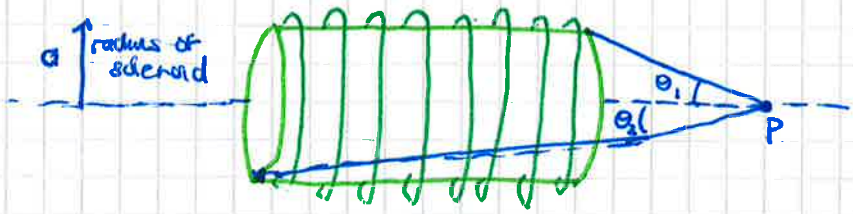
The field looks something like:



see: see purcell Fig. 6.15

↑ this line is for pts along z

EXAMPLE: Solenoid. Find the magnetic field at point P on the axis of a tightly wound solenoid consisting on n turns / unit length wrapped around a cylindrical tube of radius a and carrying a current I.



A solenoid is a cylindrical coil of wire.
To calculate the field we will assume the wire is closely and evenly spaced so that the number of turns in the winding per unit ~~leng~~ length n, is constant.

Current path is actually helical, we will ignore this and consider it as a stack of loops.

We can generalize Biot-Savart's law for surface & steady currents!

$$\boxed{\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(\mathbf{r}) \times \hat{r}}{r^2}\, dA}$$ For surface currents

$$\boxed{\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}) \times \hat{r}}{r^2}\, d\tau}$$ For volume currents

Remember $\mathbf{r} = |\mathbf{r}|\,\hat{r}$ is the vector from the source to the field pt.

* The principle of superposition applies to magnetic fields, just as electric fields. If we have a collection of steady currents:

Net $\mathbf{B}$ is the sum of the field of each current.

NOTE: Can we write the field of a moving charge using Biot-Savart's law?

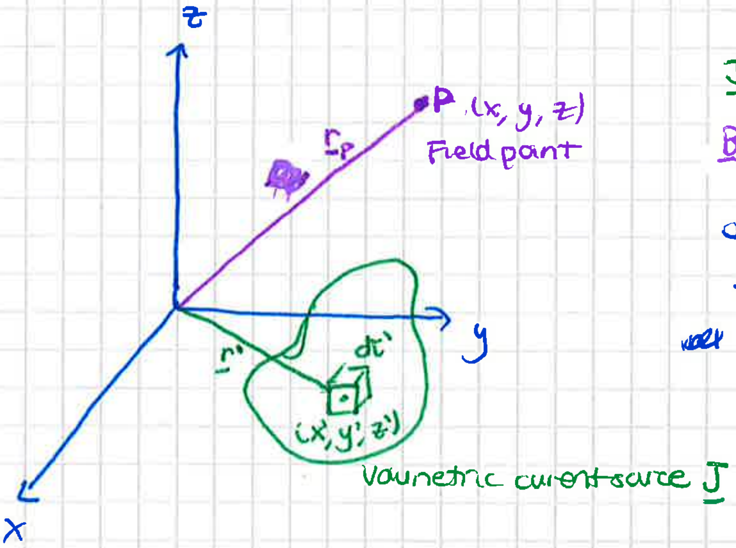
$$\cancel{\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\, q\, \frac{\mathbf{v} \times \hat{r}}{r^2}}$$ WRONG

a point charge does NOT constitute a steady current.

This equation only holds in the non-relativistic limit where $v \ll c$, where c is the speed of light, so that the effect of "retardation" can be ignored.

The divergence and the curl of $\mathbf{B}$ (Ampère's law)

We will now calculate the divergence & the curl of $\mathbf{B}$, for this we will explicitly write down Biot Savart's law using the ' and "unprimed" coordinates



$\mathbf{J}$ is a function of (x', y', z')

$\mathbf{B}$ is a function of (x, y, z)

and

$$\mathbf{r} = \mathbf{r}_p - \mathbf{r}' = (x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z}$$

we have Biot-Savart's law:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}') \times \hat{r}}{r^2}\, d\tau'$$ integration is over the primed coordinates.

Let's calculate the divergence of $\mathbf{B}$: This is with respect to the unprimed coordinates.

$$\nabla \cdot \mathbf{B} = \frac{\mu_0}{4\pi}\int \nabla \cdot \left(\mathbf{J} \times \frac{\hat{r}}{r^2} \right) d\tau'$$

And using the identity:

$$\nabla \cdot \left(\underline{J} \times \frac{\hat{r}}{r} \right) = \frac{\hat{r}}{r^2} \cdot (\nabla \times \underline{J}) - \underline{J} \cdot \left(\nabla \times \frac{\hat{r}}{r^2} \right)$$

algebra, you can calculate it & verify this.
prod rule #4 p.21 griffiths

we can subs this in the eq. for the divergence of $\underline{B}$ and get:

$$\nabla \cdot \underline{B} = \frac{\mu_0}{4\pi} \int \left[\frac{\hat{r}}{r^2} \cdot \underbrace{(\nabla \times \underline{J})}_{\text{this is zero}} - \underline{J} \cdot \underbrace{\left(\nabla \times \frac{\hat{r}}{r^2} \right)}_{\text{this is zero, you can do the algebra & check}} \right]$$

$\nabla \times \underline{J} = 0$ because $\underline{J}$ is independent of the unprimed coord

$$\Rightarrow \boxed{\nabla \cdot \underline{B} = 0}$$

the divergence of a magnetic field is zero.

Now we can calculate the curl:

$$\nabla \times \underline{B}(\underline{r}) = \frac{\mu_0}{4\pi} \int \nabla \times \left(\underline{J} \times \frac{\hat{r}}{r^2} \right) d\tau'$$

Again we will use the appropriate identity to expand the integrand:

$$\nabla \times \left(\underline{J} \times \frac{\hat{r}}{r^2} \right) = \underbrace{\underline{J} \left(\nabla \cdot \frac{\hat{r}}{r^2} \right)}_{\text{orange term}} - \underbrace{(\underline{J} \cdot \nabla) \frac{\hat{r}}{r^2}}_{\text{this term integrates to zero}} \underbrace{- \frac{\hat{r}}{r^2} (\nabla \cdot \underline{J}) + \left(\frac{\hat{r}}{r^2} \cdot \nabla \right) \underline{J}}_{\text{these terms are zero}}$$

product rule (vi) griffiths p.21

these terms are zero since the derivative is with respect to the unprimed coord & J doesn't depend on them

Demonstration that the green term ~~goes~~ integrates to zero:

$$-(\underline{J} \cdot \nabla) \frac{\hat{r}}{r^2} = (\underline{J} \cdot \nabla') \frac{\hat{r}}{r^2} \quad (*)$$

we can switch from ∇ to ∇' by adding a minus sign since the derivative only acts on $\frac{\hat{r}}{r^2}$

this is equivalent to saying:

$$\frac{\partial}{\partial x} f(x - x') = - \frac{\partial}{\partial x'} f(x - x')$$

the chain rule

Now see how this looks for the x component in particular:

$$(\underline{J} \cdot \nabla') \left(\frac{x - x'}{r^3} \right) = \nabla' \cdot \left[\frac{(x - x')}{r^3} \underline{J} \right] - \left(\frac{x - x'}{r^3} \right) \underbrace{(\nabla' \cdot \underline{J})}_{\text{this is zero for steady currents}}$$

using product rule #5

$$= \nabla' \cdot \left[\frac{(x - x')}{r^3} \underline{J} \right]$$ subs this into eq (*) we get:

$$-(\underline{J}\cdot\nabla)\frac{\hat{r}}{r^2} = \nabla'\cdot\left[\frac{(x-x')}{r^3}\underline{J}\right]$$

so ~~dt~~ the contribution to the integral is:

$$\int_V \nabla'\cdot\left[\frac{x-x'}{r^3}\underline{J}\right]d\tau' = \oint_S \frac{(x-x')}{r^3}\underline{J}\cdot d\underline{a}'$$

on the boundary of the surface the current is zero so this integral is zero.

so the contribution of the green term is zero and we are left with the orange term:

$$\underline{J}\left(\nabla\cdot\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\underline{r})$$ (see p.50 Griffiths)

where δ^3 is the 3D dirac delta function

With this we have that the curl of $\underline{B}$ is:

$$\nabla\times\underline{B} = \frac{\mu_0}{4\pi}\int \underline{J}(\underline{r}')4\pi\delta^3(\underline{r}-\underline{r}')\,d\tau' = \mu_0\underline{J}(\underline{r})$$

this is just the $\int$ of the delta function

$$\therefore \boxed{\nabla\times\underline{B} = \mu_0\underline{J}}$$ Ampere's law in differential form.

We can use Stoke's theorem to convert it into integral form:

$$\int(\nabla\times\underline{B})\cdot d\underline{a} = \oint\underline{B}\cdot d\underline{l} = \mu_0\underbrace{\int\underline{J}\cdot d\underline{a}}_{\text{total current passing through the surface}}$$

(Remember $I = \int\underline{J}\cdot d\underline{a}$ by definition)

$$\therefore \boxed{\oint\underline{B}\cdot d\underline{l} = \mu_0 I_{enc}}$$ Ampère's law in integral form

I_{enc} is the current enclosed by an Amperian loop.

The direction of integration around the boundary is given by the right hand rule.

If ~~you~~ the fingers of your right hand indicate the direction of integration around the boundary, your thumb defines the direction of positive current.

EXAMPLE: Calculate the magnetic field a distance s from a long straight wire carrying a steady current I.

We know that the direction of $\underline{B}$ is circumferential, circling around the wire as indicated by the right hand rule

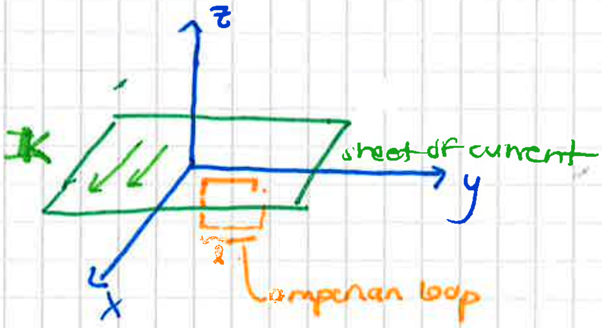
By symmetry the magnitude of $\underline{B}$ is constant around the loop so: this is key to take it out of the integral just like for Gauss's law.

$$\oint \underline{B} \cdot d\ell = B \oint d\ell = B(2\pi s) = \mu_0 I_{enc}$$

$$\Rightarrow B = \frac{\mu_0 I}{2\pi s}$$

EXAMPLE: Calculate the magnetic field of an infinite uniform surface current $\underline{K} = K\hat{x}$ flowing over the x,y plane.

K is flowing in the x direction



First of all we want to determine the direction of $\underline{B}$? we know that it can't have an x component, since the source is // to x

$$B(\underline{r}) = \frac{\mu_0}{4\pi} \int \frac{\underline{K}(\underline{r}) \times \hat{r}}{r^2}\, da$$

then the x component in the cross prod is zero

The field is $\perp$ to K, so in this case it is $\perp$ to x

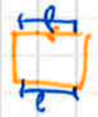
Can the field have a z component? No, because of symmetry.

Any vertical contribution from a filament at $+y$ will be cancelled by a filament at $-y$

Now let's draw a loop as shown in the figure parallel to the x,z plane. Applying Ampère's law:

$$\oint \underline{B} \cdot d\underline{\ell} = B(2\ell) = \mu_0 I_{enc} = \mu_0 K \ell$$

↑ by definition of K



$$\Rightarrow B = \frac{\mu_0 K}{2}$$ and more precisely

$$B = \begin{cases} \frac{\mu_0}{2} K\hat{y}\ ; & z < 0 \\ -\frac{\mu_0}{2} K\hat{y}\ ; & z > 0 \end{cases}$$