Final Exam

24.01.2024

The time available for the exam is 3 hours. No calculators, books or scripts are allowed, only one doublesided A4 handwritten paper with notes.
Use the space provided at each question and clearly mark your final answer.
Scrap paper is available at the end of the exam sheet.
SI units are implied throughout the exam.

NAME Solution

N°SCIPER 123456

$$\text{cartesian} \quad \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$\text{cylindrical} \quad \nabla = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{\partial}{\partial z}\right)$$

$$\text{spherical} \quad \nabla = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)$$

Problem 1

Consider a long, hollow cylinder of radius R_1 coaxial with a larger hollow cylinder of radius R_2. The inner cylinder is moving with a velocity v_0 along the direction of its axis ($\hat{z}$), whereas the outer cylinder is fixed. A viscous (η) and incompressible fluid with density ρ occupies both the space between the cylinders ($R_1 \leq r \leq R_2$) and inside the inner cylinder ($r < R_1$). No pressure gradient is applied.

In the following the flow can be considered fully established and steady, gravitational forces can be neglected, and the no-slip condition can be assumed to apply.

a) What is the pressure difference between two points separated by Δz and at the same radius $R_1 < r < R_2$? (1 point)

only $v_z(r)$. Incompressible $\rightarrow v(z_1) = v(z_2) \rightarrow$ Bernoulli $\Delta p = 0$ (No dP/dz applied).

b) Determine the velocity field of the flow between the cylinders ($R_1 \leq r \leq R_2$). (3 points)

Navier Stokes (steady, incomp. only $V_z(r)$, $\frac{dP}{dz} = 0$, ignore gravity)

$$\eta \nabla^2 \vec{v} = 0 \xrightarrow{V_z(r)} \eta \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial V_z}{\partial r}\right) = \eta\left(\frac{\partial^2 V_z}{\partial r^2} + \frac{1}{r}\frac{\partial V_z}{\partial r}\right) = 0$$

$\Rightarrow V_z(r) = C_1 \ln(r) + C_2$ $\quad r = R_1 \rightarrow V_z = V_0$ $\quad r = R_2 \rightarrow V = 0$

$C_1 \ln R_1 + C_2 = V_0$ and $C_1 \ln R_2 + C_2 = 0$ $\quad C_2 = -C_1 \ln R_2$

$$\Rightarrow C_1 = \frac{V_0}{\ln \frac{R_1}{R_2}} \quad \text{and} \quad C_2 = \frac{-V_0 \ln R_2}{\ln \frac{R_1}{R_2}}$$

$$V_z(r) = \frac{V_0}{\ln \frac{R_1}{R_2}} \ln \frac{r}{R_2}$$

c) Determine the velocity field for $r < R_1$. (2 points)

$V_z(r) = C_1 \ln(r) + C_2$ $\quad r = 0 :\ V_z$ finite $\rightarrow C_1 = 0$

$r = R_1 \quad V_z = V_0 \rightarrow C_2 = V_0$

$\Rightarrow V_z(r) = V_0$

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d) Find the vorticity Ω of the flow for $0 \leq r \leq R_2$. (3 points)

FOR $r < R_1$ $\quad \Omega = 0$

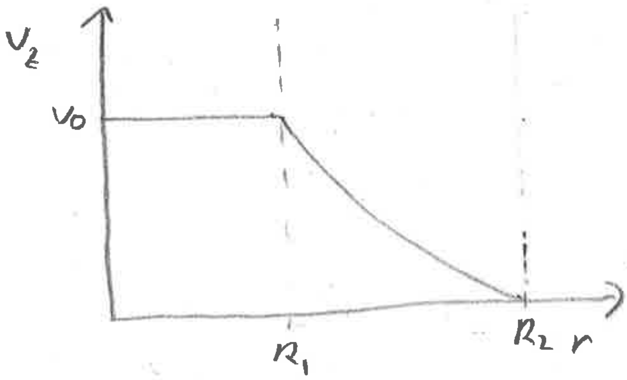
FOR $R_1 < r < R_2$ $\quad \Omega = \nabla \times \left(\frac{V_0}{\ln \frac{R_1}{R_2}} \ln \frac{r}{R_2} \right) = \nabla \times C \ln \frac{r}{R_2}$

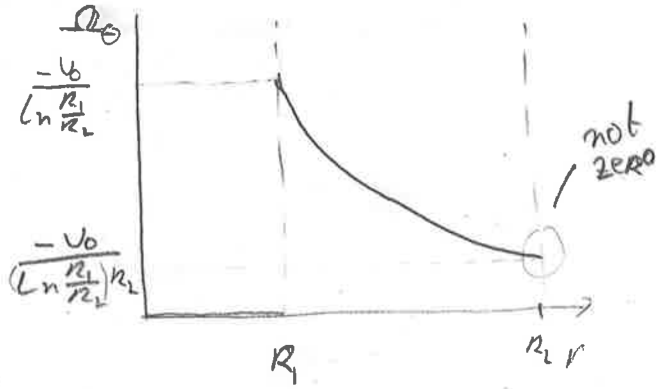
$$\nabla \times \begin{bmatrix} 0 \\ 0 \\ v_z(r) \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{\partial v_z}{\partial r} \\ 0 \end{bmatrix} \rightarrow -\frac{\partial}{\partial r}\left(C \ln \frac{r}{R_2} \right) = \frac{-C}{r}$$

$$\Omega(r) = \frac{-V_0}{\left(\ln \frac{R_1}{R_2}\right) r} \hat{\theta}$$

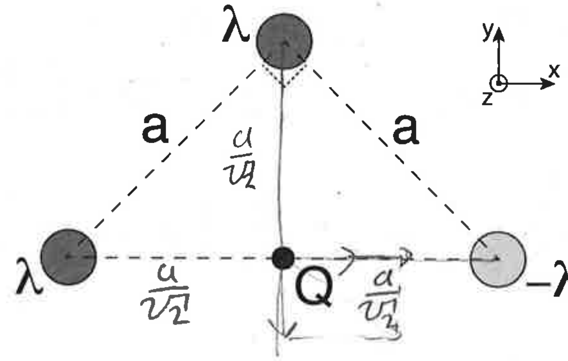
values not necessary

e) Qualitatively plot $v(r)$ and $\Omega(r)$ for $0 \leq r \leq R_2$. (2 points)





Problem 2



a) Three line charges λ and a point charge Q are placed as indicated in the figure. Determine the force $\vec{F}$ on the point charge (2 points).

Line charge $\quad \vec{E} = \frac{\lambda \hat{r}}{2\pi\varepsilon_0 r} \qquad \vec{F} = \frac{\lambda Q \hat{r}}{2\pi\varepsilon_0 \frac{a}{\sqrt{2}}} = \frac{\lambda Q \hat{r}}{\sqrt{2}\pi\varepsilon_0 a}$

$F_x = \frac{2\lambda Q \hat{x}}{\sqrt{2}\pi\varepsilon_0 a} = \frac{\sqrt{2}\lambda Q \hat{x}}{\pi\varepsilon_0 a} \qquad F_y = \frac{-\lambda Q \hat{y}}{\sqrt{2}\pi\varepsilon_0 a}$

b) A rectangular box has two opposite faces of conducting material so that it forms a parallel plate capacitor. The box is filled for one-third with a dielectric liquid with relative permittivity $\varepsilon_r = 4$. What is the ratio of the capacitance when the conducting faces are horizontal compared to when the box is rotated to have them vertical? (Ignore edge effects) (2 points)

$C = \varepsilon_0 \varepsilon_r \frac{A}{d} \qquad \varepsilon_r = 4 \qquad$ (leave out ε_0)

Horizontal : C in series : $\frac{1}{C_H} = \frac{1}{4\frac{A}{d/3}} + \frac{1}{\frac{A}{2d/3}} = \frac{1}{\frac{4}{3}\frac{A}{d}} \rightarrow C_H = \frac{4}{3}\frac{A}{d}$

Vertical : C parallel : $C_V = 4\frac{A}{3d} + \frac{2A}{3d} = \frac{6}{3}\frac{A}{d}$

Ratio $C_H : C_V = 6 : 4 = \underline{3 : 2}$

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Problem 3

A hollow conducting sphere with radius R is placed in a uniform electric field $\vec{E}_0 = E_0\hat{x}$

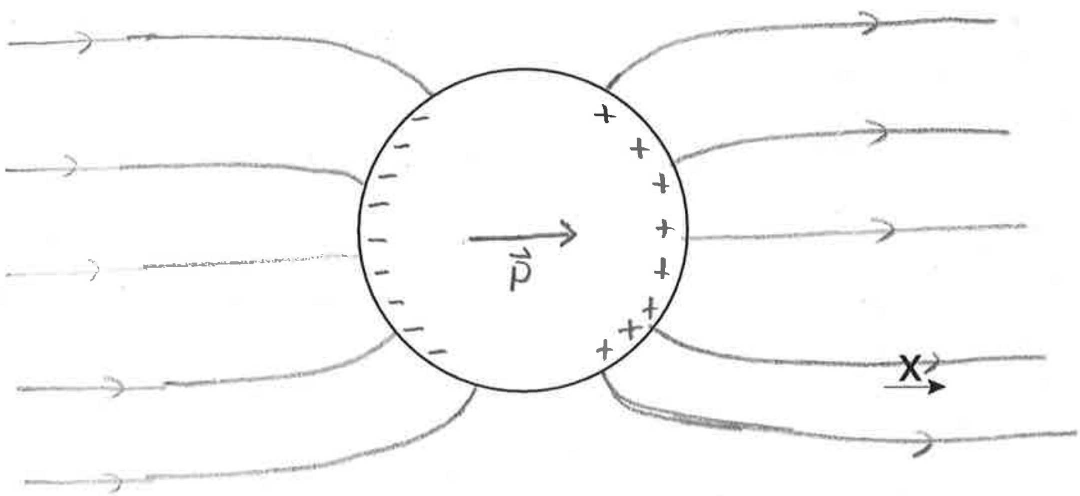
a) What is the electric field inside the sphere? (1 point)

$E = 0$

b) Using the figure below, make a sketch of the situation, paying special attention to three points : 1) The E-field lines close to the surface of the sphere. 2) The induced surface charge distribution on the sphere. 3) Where to place an electric dipole $\vec{p}$ to create the same E-field (outside the sphere) as the one created by these charges. (3 points)

$E \perp$ sphere

$\vec{p}$ in centre to right



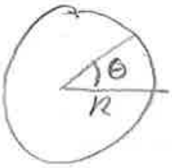
c) From the combination of $\vec{E}_0$ and the dipole field, and taking into account the boundary conditions for this total E-field at the surface of the sphere, determine the magnitude of the electric dipole p. (3 points)

Components of $\vec{E}_0$ and dipole field tangential to sphere cancel

$$E_0 \sin\theta = \frac{p \sin\theta}{4\pi\varepsilon_0 R^3} \longrightarrow p = 4\pi\varepsilon_0 R^3 E_0$$

d) Find an expression for the total E-field just outside the sphere ($E(\theta)$) and use this to calculate the surface charge distribution on the sphere $\sigma(\theta)$. (3 points)

only perpendicular (radial) component



$$E_{tot}(\theta) = E_0 \cos\theta + \frac{2p\cos\theta}{4\pi\varepsilon_0 R^3}$$ with $p = 4\pi\varepsilon_0 R^3 E_0$

$$\rightarrow E_{tot}(\theta) = E_0 \cos\theta + \frac{2\cdot 4\pi\varepsilon_0 R^3 E_0 \cos\theta}{4\pi\varepsilon_0 R^3} = 3 E_0 \cos\theta$$

$$\sigma(\theta) = \varepsilon_0 E_{tot}(\theta) = 3\varepsilon_0 E_0 \cos\theta$$

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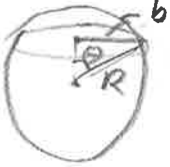
Problem 4
Consider a spherical shell with radius R and uniform surface charge density σ. The spherical shell is rotating around its z-axis with angular velocity ω.

a) Calculate the magnetic field $\vec{B}$ at the centre of the rotating spherical shell. (3 points)
Hint : $\int_0^\pi \sin^3\theta d\theta = \frac{4}{3}$

from lecture



$b = R\sin\theta$

$$dB = \frac{\mu_0 I b^2}{2R^3} = \frac{\mu_0}{2}\frac{dq}{dt}\frac{R^2\sin^2\theta}{R^3}$$

$$dq = \sigma dA = \sigma(2\pi b\, r\, d\theta) = \sigma(2\pi R\sin\theta\, R\, d\theta)$$

$$dt = \frac{2\pi}{\omega}$$

$$dB = \frac{\mu_0}{2}\frac{\sigma 2\pi R\sin\theta\, R\, d\theta}{\frac{2\pi}{\omega}}\frac{\sin^2\theta}{R} = \frac{\mu_0\sigma\omega}{2}R\sin^3\theta\, d\theta$$

$$B = \int_0^\pi dB = \frac{\mu_0\sigma\omega}{2}R\int_0^\pi \sin^3\theta\, d\theta = \frac{2}{3}\mu_0\sigma\omega R$$

Direction (right hand rule): $+\hat{z}$

$$\vec{B} = \frac{2}{3}\mu_0\sigma\omega R\,\hat{z}$$

b) Determine the total magnetic moment $\vec{m}$ of the rotating spherical shell. (3 points)

$$d\vec{m} = I\, A\,\hat{z} \qquad A = \pi b^2 = \pi R^2\sin^2\theta$$

$$I = \frac{dq}{dt} = \omega\sigma R^2\sin\theta\, d\theta$$

$$d\vec{m} = \pi\omega\sigma R^4\sin^3\theta\, d\theta$$

$$\vec{m} = \int_0^\pi d\vec{m} = \pi\omega\sigma R^4\int_0^\pi \sin^3\theta\, d\theta = \frac{4}{3}\pi\omega\sigma R^4\,\hat{z}$$

c) We now want to replace the rotating shell by a (stationary) solid sphere made of a ferromagnetic material with uniform magnetisation $\vec{M}$. For what value of $\vec{M}$ will the two spheres produce the same B-field for $r > R$?
The final value should be in terms of values given in the problem (R, σ, and ω), but if you did not manage to solve (b) you can express it in terms of $\vec{m}$. (2 points)

$\vec{m} = \iiint \vec{M}\, d\tau$ uniform $\vec{M} \rightarrow \vec{m} = \vec{M} \iiint d\tau = \frac{4}{3}\pi R^3 \vec{M}$

$$\vec{M} = \frac{\vec{m}}{\frac{4}{3}\pi R^3} = \frac{\frac{4}{3}\pi R^3 \omega \sigma R \hat{z}}{\frac{4}{3}\pi R^3} = \omega \sigma R \hat{z}$$

(value asked $M = \omega \sigma R$)

Problem 5

A spring with spring constant κ and equilibrium length S has a total of N windings with area A. The left side of the spring is fixed, whereas the right side can move freely along the x-axis. The spring is connected to flexible wires that allow to pass a current through it.

a) Determine the (self-)inductance of the spring. (1 point)

$B = \mu_0 n I = \mu_0 \frac{N}{S} I$ $\varphi = NAB = \mu_0 \frac{N^2}{S} A I$

$L = \frac{\varphi}{I} = \mu_0 \frac{N^2}{S} A$ (ok)

Better: replace S with $S + x$

$$L = \mu_0 \frac{N^2}{S+x} A$$

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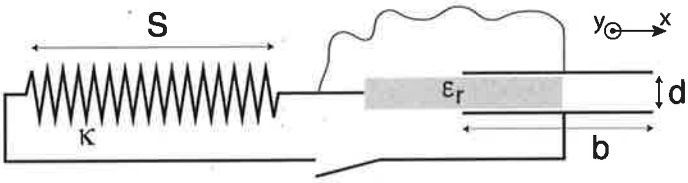
b) What is the total force on the right side of the spring as a function of current I passing through it ? (2 points)
Hint : set $x = 0$ at the right side for $I = 0$.

$$U_m = \tfrac{1}{2} L I^2 = \tfrac{1}{2} \mu_0 \frac{N^2}{S+x} A I^2$$

$$F_L = \nabla U_m = \frac{\partial}{\partial x} U_m = -\tfrac{1}{2} \mu_0 \frac{N^2}{(S+x)^2} A I^2 \qquad F_s = -kx$$

$$F_x = -kx - \tfrac{1}{2} \mu_0 \frac{N^2}{(S+x)^2} A I^2$$

c) As illustrated in the figure, a dielectric slab with ε_r and thickness $\lesssim d$ is attached to the spring. This slab is placed between the plates of a capacitor with plate dimensions a along the y-direction and b along the x-direction, and distance d between the plates. When there is no charge on the capacitor, the dielectric fills the capacitor exactly half way up to $b/2$. What part of the dielectric will be inside the capacitor if the latter is charged to Φ_0 ? Assume that no current is running through the spring. (2 points)



$$C_c = \varepsilon_0 \frac{a(\frac{1}{2}b - x)}{d} \qquad C_r = \varepsilon_r \varepsilon_0 \frac{a(\frac{1}{2}b + x)}{d}$$

$$C = C_c + C_r = \varepsilon_0 \frac{a}{d}\left(\left(\frac{b}{2} - x\right) + \varepsilon_r\left(\frac{b}{2} + x\right)\right) = \varepsilon_0 \frac{a}{d}\left(\frac{b}{2}(\varepsilon_r + 1) + x(\varepsilon_r - 1)\right)$$

$$\frac{\partial C}{\partial x} = \varepsilon_0 \frac{a}{d}(\varepsilon_r - 1) \implies F_x = \tfrac{1}{2} \Phi^2 \frac{\partial C}{\partial x} = \tfrac{1}{2} \Phi^2 \varepsilon_0 \frac{a}{d}(\varepsilon_r - 1)$$

$$F_s = -kx \to F_{tot} = 0 \to kx = \tfrac{1}{2} \Phi^2 \varepsilon_0 \frac{a}{d}(\varepsilon_r - 1) \to x = \frac{\Phi^2 \varepsilon_0 a (\varepsilon_r - 1)}{2kd}$$

part inside:

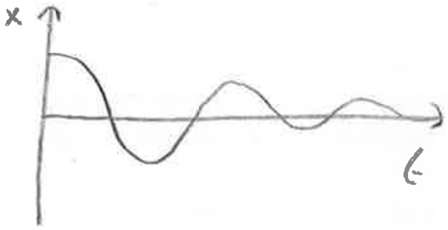
$$\frac{b}{2} + x = \frac{b}{2} + \frac{\Phi^2 \varepsilon_0 a(\varepsilon_r - 1)}{2kd}$$

(Final questions on next page)

d) The charged capacitor is connected in series to the spring via wires with total resistance R. At $t = 0$ a switch is closed to connect the capacitor with the spring. Describe qualitatively what will happen when the switch is closed and sketch (no exact values) what part of the dielectric is inside the capacitor as a function of time for $t \geq 0$. (2 points)

The capacitor will discharge to the spring. The force to pull the slab inside will reduce and the spring will pull harder due to the current. Then the process inverses. Thus an oscillation damped by R.

(Forced damped oscillation)



e) What is the total impedance of the circuit? (2 points)

LCR in series: $Z = R + j(\omega L - 1/\omega C)$

$L = \mu_0 \frac{N^2}{S+x} A$ $\quad C = \varepsilon_0 \frac{a}{d}\left(\frac{b}{2}(\varepsilon_r + 1) + x(\varepsilon_r - 1)\right)$

$$Z = R + j\left(\frac{\mu_0 \omega N^2 A}{S+x} - 1/\omega\varepsilon_0 \frac{a}{d}\left(\frac{b}{2}(\varepsilon_r+1) + x(\varepsilon_r - 1)\right)\right)$$