Solution Sheet 7

Discussion 30.10.2024

Discussion 1 - Gauss's law

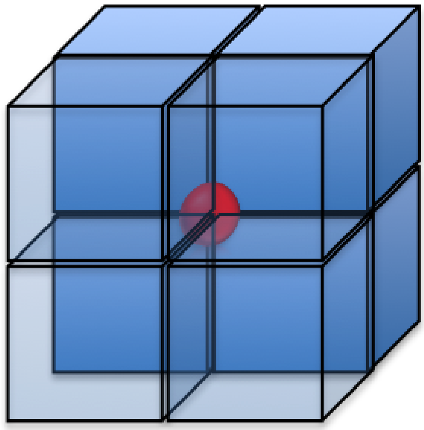
a) If the charge is in the centre of the regular tetrahedron, the value of the electric field flux through each face φ_{Te} is the same.
Using Gauss's law we obtain:

$$4\varphi_{Te} = \frac{Q}{\epsilon_0} \Rightarrow \varphi_{Te} = \frac{Q}{4\epsilon_0}$$

b) If the charge is at the corner of a cube, three of the faces of the cube have zero flux because the surface normal is perpendicular to the E-field and the effective areas are thus 0. The other three faces each have the same magnitude of electric flux φ_C. Moreover we know that 8 cubes make a big cube which encloses the charge in the centre (see figure below). Thus,

$$3\varphi_C \times 8 = \frac{Q}{\epsilon_0} \Rightarrow \varphi_C = \frac{Q}{24\epsilon_0}$$

One can also argue that the flux through each of the 6 faces of the large cube is the same, and that 4 faces of a small cube make up a face of the large cube, yielding also the factor 24.



Solution 1 - Electron Beam

The infinite length cylinder has translational symmetry along the longitudinal direction. We thus use the same geometry for the Gaussian surface as explained in the lecture; a cylinder with length l and radius r. We only need to consider the electrons at positions $\leq r$ because only these are enclosed by our Gaussian surface. The flux through this surface becomes:

$$\varphi = \oiint \vec{E}(r) \cdot d\vec{S} = \frac{Q(r)}{\epsilon_0} \tag{1}$$

The flux through the two disk-shaped sides is 0, since the E-field is orthogonal to the surface normal vector.

$$\begin{aligned} \implies 2\pi r l E(r) &= \frac{Q(r)}{\epsilon_0} && (2) \\ \implies E(r) &= \frac{Q(r)}{2\pi\epsilon_0 r l} && (3) \end{aligned}$$

The total charge enclosed by the Gaussian cylinder is:

$$Q(r) = \begin{cases} \lambda \frac{r^2}{a^2} l & , \quad r \leq a \\ \lambda l & , \quad r > a \end{cases} \tag{4}$$

hence

$$E(r) = \begin{cases} \frac{\lambda r}{2\pi\epsilon_0 a^2} & , \quad r \leq a \\ \frac{\lambda}{2\pi\epsilon_0 r} & , \quad r > a \end{cases} \tag{5}$$

The direction of the electric field points toward the center.

Solution 2 - Charged slab

On the xz plane the $E = 0$ by symmetry; also by symmetry we can affirm that $\vec{E}$ is along y. Set up a gaussian surface consisting of a rectangular box with one face on the xz plane (i.e at the center of the slab in the y direction) and the other at y. The faces parallel to the slab have area A.
Gauss' law:

$$\oiint \vec{E} \cdot d\vec{S} = \frac{Q_{inside}}{\epsilon_0}$$

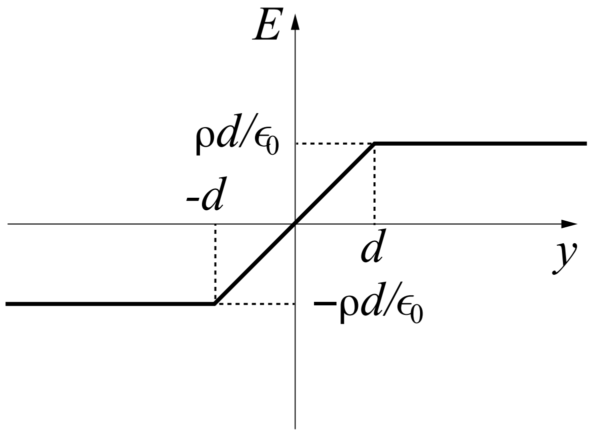
Applied to our situation:

$$E\,A = \frac{\rho\,A\,y}{\epsilon_0} \implies \vec{E}(y) = \frac{\rho}{\epsilon_0} y\hat{y} \quad \text{for } |y| < d$$

$$E\,A = \frac{\rho\,A\,d}{\epsilon_0} \implies \vec{E}(y) = \frac{\rho}{\epsilon_0} d\hat{y} \quad \text{for } y > d$$

and

$$\vec{E}(y) = -\frac{\rho}{\epsilon_0} d\hat{y} \quad \text{for } y < -d$$



Discussion 2 - Cavities in a conductor

a) In the presence of an electric field, the mobile charges of a conductor distribute themselves in such a way as to cancel the electric field inside the conductor. The fields created by q_a and q_b are cancelled in the conductor by a charge induced at the surface of each cavity. The total charge at the surface of each cavity is equal and opposite to the total charge in each cavity. For the same reason, a charge is induced on the outer surface of the sphere which is equal and opposite to the total charge induced on the surface of each cavity. The induced charge is distributed on each respective surface:

$$\sigma_a = -\frac{q_a}{4\pi r_a^2}; \qquad \sigma_b = -\frac{q_b}{4\pi r_b^2}; \qquad \sigma_R = \frac{q_a + q_b}{4\pi R^2}$$

b) From Gauss' law, with the gaussian surface outside the sphere and $\vec{r}$ relative to the center of the sphere:

$$\oiint_S \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} Q_{inside} \rightarrow \vec{E}_{out} = \frac{1}{4\pi\epsilon_0} \frac{q_a + q_b}{r^2} \hat{r} = \frac{R^2}{\epsilon_0} \frac{\sigma_R}{r^2} \hat{r}$$

The field does not depend on the size, shape, and position of the cavities because the conductor conceals all information about the cavities, revealing only the contained total charge. Since the conductor has a spherical shape, the electric field outside the conductor is the same as the one generated by an equivalent point charge located at the origin.

c) There is no force on the charges since the electric field is zero in the conductor. The field generated by q_a does not penetrate the cavity b, and vice versa. So there can be no force acting on the charges.

Solution 3 - Corona Discharge

a)To avoid corona discharge the field has to be lower than $E_{ionize} = 30$ kV/cm. Thus,

$$\frac{Q}{4\pi\epsilon_0(d/2)^2} \leq E_{ionize} \implies Q \leq \pi\epsilon_0 d^2 E_{ionize} \implies Q_{max} = \pi\epsilon_0 d^2 E_{ionize}$$

$$\begin{aligned}\implies Q_{max} &= \pi\epsilon_0 d^2 E_{ionize} \\ &= 3.14 * 8.85 \times 10^{-12} \times (0.18)^2 \times 3 \times 10^6 C \\ &= 2.7 \times 10^{-6} C\end{aligned}$$

b) Locally the following is valid

$$\begin{cases}\Phi(r) = \frac{Q}{4\pi\epsilon_0 r} \\ E(r) = \frac{Q}{4\pi\epsilon_0 r^2}\end{cases}$$

$$\Rightarrow r = \Phi(r)/E(r)$$

Now, we know the maximal electric field is $E_{ionize}(r) = 30\text{kV/cm}$ and $\Phi(r) = 90\text{kV}$.
So

$$r_{\text{smallest}} = \Phi(r)/E_{ionize}(r) = 3\text{cm}$$

Solution 4 - Atmosphere charge density

We apply Gauss's law to a closed surface just outside the surface of the earth and to the edge of the ionosphere:

$$4\pi R_1^2 E_1 = \frac{Q_E}{\epsilon_0} \tag{6}$$

$$4\pi R_2^2 E_2 = \frac{Q_E}{\epsilon_0} + \frac{4}{3\epsilon_0}\pi\rho(R_2^3 - R_1^3) \tag{7}$$

Where ρ is the volume charge density of the atmosphere. Substituting these expressions we obtain:

$$4\pi R_2^2 E_2 = 4\pi R_1^2 E_1 + \frac{4}{3\epsilon_0}\pi\rho(R_2^3 - R_1^3) \tag{8}$$

$$\frac{1}{3\epsilon_0}\rho(R_2^3 - R_1^3) = R_2^2 E_2 - R_1^2 E_1 \tag{9}$$

Solved for ρ this yields:

$$\rho = \frac{3\epsilon_0\,(R_2^2 E_2 - R_1^2 E_1)}{R_2^3 - R_1^3} \tag{10}$$