Final Exam

24.01.2023

The time available for the exam is 3 hours. No calculators, books or scripts are allowed, only one doublesided A4 handwritten paper with notes.
Use the space provided at each question and clearly mark your final answer.
Scrap paper is available at the end of the exam sheet.
SI units are implied throughout the exam.

NAME S. Olution

N°SCIPER 123456

P1 : 12
P2 : 10
P3 : 6
P4 : 2
P5 : 10

40

cartesian $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

cylindrical $\nabla = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{\partial}{\partial z}\right)$

spherical $\nabla = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)$

Problem 1

A tornado can be modelled as a steady air flow with a vertical (along $\hat{z}$) axis of revolution and a velocity field orthogonal to this axis, invariant by rotation and translation along z. Outside a vertical cylinder centred at $r = 0$ and with radius a, the flow is irrotational; inside this cylinder the vorticity is uniform and given by $\vec{\Omega} = \Omega_0 \hat{z}$.

Air can be assumed incompressible and non-viscous and the transition in vorticity can be considered instantaneous. For $r \to \infty$ the pressure $p = p_0$ at ground level.

a) Taking into account the symmetry of the model, indicate which vector components of the velocity field are non-zero ($\neq 0$) and on which coordinates they depend. (1 point)

$\vec{v} = v(r)\,\hat{\theta} \qquad (v_r = v_z = 0)$

b) Determine the expression for $v(r)$. (3 points)

$$\nabla \times \vec{v} = \Omega \rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\,v(r)\right) = \Omega(r) \begin{cases} 0 & r > a \\ \Omega_0 & r < a \end{cases}$$

$r < a$: $r v(r) = \frac{1}{2}\Omega_0 r^2 + C_1 \rightarrow v(r) = \frac{1}{2}\Omega_0 r + \frac{C_1}{r}$ Finite for $r = 0$ $\rightarrow C_1 = 0$

$r > a$: $r v(r) = C_2 \rightarrow v(r) = \frac{C_2}{r}$

$r = a \rightarrow v(a) = \frac{1}{2}\Omega_0 a = \frac{C_2}{a} \rightarrow C_2 = \frac{1}{2}\Omega_0 a^2$

$v(r) = \frac{\Omega_0 a^2}{2r} \qquad r > a$

$v(r) = \frac{1}{2}\Omega_0 r \qquad r < a$

c) Find the expression of the pressure at ground level ($z = 0$) at any point **outside** the cylinder of radius a. (2 points)

Irrotational $\rightarrow$ Bernoulli valid for any 2 points

$r = r$ and $r \rightarrow \infty$

$\hookrightarrow v = 0 \quad p = p_0$

$$\frac{p(r)}{\rho} + \frac{1}{2}v^2(r) = \frac{p_0}{\rho}$$

$$p(r) = p_0 - \frac{1}{8}\rho\,\frac{a^4 \Omega_0^2}{r^2}$$

d) Find the expression of the pressure at ground level ($z = 0$) at any point **inside** the cylinder of radius a. (4 points)

$$\Omega \times \vec{v} + \frac{1}{2} \nabla v^2 = \frac{-\nabla p}{\rho}$$ only dependency on r

$$\rightarrow \underbrace{-\Omega_0 v + \frac{1}{2} \frac{\partial}{\partial r} v^2}_{} = -\frac{\partial p}{\partial r} \frac{1}{\rho}$$

$$\hookrightarrow -\frac{1}{2}\Omega_0^2 r + \frac{1}{2} \frac{\partial}{\partial r}\left(\frac{1}{4} \Omega_0^2 r^2\right) = -\frac{1}{2} \Omega_0^2 r + \frac{1}{4} \Omega_0^2 r = -\frac{1}{4} \Omega_0^2 r$$

$$\frac{\partial p}{\partial r} = \frac{\rho \Omega_0^2}{4} r \quad \rightarrow \quad p(r) = \frac{1}{8} \rho \Omega_0^2 r^2 + c$$

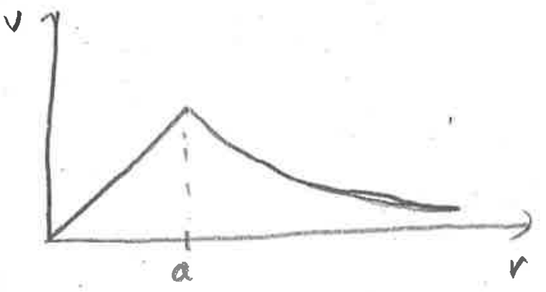
FROM (c)

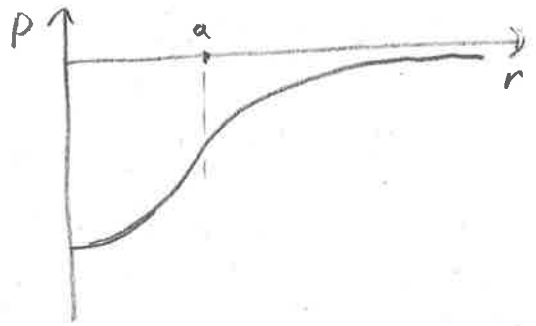
continuity at $r = a$: $p(a) = \frac{1}{8} \rho \Omega_0^2 a^2 + c = P_0 - \frac{1}{8} \rho \Omega_0^2 a^2$

$$\rightarrow c = P_0 - \frac{1}{4} \rho \Omega_0^2 a^2$$

$$p(r) = P_0 - \frac{1}{4} \rho \Omega_0^2 a^2 + \frac{1}{8} \rho \Omega_0^2 r^2$$

e) Qualitatively plot $v(r)$ and $p(r)$ (at $z = 0$) for $0 < r < 4a$. (2 points)

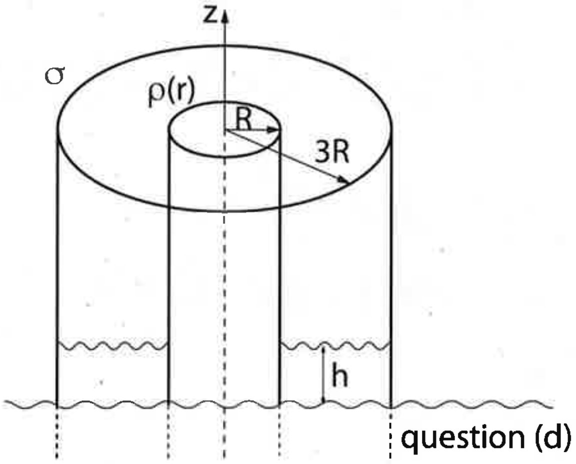




(general trend)

Problem 2

Consider a cylindrical wire with radius R and a thin outer cylindrical shell with radius $3R$ as illustrated in the figure. The wire has a (negative) volume charge density $\rho(r) = -3r^2$ and the shell a surface charge density σ. Ignore effects due to the end of the wire and shell.



a) Determine the electric field inside the wire ($r < R$). (2 points)

Gauss   flux: $\Phi_E = 2\pi r \ell E$

enclosed $Q = \ell \int_0^r \rho(r)\, 2\pi r\, dr = -\ell \int_0^r 6 r^3 \pi\, dr = -\frac{3\pi}{2} r^4 \ell$

$\Rightarrow 2\pi r E = -\frac{3\pi}{2\varepsilon_0} r^4$ $\vec{E} = -\frac{3}{4\varepsilon_0} r^3 \hat{r}$

b) Find the value for σ to ensure that $\vec{E} = 0$ for $r > 3R$. (1 point)

$\int \sigma = -\int \rho$ $6\pi R \ell \sigma = +\frac{3\pi}{2} R^4 \ell$

$\sigma = \frac{R^3}{4}$

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c) An electric dipole $\vec{p} = p\hat{r}$ is placed between the wire and the shell ($R < r < 3R$). Determine the force on the dipole. (3 points)

$$\vec{E} = \frac{-3R^4\,\hat{r}}{4\varepsilon_0\, r}$$

$$\vec{F} = (\vec{p}\cdot\nabla)\vec{E} = p_r \frac{\partial}{\partial r}\vec{E} = +\frac{3pR^4\hat{r}}{4\varepsilon_0\, r^2}$$

d) The wire and shell (without the dipole) are partially vertically immersed in a tank of dielectric liquid (susceptibility χ_e and mass density ρ_m). A potential difference Φ is maintained between the wire and shell. To what height h, with respect to the level in the tank, does the oil rise in the space between the wire and shell? (Ignore hydrostatic and hydrodynamic (capillary) effects.) (4 points)

$$C = \frac{2\pi\varepsilon_0 L}{\ln\left(\frac{3R}{R}\right)} = \frac{2\pi}{\ln 3}\varepsilon_0 L$$

empty : $C_e = \frac{2\pi}{\ln 3}\varepsilon_0 (L-h)$

Filled : $C_r = \frac{2\pi}{\ln 3}\varepsilon_0\varepsilon_r h = \frac{2\pi}{\ln 3}\varepsilon_0(1+\chi_e)h$

parallel : $C = C_e + C_r = \frac{2\pi\varepsilon_0}{\ln 3}\left(L-h+(1+\chi_e)h\right) = \frac{2\pi\varepsilon_0}{\ln 3}(L+\chi_e h)$

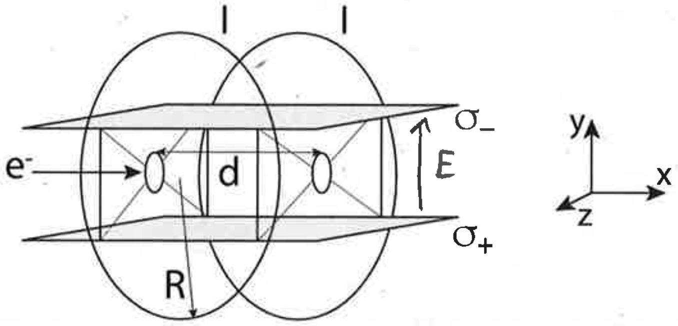
$$F = \tfrac{1}{2}\Phi^2\frac{\partial C}{\partial h} = \Phi^2\frac{\pi\varepsilon_0\chi_e}{\ln 3}$$

$$h = \frac{\varepsilon_0\chi_e\Phi^2}{\ln 3\,\rho_m\, 8R^2 g}$$

gravity : $F = mg = \rho_m \pi 8R^2 g h$

Problem 3

Consider the device sketched in the figure below. The two plates (xz-plane) are charged with an equal and opposite surface charge density $\sigma_\pm$ and the two single winding coils (xy-plane) with radius R are placed in a Helmholtz configuration. A collimated (only $v_x \neq 0$) beam of electrons, with large velocity spread, enters from the left through a narrow opening and can exit again through a similar opening after a distance d. The electrons pass through the centre of the device and over this distance all fields can be considered homogeneous. Relativistic effects can be ignored.



a) What current, and in which direction, should be sent through the coils to let only electrons with kinetic energy E_p exit through opening on the right? (3 points)

Helmholtz : distance coil → centre $= \frac{R}{2}$

$E_y = \frac{\sigma}{\varepsilon_0}\hat{y}$ need B along $\hat{z}$

$$B_z = \frac{\mu_0 I R^2}{\left(R^2 + \frac{R^2}{4}\right)^{3/2}} = \frac{\mu_0 I R^2}{\left(\frac{5}{4}R^2\right)^{3/2}} = \frac{\mu_0 I R^2}{\left(\frac{5}{4}\right)^{3/2} R^3} = \frac{\mu_0 I}{\left(\frac{5}{4}\right)^{3/2} R}$$

$$E_p = \tfrac{1}{2} m_e v_x^2 \rightarrow v_x = \sqrt{\frac{2E_p}{m_e}} = \frac{E_y}{B_z}$$

$$\frac{\mu_0 I}{\left(\frac{5}{4}\right)^{3/2} R} = \frac{\sigma}{\varepsilon_0}\sqrt{\frac{m_e}{2E_p}} \rightarrow I = \frac{\left(\frac{5}{4}\right)^{3/2}\sigma R}{\varepsilon_0 \mu_0}\sqrt{\frac{m_e}{2E_p}}$$

counter clock wise

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b) When entering the device, the spin of all electrons points along the $\hat{x}$ direction ($\theta = 0, \phi = 0$). Over what angle will the spin of the electrons from (a) have rotated when exiting the device? Take θ in the xy-plane and ϕ in the xz-plane. (3 points)

$$B_z = \frac{\sigma}{\varepsilon_0 v_x} \qquad \omega_L = -\gamma B = \frac{e\sigma}{m_e \varepsilon_0 v_x} \qquad \text{in } xy\text{-plane}$$

$$dt = \frac{d}{v_x} \rightarrow \theta = \omega_L dt = \frac{e\sigma d}{m_e \varepsilon_0 v_x^2} \qquad v_x^2 = \frac{2E_p}{m_e}$$

$$\theta = \frac{e\sigma d}{2E_p \varepsilon_0} \qquad (\varphi = 0)$$

In terms of I

$$\theta = \frac{e\mu_0 I d\sqrt{2E_p}}{\left(\frac{5}{4} m_e\right)^{3/2} R}$$

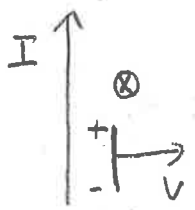
Problem 4

A current $I(t) = a_0 t$ is flowing through a long wire along the positive z-direction. A short wire with length L is placed parallel to this wire such that they initially overlap, but don't touch. At $t = 0$ the short wire starts moving away from the long wire with $\vec{v} = v_0\hat{r}$. Determine the potential at the bottom (z) of the small wire with respect to its top ($z + L$). (2 points)

$$d\varphi = \vec{B} \cdot d\vec{A} = L v_0 B\, dt \qquad B = \frac{\mu_0 a_0 t}{2\pi r} \qquad r = v_0 t$$

$$d\varphi = L\frac{v_0 \mu_0 a_0 t\, dt}{2\pi v_0 t} = \frac{L\mu_0 a_0}{2\pi} dt$$

$$\Delta\Phi = -\frac{d\varphi}{dt} = -\frac{L\mu_0 a_0}{2\pi}$$



Potential at bottom lower

Problem 5

A particle with rest mass m_0 and charge q is released at the origin $(x, y, z, t) = (0, 0, 0, 0)$ of reference system S in the presence of extended and uniform fields $\vec{E} = E_0\hat{y}$ and $\vec{B} = B_0\hat{z}$, with $E_0 < cB_0$.

a) Find the inertial system S' in which $\vec{E}' = 0$. (2 points)

$E_x' = E_x \quad E_y' = \gamma(E_y - vB_z) \quad E_z' = \gamma(E_z + vB_y) \quad \vec{E} = (0, E_0, 0)$

$B_x' = B_x \quad B_y' = \gamma(B_y + \frac{v}{c^2}E_z) \quad B_z' = \gamma(B_z - \frac{v}{c^2}E_y) \quad \vec{B} = (0, 0, B_0)$

$E_x' = 0, \; E_z' = 0 \quad E_y' = \gamma(E_0 - vB_0) \rightarrow E_0 - vB_0 = 0 \rightarrow v = \frac{E_0}{B_0}$

S' moves with $\vec{v} = \frac{E_0}{B_0}\hat{x}$ w.r.t S

b) Determine $\vec{B}'$ in system S'. (1 point)

$B_x' = 0 \quad B_y' = 0 \quad B_z' = \gamma(B_0 - \frac{v}{c^2}E_0) = \gamma B_0(1 - \frac{v^2}{c^2}) = \gamma\frac{B_0}{\gamma^2} = \frac{B_0}{\gamma}$

$\vec{B}' = \frac{B_0}{\gamma}\hat{z}$

c) Determine the trajectory of the particle in the moving system S'. (Hint: Similar to classical mechanics the centripetal force is $F_c = p\frac{v}{r}$, but with p the relativistic momentum.) (3 points)

In S' particle with initial velocity $-v\hat{x}$

only $B' \longrightarrow$ circle with radius $r = \frac{mv}{qB} = \frac{\gamma m_0 v}{qB}$

$r = \frac{\gamma^2 m_0 E_0}{qB_0^2}$ in $x'y'$-plane

$\Rightarrow x' = -r\sin\omega t' \quad y' = r(1 - \cos\omega t') \quad z' = 0$

$\omega = \frac{v}{r} = \frac{qB_0}{\gamma^2 m_0}$

ok if this is only given in d)

(Final question on next page)

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d) Determine the trajectory of the particle in the original reference system S. (4 points)

Lorentz transformation of coordinates

$x' = \gamma(x - vt) \qquad y' = y \qquad z' = z \qquad t' = \gamma\left(t - \frac{v}{c^2}x\right)$

$\gamma(x - vt) = -r\sin\left[\omega\left(\gamma\left(t - \frac{v}{c^2}x\right)\right)\right]$

$x = vt - \frac{r}{\gamma}\sin\left(\omega\gamma\left(t - \frac{v}{c^2}x\right)\right)$

$y = r\left[1 - \cos\left(\omega\gamma\left(t - \frac{v}{c^2}x\right)\right)\right]$

$z = 0$

(This is a stretched cycloid with

$(\gamma(x - vt))^2 + (y - r)^2 = r^2$)

[In grading, give points if possible. Maybe some will start from the given $\vec{E}$ and $\vec{B}$]