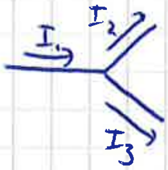
Lecture 13. Magnetostatics. Lorentz force law

Last time: Circuits

* EMF: source of energy to maintain a constant current in a closed circuit.

$$\mathcal{E} \equiv \frac{dW}{dq}$$

* Kirchhoff's rules:



$\sum I_{in} = \sum I_{out}$ Junction rule

$I_1 = I_2 + I_3$

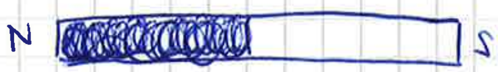
$\sum_{\text{closed loop}} \Delta V = 0$ loop rule

Today: Magnetism

How it started:

* Ancient greece: Pieces of magnetite → can attract irons but no effect on Au, Ag, Cu, etc.
  ↳ can attract or repel ~~another~~ another piece of magnetite depending on the relative orientation.

* XII century → magnetic compass
  ↳ small magnetic needle will always come to rest with one end pointing "North" and the other "South"



Like poles repel
Unlike poles attract

Demo 567 and 564 circular magnets + magnetized needles

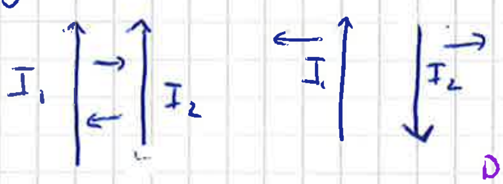
When things got interesting:

* In 1820 Oersted realized that current flowing on a wire made the needle of a compass swing

Revisit demo 564 w/ Helmholtz coil

BIG DEAL: Electricity + magnetism are related

* Soon after came the definite proof w/ Ampère's experiment with parallel wires carrying current.



* If currents are // wires attract
  if currents are anti // wires repel
* If we place a stationary charge nearby it doesn't move → wires are overall neutral

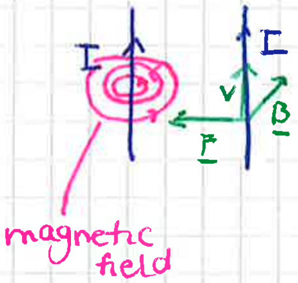
Demo 592 Ampère's experiment.

whatever F accounts for the attraction of // currents & repulsion of antiparallel is NOT electrostatic.

A stationary charge produces only an electric field $\underline{E}$. A moving charge generates in addition a magnetic field $\underline{B}$.

Take a compass & you can't find magnetic fields anywhere. If you take a compass in the vicinity of a current carrying wire you'll see the magnetic field "curls" around the source.

And here is where the right hand rule comes back to haunt us!



More refined observations of Ampere's experiment:

$F \sim I_1 I_2$ Force is proportional to the velocity of the charges in motion

Direction of $\underline{F}$ is perpendicular to the velocity

Interpretation → Some field is created by the charges in motion.
↳ magnetic $\underline{F}$ is proportional to the cross product:

$$\boxed{\underline{F} = Q(\underline{v} \times \underline{B})}$$ magnetic force on a charge Q moving at velocity $\underline{v}$

In the presence of both electric and magnetic fields, the total force will be:

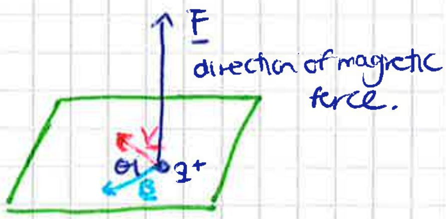
$$\boxed{\underline{F} = Q[\underline{E} + (\underline{v} \times \underline{B})]}$$ Lorentz force law

Thoughts on Lorentz force law:

* So far the Lorentz force law is an empirical law, put together from experimental observations.
* It defines the magnetic field $\underline{B}$
* Units of $\underline{B}$:

$$[B] = \frac{[F]}{[qv]} = \frac{N}{m/s\, C} = \frac{N}{A \cdot m} = \text{Tesla } (T)$$

where m meters
N newtons
s seconds
C coulomb
A ampere

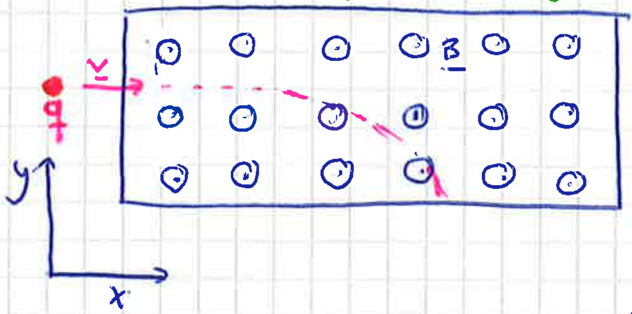


* Magnitude of the magnetic force is proportional to v and q.
* Because $\underline{F} \propto \underline{v} \times \underline{B}$, when $\underline{v} \parallel \underline{B}$ the magnetic force vanishes.
* Remembering the geometrical interpretation of the x product. If $\underline{v}$ makes an angle θ with $\underline{B}$, the direction of $\underline{F}$ is perpendicular to the plane formed by $\underline{v}$ and $\underline{B}$ and it's magnitude is proportional to sin θ.
* when we change the sign of Q, the direction of $\underline{F}$ also reverses.

EXAMPLE:
Trajectory of a particle in a magnetic field

A particle of charge q and mass m moves with a velocity $\underline{v} \parallel +\hat{x}$ axis in a magnetic field $\underline{B} \parallel -\hat{z}$ axis (into the page):

What is the trajectory of q in the magnetic field?



$$\underline{F}_{mag} = q(\underline{v} \times \underline{B})$$

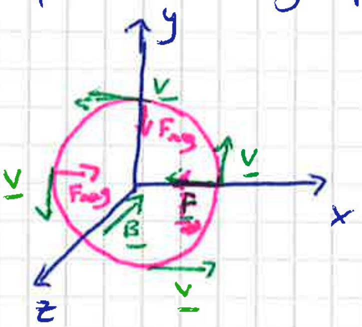
$\underline{B}$ points out of the page and $\underline{v}$ is $\parallel$ to $\hat{x}$

$\Rightarrow \underline{B} \perp \underline{v}$

And we saw that by definition $\underline{F}_{mag} \perp$ the plane that v & B are in

$\therefore$ $\underline{v}$, $\underline{B}$ and $\underline{F}_{mag}$ are all perpendicular

So then the path of the charged particle will be:



particle moves in a counterclockwise circle!

Magnetic force has a fixed magnitude

$$|F_{mag}| = qvB$$

that allows it to sustain circular motion

$$QvB = \frac{mv^2}{R}$$

where R is the radius of the circle its trajectory describes

$$R = \frac{mv}{qB}$$

the time it takes for the particle to complete a revolution is:

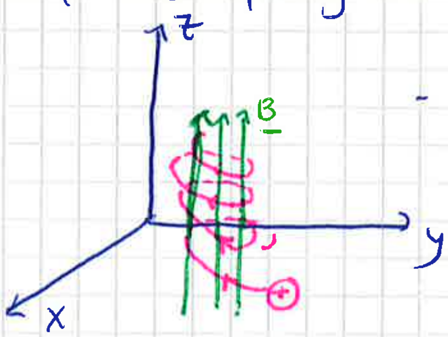
$$T = \frac{2\pi R}{v} = \frac{2\pi\, mv}{v\, qB} = \frac{2\pi m}{qB}$$

using the expression of R we just derived

Similarly we can obtain the angular speed of the particle:

$$\omega = 2\pi f = \frac{v}{R} = \frac{qB}{m}$$

If the initial velocity of the charged particle has a component parallel to the magnetic field $\underline{B}$, instead of a circle, the particle's trajectory will be a helical path:



- particle velocity has a non-zero component in the direction of $\underline{B}$

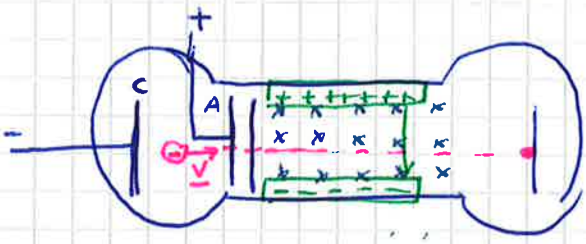
Demo 415. Deflection of an electron beam by $\underline{B}$

J.J Thompson's experiment

Discovery of the electron and measurements of e/m_e in 1897.
this experiment showed that atoms are not indivisible, rather contain smaller charged particles.

The idea:
→ A beam of "cathode rays" crosses a region with $\underline{E}$ & $\underline{B}$ present.
→ Choosing $v_e \parallel x$ axis, $B \parallel z$ axis, $E \parallel y$ axis $\Rightarrow \underline{F}_{lorentz} \parallel \underline{F}_{electric}$

→ E and B can be adjusted such that $F_{mag} = -F_{electric}$ so the e will go straight



Thompson's apparatus

× indicates magnetic field

▭ charged plates with + or - ve charge

$\underline{E}$ field points from the top plate to the bottom

* electrons with charge $q = -e$ and mass m are emitted from the cathode C and then accelerated towards slit A.
* Let the potential difference between A and C be $\Delta V = V_A - V_C$
* Change in potential energy is equal to the work done to accelerate the electrons:

$$\Delta U = W_{ext} = q\Delta V = -e\Delta V$$

* By conservation of energy, the kinetic energy gained is:

$$\Delta K = -\Delta U = \frac{mv^2}{2} = -e\Delta V$$

subs ΔV

$$\Rightarrow v = \left[\frac{2e\Delta V}{m}\right]^{1/2} \quad (*)$$ is the speed of the electrons.

* After this, the $\bar{e}$ will pass through a region with an electric field pointing downwards, so they should be deflected upwards.

However if we also add a magnetic field directed into the page, then the $\bar{e}$ will experience a downward magnetic force $-e\underline{v} \times \underline{B}$.

If the two forces exactly cancel, then the electrons will move in a straight path.

This condition is met when:

$$\underline{F} = q(\underline{E} + \underline{v} \times \underline{B}) = 0$$

$$\Rightarrow q\underline{E} = q\,\underline{v} \times \underline{B}$$ in this case $q = -e$

$$\Rightarrow eE = evB \Rightarrow v = \frac{E}{B} \quad (**)$$

so only particles with speed $v = \frac{E}{B}$ will move in a straight line.

Combining equations (*) and (**) we get:

$$\frac{E}{B} = \left[\frac{2e\Delta V}{m}\right]^{1/2} \Rightarrow \frac{e}{m} = \frac{E^2}{2\Delta V B^2}$$

so if we control experimentally E, B & ΔV then we can measure $\frac{e}{m} = 1.7588 \times 10^{11}\,C/kg$

Important note: Magnetic forces and work

* Moving a charge in an electric field requires work

$$W_{12} = -q\int_1^2 \underline{E}\cdot d\underline{s}$$

* How much work does it take to move a charge in a magnetic field?

$$dW = \underline{F}\cdot d\underline{s} = \underline{F}\cdot\underline{v}\,dt = q(\underline{v}\times\underline{B})\cdot\underline{v}\,dt = 0$$

$\underline{F}_{mag} = q(\underline{v}\times\underline{B})$

this follows because $(\underline{v}\times\underline{B})$ is $\perp$ to $\underline{v}$ so

$(\underline{v}\times\underline{B})\cdot\underline{v} = 0$

because the dot prod of 2 $\perp$ vectors is zero

$\therefore$ $\boxed{\text{Magnetic forces DO NOT work}}$

No work is needed to move a particle in a magnetic field because $\underline{v}$ & $\underline{F}$ are always $\perp$.

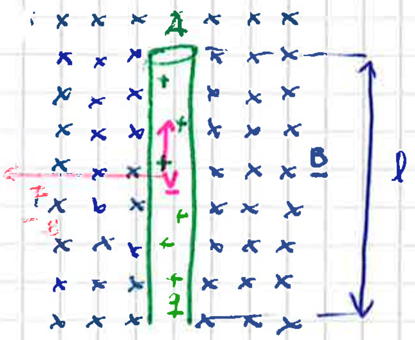
Magnetic forces may alter the direction in which a particle moves but the can't speed it up or slow it down.

Magnetic force on a current carrying wire

We have just seen that a charged particle placed in a magnetic field experiences a force $\underline{F}_{mag}$.

Since an electric current consists of a collection of charged particles in motion, when placed in a magnetic field $\underline{B}$, a current carrying wire will also experience a magnetic force.

To calculate the force exerted on the wire consider a segment of wire of length ℓ and cross sectional area A. The magnetic field points into the page represented by x's:



A crossectional area of wire

$\underline{B}$ is the magnetic field that points into the page represented by x

ℓ length of wire

$\underline{v}$ charge move at this av. drift velocity

The total magnetic force on this wire segment is:

$Q_{tot} = q(nA\ell)$ total amount of charge in the wire segment

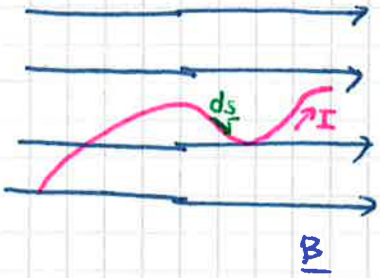
$n \equiv$ # of charges/unit volume

$$\underline{F}_{mag} = Q_{tot}(\underline{v}\times\underline{B}) = qnA\ell(\underline{v}\times\underline{B}) = I(\underline{\ell}\times\underline{B})$$

because $I = nqv_d A$ current definition

with $\underline{\ell}$ = length vector of magnitude ℓ directed along the direction of the current.

So this is for a small segment of wire, for a wire of arbitrary shape, the magnetic force can be obtained by summing over the forces acting on the small segments that make up the wire:



As usual, we have an infinitesimal wire element: $\underline{ds}$

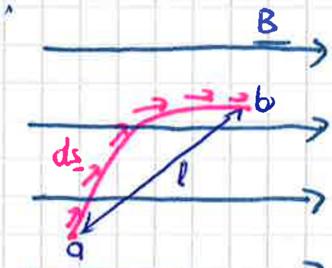
The magnetic $\underline{F}$ acting on this element is:

$$d\underline{F}_{mag} = I\, d\underline{s} \times \underline{B}$$

$$\Rightarrow \boxed{\underline{F}_{mag} = I \int_a^b d\underline{s} \times \underline{B} \quad \text{where } a \,\&\, b \text{ are the end points of the wire}}$$

we have taken I out of the integral assuming the current has constant magnitude along the wire.

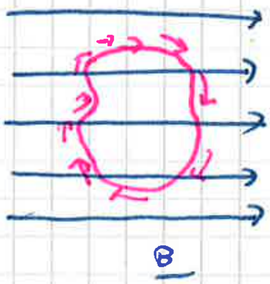
Now if we consider the following wire as an example:



In this case:

$$\underline{F}_{mag} = I \int_a^b d\underline{s} \times \underline{B} = I\, \underline{\ell} \times \underline{B}$$

Now if we consider a closed loop of arbitrary shape:



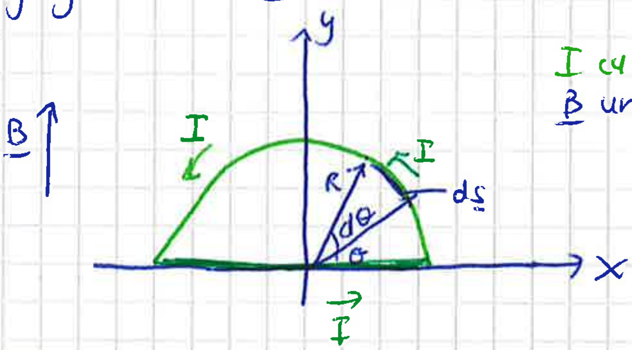
$$\underline{F}_{mag} = I \oint d\underline{s} \times \underline{B}$$

Since the set of $d\underline{s}$ form a closed polygon and their vector sum is zero $\Rightarrow \oint d\underline{s} = 0$

$\therefore$ $\boxed{\text{For any closed loop of current} \quad \underline{F}_{mag} = 0}$

EXAMPLE: Magnetic force on a semicircular loop

Consider a closed semi-circular loop lying in the x, y plane carrying a current I:



I current flowing counterclockwise
$\underline{B}$ uniform magnetic field in y direction

Q. What is the magnetic force acting on the straight segment of the semicircular arc?

this part

Solution:
Let $\underline{B} = B\hat{j}$ the magnetic field (mag B & oriented along y axis)

Let $\underline{F}_1$ & $\underline{F}_2$ be the forces acting on the wire segments ~~ca~~ & $\widehat{ca}$ respectively

Using the eq. we derived above:

$$\underline{F}_{mag} = I \int_a^b d\underline{s} \times \underline{B}$$

* For the straight segment we have that:

$\underline{F}_1 = I\underbrace{(2R)}_{\substack{\text{length of the straight} \\ \text{segment is } 2R\text{, where } R \\ \text{is the radius of the circle}}}\hat{\imath} \times B\hat{\jmath} = 2IRB\hat{k}$, where $\hat{k}$ is directed out of the page.

$\hat{\imath}$: segment is oriented along x axis

* To evaluate $\underline{F}_2$:

As usual we first note that the differential length element $d\underline{s}$ on the semicircle can be written as:

$$d\underline{s} = ds\,\hat{\theta} = R\,d\theta(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath})$$

$ds\,\hat{\theta}$: definition of arclength

with this the magnetic force is:

$$d\underline{F}_2 = I\,d\underline{s} \times \underline{B} = IR\,d\theta(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}) \times B\hat{\jmath}$$

$$= -IBR\sin\theta\,d\theta\,\hat{k}$$ so $\underline{F}_2$ points into the page.

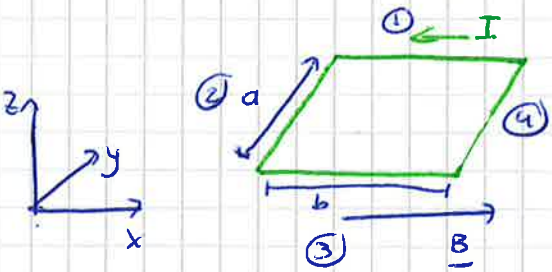
And now we have to integrate over the circular arc:

$$\underline{F}_2 = -IBR\,\hat{k}\int_0^{\pi} \sin\theta\,d\theta = -2IBR\,\hat{k}$$

And we can see that the net force on the loop is actually zero. As it should be for a closed current loop.

Torque on a current loop

what happens when we place a rectangular loop carrying a current I in the xy plane and switch on a uniform $\underline{B}$ // to the plane of the loop?



loop has sides w/ length a & b

let's consider 4 regions of the loop

What is the magnetic $\underline{F}$ experienced by each region of the loop?

For 1 & 3 $\underline{F}_{mag} = 0$ (since $I \parallel \underline{B}$ therefore $\underline{F}_{mag} = I(\underline{\ell} \times \underline{B}) = 0$)

For regions ② & 4 there is a magnetic force. Using the same eq we see:

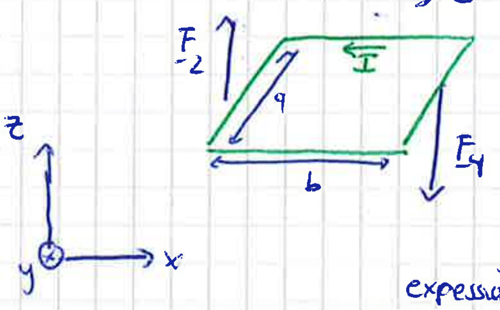
$\underline{F}_2 = I(-a\hat{\jmath}) \times (B\hat{\imath}) = IaB\hat{k}$ ← points out the page

$\underline{F}_4 = I(a\hat{\jmath}) \times (B\hat{\imath}) = -IaB\hat{k}$ ← points into the page

As expected the net force on the loop is zero. $\underline{F}_{net} = \underline{F}_1 + \underline{F}_2 + \underline{F}_3 + \underline{F}_4 = 0$

However, the forces $\underline{F}_2$ and $\underline{F}_4$ will produce a torque, which causes the loop to rotate around the y axis. Let's calculate the torque w/ respect to the center of the loop:

Remembering $\underline{\tau} = \underline{r} \times \underline{F}$ ← torque



$$\underline{\tau} = \left(-\frac{b}{2}\hat{\imath}\right) \times \underline{F}_2 + \left(\frac{b}{2}\hat{\imath}\right) \times \underline{F}_4$$

$$= \left(-\frac{b}{2}\hat{\imath}\right) \times \left(IaB\hat{k}\right) + \left(\frac{b}{2}\hat{\imath}\right) \times \left(-IaB\hat{k}\right)$$

subs the expressions we derived for F_2 & F_4

$$= \left(\frac{IabB}{2} + \frac{IabB}{2}\right)\hat{\jmath} = IAB\hat{\jmath}$$

where we defined $A = ab$ area of loop

$\therefore \underline{\tau} = IAB\hat{\jmath}$ since it's positive the rotation is clockwise ~~around the y axes~~ and we see it rotates around the y axis ($\hat{\jmath}$)

We can re-write this in terms of the area vector:

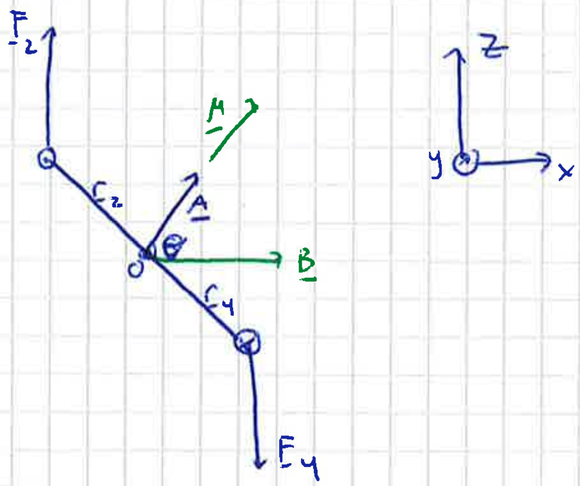
$\underline{A} = A\hat{n}$ ← direction normal to the loop plane

In this case we have $\hat{n} = +\hat{k}$ so we can re-write the torque as:

$$\boxed{\underline{\tau} = I\underline{A} \times \underline{B}}$$

Notice this is maximum when $\underline{B}$ is ∥ to the plane of the loop (or ⊥ to $\underline{A}$)

Now let's consider the general case where the loop (or the area vector $\underline{A}$) makes an angle θ with respect to $\underline{B}$:



From this figure we can see:

$$\underline{r}_2 = \frac{b}{2}\left(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{k}\right) = -\underline{r}_4$$ and the net torque becomes:

$$\underline{\tau} = \underline{r}_2 \times \underline{F}_2 + \underline{r}_4 \times \underline{F}_4 = 2\underline{r}_2 \times \underline{F}_2 = 2\frac{b}{2}\left(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{k}\right) \times \left(IaB\hat{k}\right)$$

$$= IabB\sin\theta\,\hat{\jmath} = I\underline{A} \times \underline{B}$$

And if we have a loop with N "turns":

$$\tau = NIAB\sin\theta$$

The quantity $NI\underline{A}$ is called the magnetic dipole moment $\underline{\mu}$:

$$\boxed{\underline{\mu} = NI\underline{A}}$$

← note direction of $\underline{\mu}$ is the same as $\hat{n}$. Determined by the right hand rule.

EXAMPLE: Calculate the magnetic field a distance s from a long straight wire carrying a steady current I.

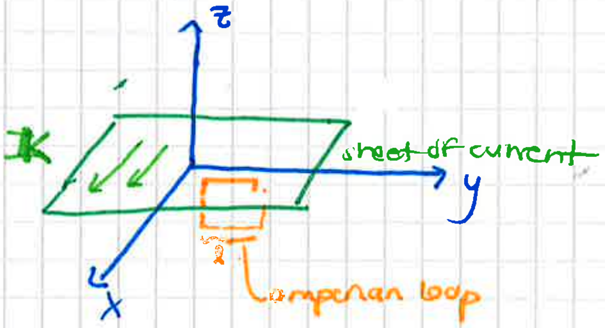
We know that the direction of $\underline{B}$ is circumferential, circling around the wire as indicated by the right hand rule

By symmetry the magnitude of $\underline{B}$ is constant around the loop so: — this is key to take it out of the integral Just like for Gauss's law.

$$\oint \underline{B}\cdot d\underline{l} = B\oint dl = B(2\pi s) = \mu_0 I_{enc}$$

$$\Rightarrow B = \frac{\mu_0 I}{2\pi s}$$

EXAMPLE: Calculate the magnetic field of an infinite uniform surface current $\underline{K} = K\hat{x}$ flowing over the x,y plane.



K is flowing in the x direction

First of all we want to determine the direction of $\underline{B}$: we know that it can't have an x component, since the source is $\parallel$ to x

$$\underline{B}(\underline{r}) = \frac{\mu_0}{4\pi}\int \frac{\underline{K}(\underline{r}) \times \hat{r}}{r^2}\, da$$

then the x component in the cross prod is zero

The field is $\perp$ to K, so in this case it is $\perp$ to x

Can the field have a z component? No, because of symmetry.

Any vertical contribution from a filament at $+y$ will be cancelled by a filament at $-y$

Now let's draw a loop as shown in the figure parallel to the x,z plane. Applying Ampère's law:

$$\oint \underline{B}\cdot d\underline{l} = B(2l) = \mu_0 I_{enc} = \mu_0 K l$$

↑ by definition of K

$$\Rightarrow B = \frac{\mu_0 K}{2}$$ and more precisely

$$B = \begin{cases} \frac{\mu_0}{2} K\hat{y}\,; & z<0 \\ -\frac{\mu_0}{2} K\hat{y}\,; & z>0 \end{cases}$$