Lecture 10. Electric fields in matter II. Electric displacement. Linear dielectrics.

Last time:

* Dielectrics
  - material made of Non-polar atoms → $\underline{E}_{ext}$ will induce $\underline{p}$ in the direction of $\underline{E}_{ext}$
  - material made of Polar molecules → Each dipole in the material will experience a torque $\underline{N} = \underline{p} \times \underline{E}$ that will align it w/ $\underline{E}_{ext}$.

  material becomes polarized in $\underline{E}_{ext}$

* Polarization $\underline{P} \equiv$ dipole moment / unit volume

* Bound charges
  - $\sigma_b = \underline{P} \cdot d\hat{n}$ surface bound charge density
  - $\rho_b = -(\nabla \cdot \underline{P})$ Volume bound charge density

TODAY: The electric displacement. What is $\underline{E}$ generated by a polarized material?

* $\underline{E}$ generated by a polarized material → Due to its bound charges

* If "free charges" are also present

Total $\underline{E}$ is the sum of $\underline{E}_{bound} + \underline{E}_{free}$

So in other words:

$$\nabla \cdot \underline{E} = \frac{\rho}{\varepsilon_0} \quad (*)$$

gauss's law in differential form (Max. eq #1)

in the case of a polarized dielectric → $\rho = \rho_b + \rho_f$

So eq (*) becomes:

$$\nabla \cdot \underline{E} = \frac{\rho_b + \rho_f}{\varepsilon_0} \Rightarrow \nabla \cdot \underline{E} = \frac{1}{\varepsilon_0}\left(-\nabla \cdot \underline{P} + \rho_{free}\right)$$

From our definition of ρ_{bound}.

We can do some algebra to rewrite the equation as:

$$\nabla \cdot \underbrace{(\varepsilon_0 \underline{E} + \underline{P})}_{\underline{D}\text{ electric displacement}} = \rho_{free}$$

$$\nabla \cdot \underline{D} = \rho_{free} \quad \text{where } \underline{D} \equiv (\varepsilon_0 \underline{E} + \underline{P})$$

Then we have that Gauss's law for dielectrics can be written as:

$$\nabla \cdot \underline{D} = \rho_{free} \quad \text{differential form}$$

or

$$\oint \underline{D} \cdot d\underline{a} = Q_{free} \quad \text{integral form}$$

From these expressions it would seem that $\underline{D}$ is the same as $\underline{E}$ but there are significant differences.

For example, we know that the curl of $\underline{E}$ is zero (no circulation), but for $\underline{D}$:

$$\nabla \times \underline{D} = \nabla \times (\varepsilon_0 \underline{E} + \underline{P}) = \varepsilon_0 \cancelto{0}{\nabla \times \underline{E}} + \nabla \times \underline{P} = \nabla \times \underline{P}$$

Remember that the condition of $\nabla \times \underline{E} = 0 \Leftrightarrow \oint \underline{E} \cdot d\underline{l} = 0$ so we needed this to define the electric potential. The fact that $\underline{D}$ does not satisfy this means that:
there is NO potential function that generates $\underline{D}$.

Also, the displacement vector $\underline{D}$ is not uniquely determined by ρ_{free}, it requires additional information, for example $\underline{P}$.

EXAMPLE:
Let's look at the problem we solved last week and see if we can get a quicker answer considering Gauss's law for $\underline{D}$.

A thick spherical shell is made of some material w/ frozen in polarization

$$\underline{P}(\underline{r}) = \frac{k}{r}\hat{r} \quad \text{where } k \text{ is a const}$$
r is the distance from the center

Find the electric field in all regions using $\oint \underline{D} \cdot d\underline{a} = Q_{fenc}$ and then get $\underline{E}$ from the definition of $\underline{D}$

We have that $\oint \underline{D} \cdot d\underline{a} = Q_{free}$ there is no Q_{free} $\Rightarrow D = 0$ everywhere.

Now to find $\underline{E}$:

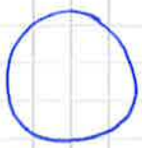
$$\underline{D} = \varepsilon_0 \underline{E} + \underline{P} \underset{\underline{D}=0}{\Rightarrow} \varepsilon_0 \underline{E} = -\underline{P} \Rightarrow \underline{E} = -\frac{k}{\varepsilon_0 r}\hat{r}$$

for inside the sphere, which is what we had found last time!

Another EXAMPLE:

* Suppose the field inside a large piece of dielectric is E_0 so that the electric displacement is $\underline{D}_0 = \varepsilon_0 \underline{E}_0 + \underline{P}$

Let's suppose we hollow out the following cavities from the material:



sphere

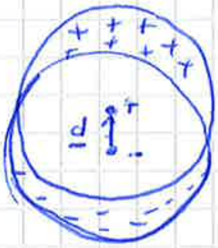
(i.e. it doesn't change when the cavity is excavated.)

Find $\underline{E}$ inside the cavity in terms of $\underline{D}_0$ & $\underline{P}$. Assume the polarization is "frozen" let's start with the sphere:

Making a spherical cavity is equivalent to superimposing a material with polarization $-\underline{P}$.

Let's calculate the electric field for a sphere w/ polarization $-\underline{P}$ uniform

This problem is equivalent to:
We have 2 spheres of charge: a positive & a negatively charged one. If we shift them slightly:

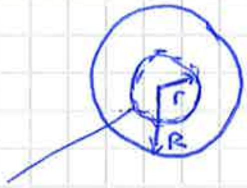


we need to calculate $\underline{E}$ in the region where the spheres overlap:

how do we do this?

superposition, let's first calculate it for one sphere:

we did this back in lecture 3 or 4



gaussian surface

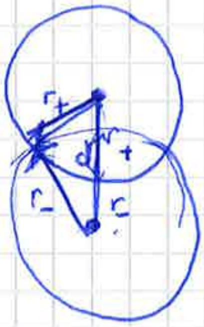
$$\oint \underline{E}\cdot d\underline{a} = E(4\pi r^2) = \frac{1}{\varepsilon_0} Q_{enc} = \frac{1}{\varepsilon_0}\frac{4}{3}\pi r^3 \rho$$

$$\Rightarrow \underline{E}_+ = \frac{1}{3\varepsilon_0}\rho r_+ \hat{r}$$ for the sphere w/ + charge

& it should be the same for a sphere of negative charge.

$$\underline{E}_- = \frac{-\rho}{3\varepsilon_0} r \hat{r}$$

Now if we overlap the spheres:



For 2 spheres:

$$\underline{E}_{tot} = \underline{E}_- + \underline{E}_+ = \frac{\rho}{3\varepsilon_0}(\underline{r}_+ - \underline{r}_-) = \frac{-\rho}{3\varepsilon_0}\underline{d} \quad (*)$$

and remembering $\rho = \frac{q}{V} = \frac{q}{\frac{4}{3}\pi R^3}$ $\Rightarrow \underline{E}_{tot} = \frac{-1}{4\pi\varepsilon_0}\frac{q\,\underline{d}}{R^3}$ (**)

So now let's get back to our overlapping spheres:

q charge of sphere
$\underline{d}$ distance between sphere centers
R radius of the sphere

Now all that we have left is express eq (*) in terms of the polarization of the sphere $\underline{P} = \frac{\underline{p}}{\text{volume}} \Rightarrow \left(\frac{4}{3}\pi R^3\right)\underline{P} = \underline{p} = q\underline{d}$

So subs this in eq (**) we have:

$$\underline{E} = -\frac{1}{3\varepsilon_0}\underline{P}$$ field for a sphere w/ polarization $\underline{P}$ $\left(\underline{E} = \frac{\underline{P}}{3\varepsilon_0} \text{ if pol is negative} \right)$

→ Now we can answer the original question. $\underline{E}$ inside will be ~~[illegible]~~ $\underline{E}_0$ ~~[illegible]~~ plus $\underline{E}$ due to the sphere

$$\Rightarrow \underline{E} = \underline{E}_0 + \frac{1}{3\varepsilon_0}\underline{P}$$ and at the center $\underline{P} = 0 \Rightarrow \underline{D} = \varepsilon_0\underline{E} = \varepsilon_0\underline{E}_0 + \frac{\varepsilon_0}{3\varepsilon_0}\underline{P} = \underline{D}_0 - \frac{2}{3}\underline{P}$ □

...electrics

A material is called a linear dielectric if the polarization is linearly proportional to $\underline{E}$

$$\underline{P} = \underbrace{\varepsilon_0 \chi_e}_{} \underline{E}$$

electric susceptibility dimentionless (that's why its multiplied by ε_0)

! So for a linear dielectric:

$$\underline{D} = \varepsilon_0 \underline{E} + \underline{P} = \varepsilon_0 \underline{E} + \varepsilon_0 \chi_e \underline{E} = \varepsilon_0 (1+\chi_e) \underline{E} = \varepsilon_0 \varepsilon_r \underline{E} = \varepsilon \underline{E}$$

$$\therefore \underline{D} = \varepsilon \underline{E}$$

dielectric constant (ε_r) — dielectric permittivity (ε)

If the material is isotropic + homogeneous, there will be a relation between free charge density + bound charge density:

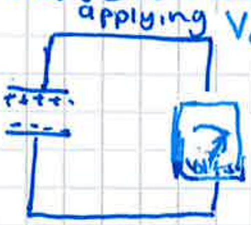
$$\rho_f = \nabla \cdot \underline{D} = \nabla \cdot \varepsilon \underline{E} = \varepsilon_0 \varepsilon_r \nabla \cdot \underline{E} = \varepsilon_0 \varepsilon_r \nabla \cdot \frac{\underline{P}}{\varepsilon_0 \chi_e} = \frac{\varepsilon_r}{\varepsilon_r - 1} \nabla \cdot \underline{P} = \frac{-\varepsilon_r}{\varepsilon_r - 1} \rho_b$$

$$\sigma_b = \underline{P} \cdot \hat{n} = (\underline{D} - \varepsilon_0 \underline{E}) \cdot \hat{n} = \left(\underline{D} - \frac{\underline{D}}{\varepsilon_r}\right) \cdot \hat{n} = \left(\frac{\varepsilon_r - 1}{\varepsilon_r}\right) \underline{D} \cdot \hat{n} = \left(\frac{\varepsilon_r - 1}{\varepsilon_r}\right) \sigma_f$$

Dielectrics without battery

What happens to an electric field due to the presence of a dielectric?

We have a capacitor that we first charge by applying V_0.

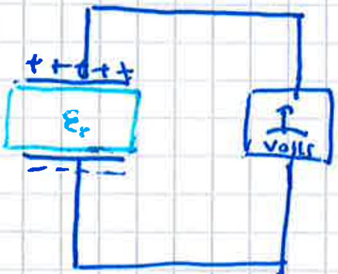


← NOT a battery just a voltmeter

The capacitor holds a charge $Q_0 = C_0 V_0$!

We disconnect the battery and leave Q_0 = constant.

We now put a dielectric inside the capacitor. What happens to $\underline{E}$? or V?



If we measure the potential while keeping the charge constant, we will see that it drops to:

$$V = \frac{V_0}{\varepsilon_r}$$

experimental fact.

This means that the capacitance changes in the presence of a dielectric, it is now:

$$C = \frac{Q}{V} = \frac{Q_0}{V_0/\varepsilon_r} = \frac{\varepsilon_r Q_0}{V_0} = \varepsilon_r C_0 \quad \therefore \boxed{C = \varepsilon_r C_0}$$

the capacitance is now increased by a factor ε_r

how do I know it is increasing and not decreasing? For it to decrease $\varepsilon_r < 1$ but by definition $\varepsilon_r = 1 + \chi_e$ so $\varepsilon_r > 1$ since $\chi_e > 0$

We saw what happens to the capacitance, what happens to the electric field?

$$E = \frac{V}{d} = \frac{V_0/\varepsilon_r}{d} = \frac{1}{\varepsilon_r}\left(\frac{V_0}{d}\right) = \frac{E_0}{\varepsilon_r} \quad \therefore \boxed{E = \frac{E_0}{\varepsilon_r}}$$

↑ for a // plate capacitor

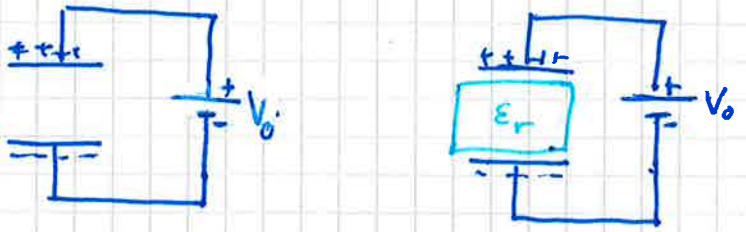
the field decreases by a factor ε_r

Dielectrics with battery

Let's consider the case now in which the battery used to charge the capacitor remains connected as the dielectric is inserted.
(i.e in this case we are holding the potential difference constant)

What happens to the charge?

Experimentally Faraday found: $Q = \varepsilon_r Q_0$ where Q_0 is the charge on the plates in the absence of dielectric.



so then the capacitance becomes:

$$C = \frac{Q}{V_0} = \frac{\varepsilon_r Q_0}{V_0} = \varepsilon_r C_0 \quad \therefore \boxed{C = \varepsilon_r C_0}$$

which is the same as we found if the charge is kept constant. But now we are keeping V constant and Q increases.

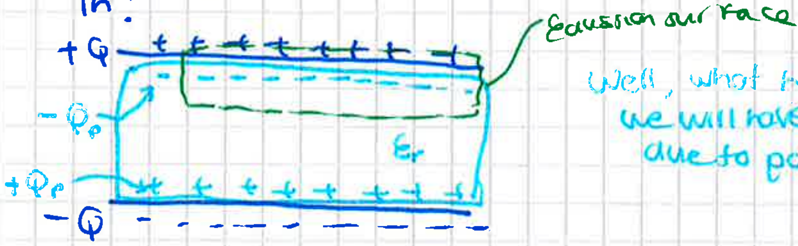
Electric field of a parallel plate capacitor

When no dielectric is present: we use Gauss's law to calculate the field and get:

$$\oint \underline{E} \cdot d\underline{a} = E_0 A = \frac{Q_{enc}}{\varepsilon_0} \Rightarrow E_0 = \frac{\sigma}{\varepsilon_0}$$

$$\sigma = \frac{Q}{A}$$

What happens now if we throw a dielectric in?



Well, what happens?
We will have some bound charges due to polarization.

We can use again Gauss's law

$$\oint \underline{E} \cdot d\underline{a} = EA = \frac{Q_{enc}}{\varepsilon_0} = \frac{Q - Q_p}{\varepsilon_0} \quad \therefore \quad E = \frac{Q - Q_p}{\varepsilon_0 A} \quad (*)$$

We just discussed that the effect of the dielectric is to 'weaken' the original field by a factor ε_r. So setting this equal to expression (*) we get:

$$E = \frac{E_0}{\varepsilon_r} = \frac{Q}{\varepsilon_r \varepsilon_0 A} = \frac{Q - Q_p}{\varepsilon_0 A} \quad (**)$$

so from here we can obtain the amount of bound charge Q_p

$$\Rightarrow Q_p = Q\left(1 - \frac{1}{\varepsilon_r}\right) \quad (1)$$

which we can write in terms of surface charge density as:

$$\sigma_p = \sigma\left(1 - \frac{1}{\varepsilon_r}\right)$$

in the limit $\varepsilon_r = 1$, $\sigma_p \to 0$ which corresponds to the case of no dielectric.

so now if we substitute the expression for Q_p ~~into~~ from (Δ) into (ΔΔ) we get:

$$\oint \underline{E} \cdot d\underline{a} = \frac{Q}{\varepsilon_r \varepsilon_0} = \frac{Q}{\varepsilon}$$

where $\varepsilon = \varepsilon_r \varepsilon_0$ the dielectric permittivity.

and for a linear dielectric
$\underline{D} = \varepsilon \underline{E}$

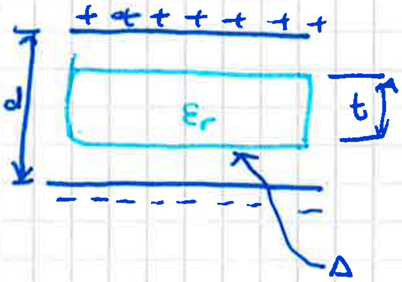
so the last equation becomes:

$$\boxed{\int \underline{D} \cdot d\underline{a} = Q \quad \text{we have arrived again at Gauss's law}}$$

EXAMPLE: Capacitance with dielectrics

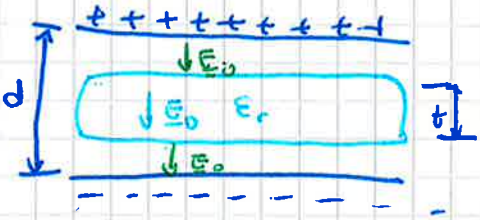
A dielectric of thickness t, area A and dielectric constant ε_r is inserted into the space between the plates of a parallel plate capacitor with spacing d, charge Q and area A.

What is the capacitance of the system?



When dealing w capacitors it is helpful to first calculate the potential difference.
We have already calculate that in the absence of dielectric, the field between plates is given by $E_0 = \frac{Q}{\varepsilon_0 A}$ (*) and when we add the dielectric: $E_D = \frac{E_0}{\varepsilon_r}$ (**)

If we draw the fields:



The potential can be found by integrating the electric field along a straight line from the top to the bottom of the plates:

potential ≠ between 2 faces of the dielectric

$$V = -\int_{+}^{-} \underline{E} \cdot d\underline{l} = -V_0 + V_D = -E_0 (d-t) - E_D t = \frac{Q}{A\varepsilon_0}(d-t) - \frac{Q}{A\varepsilon_0 \varepsilon_r} t$$

Subs the expressions * & **

$$= \frac{-Q}{A\varepsilon_0}\left[d - t\left(1 - \frac{1}{\varepsilon_r}\right)\right]$$

Now that we have the potential difference we can calculate the capacitance

$$C = \frac{Q}{V} = \frac{\varepsilon_0 A}{d - t\left(1 - \frac{1}{\varepsilon_r}\right)}$$

It is useful to check the following limits:

again:

$$C = \frac{Q}{V} = \frac{\varepsilon_0 A}{d - t\left(1 - \frac{1}{\varepsilon_r}\right)}$$

limits:

(i) As $t \to 0$, i.e the thickness of the dielectric goes to zero we get:

$$C = \frac{\varepsilon_0 A}{d} = C_0$$

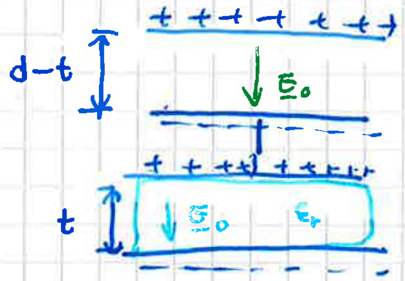
which is what we expect w/no dielectric

(ii) As $\varepsilon_r \to 1$ we again have $C \to \frac{\varepsilon_0 A}{d} = C_0$ which is again a situation that corresponds to the dielectric being absent.

(iii) In the limit $t \to d$, i.e the dielectric fills the entire space:

$$C \to \frac{\varepsilon_r \varepsilon_0 A}{d} = \varepsilon_r C_0$$

Another interesting note:
the configuration is equivalent to two capacitors connected in series!

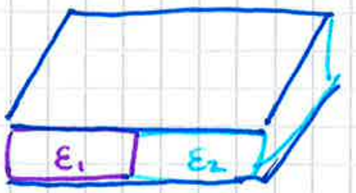


$$\frac{1}{C} = \frac{d-t}{\varepsilon_0 A} + \frac{t}{\varepsilon_r \varepsilon_0 A}$$

This is a good thing to remember when solving problems that involve capacitors and dielectrics.

EXAMPLE: capacitor filled with two dielectrics

Two dielectrics with dielectric constants ε_1 and ε_2 each fill half the space between the plates of a parallel plate capacitor:



each plate has an area A and the plates are separated by a distance d.

What is the capacitance of the system?

The potential difference in each half of the capacitor is the same
$\Rightarrow$ we can think of this system as 2 capacitors connected in parallel

For capacitors in //, total capacitance will be:

$$C = C_1 + C_2$$

so all we need to do is calculate C_1 & C_2

From the last discussion we just saw that:

$$\frac{1}{C} = \frac{t}{\varepsilon_r \varepsilon_0 A}$$ for a capacitor w/ a dielectric, where t is the thickness of the dielectric, A area

so then for this case we have

$$C_i = \frac{\varepsilon_i \varepsilon_0 (A/2)}{d}$$

because in our case each dielectric only covers half of the area and has a thickness d

so subs these expressions in (*) we get:

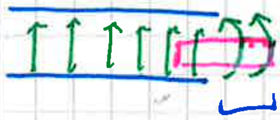
$$C = \frac{\varepsilon_1 \varepsilon_0 (A/2)}{d} + \frac{\varepsilon_2 \varepsilon_0 (A/2)}{d} = \frac{\varepsilon_0 A}{2d}(\varepsilon_1 + \varepsilon_2)$$

Forces on dielectrics

Demo. 429 Attraction of a dielectric into a capacitor

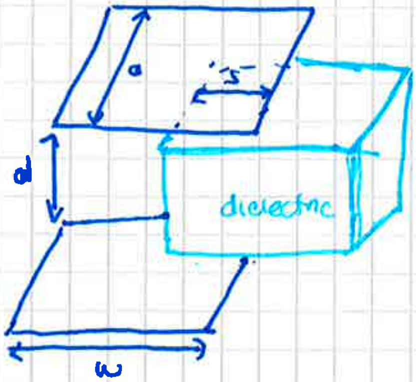
A dielectric slab placed partially between ~~places~~ the plates of a capacitor will be pulled into the capacitor.

This is a result of the ~~Ring~~ fringing fields around the edges of the capacitor



the electric field here cannot be zero. otherwise the line integral along a closed loop at the edge of the capacitor would not be zero

then we have the following situation:



A direct calculation of the F requires knowledge of the fringing fields of the capacitor, which are often not well known and difficult to calculate.

An alternative to calculate this F is to consider the change in energy of the system when the dielectric is displaced a distance ds.

The work to be done to move the dielectric an infinitesimal distance ds is:

$$dW = \underline{F}_{ext} \cdot d\underline{s}$$ where $\underline{F}_{ext}$ is the F provided by us to move the slab out of the capacitor.

so:

$$\underline{F}_{field} = -\underline{F}_{ext} = -\frac{dW}{ds} \quad (1)$$

let's consider the situation shown in the figure where the dielectric is a distance s inside the capacitor. The capacitance of the system is:

$$C = C_{vac} + C_{diel} = \frac{\varepsilon_0 (w-s) a}{d} + \frac{\varepsilon_r \varepsilon_0 s a}{d}$$

$$= \frac{\varepsilon_0 a}{d} (w - s + \varepsilon_r s)$$

$$= \frac{\varepsilon_0 a}{d} \left[w + (\varepsilon_r - 1) \right]$$

$$= \frac{\varepsilon_0 a}{d} \left[w + \chi_e s \right] \quad ☆$$

If the total charge on the top plate is Q we know that the energy stored in the capacitor is equal to

$$W = \frac{1}{2} \frac{Q^2}{C}$$

so subs this in equation (1) and considering the result of eq ☆ we get:

$$F_{field} = -\frac{dW}{ds} = \frac{1}{2} \frac{Q^2}{C^2} \frac{dC}{ds} \quad \text{and} \quad \frac{dC}{ds} = \frac{\varepsilon_0 \chi_e a}{d}$$

$$\therefore F_{field} = \frac{1}{2} \frac{Q^2}{C^2} \frac{\varepsilon_0 \chi_e a}{d} = \frac{1}{2} \frac{\varepsilon_0 \chi_e a}{d} V^2$$

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side note: Remember if you ever have questions about the expressions for $\underline{E}$, $\underline{P}$, etc.

Look at Griffiths figure 2.35