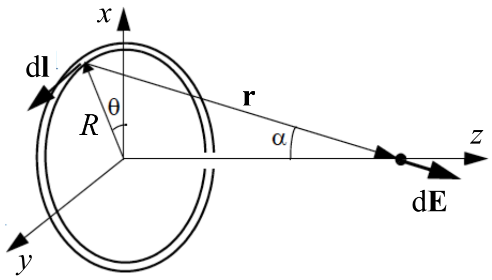
Solution Sheet 6

Discussion 16.10.2024

Solution 1 - Circular charged wire

a) An element with length $d\vec{l}$ creates an E-field $d\vec{E}$ on the point z along the central axis given by



$$d\vec{E} = \frac{dQ}{4\pi\epsilon_0 r^2}\hat{r}$$

Whereby

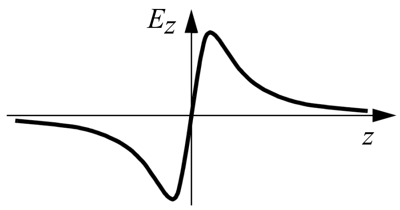
$$dQ = \frac{Qdl}{2\pi R}$$

The components of $d\vec{E}$ along x and y cancel and only the projection on z remains

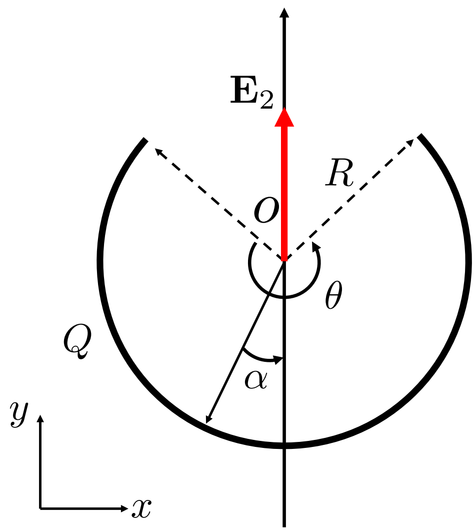
$$\vec{E} = \int_0^{2\pi R} \frac{Q}{2\pi R}\frac{1}{4\pi\epsilon_0}\frac{\cos\alpha}{r^2}dl\hat{z} = \int_0^{2\pi R} \frac{Q}{2\pi R}\frac{1}{4\pi\epsilon_0}\frac{z}{(z^2+R^2)^{3/2}}dl\hat{z}$$

where we used the fact that $r^2 = z^2 + R^2$ and $z = r\cos\alpha$. The E-field thus becomes :

$$\vec{E} = \frac{Q}{4\pi\epsilon_0}\frac{z}{(z^2+R^2)^{3/2}}\hat{z}$$



The electric field is zero in centre of the ring, changes sign with the direction of z and has the shape plotted in the figure



b) From a symmetry consideration, the E-field at the centre O is pointed in the y-direction.

In addition, we can consider the y component $E_{1,y}$ of the E-field generated by only half of the wire, as the total field will be $E_{2,y} = 2E_{1,y}$.

The line charge density λ is

$$\lambda = \frac{Q}{R\theta} \tag{1}$$

For a small charge at α with a small wire piece of length $R \cdot d\alpha$,

$$\begin{aligned} dq &= \lambda \cdot R \cdot d\alpha \\ &= \frac{Q d\alpha}{\theta} \end{aligned} \tag{2}$$

$$\begin{aligned} dE_y &= \frac{dq}{4\pi\epsilon_0 R^2} \cos\alpha \\ &= \frac{Q}{4\pi\epsilon_0 \theta R^2} \cos\alpha\, d\alpha \end{aligned} \tag{3}$$

$$\begin{aligned} E_{1,y} &= \int_0^{\theta/2} dE_y \\ &= \int_0^{\theta/2} \frac{Q}{4\pi\epsilon_0 \theta R^2} \cos\alpha\, d\alpha \\ &= \frac{Q}{4\pi\epsilon_0 \theta R^2} \int_0^{\theta/2} \cos\alpha\, d\alpha \\ &= \frac{Q}{4\pi\epsilon_0 \theta R^2} \sin\alpha \,\Big|_0^{\theta/2} \\ &= \frac{Q}{4\pi\epsilon_0 \theta R^2} \sin(\theta/2) \end{aligned} \tag{4}$$

The total field is

$$\begin{aligned} E_{2,y} &= 2E_{1,y} \\ &= \frac{Q}{2\pi\epsilon_0\theta R^2}\sin(\theta/2) \end{aligned} \tag{5}$$

For the semi circle ($\theta = \pi$) we thus obtain

$$\vec{E} = \frac{Q}{2\pi^2\epsilon_0 R^2}\hat{y}$$

For the three quarter circle ($\theta = \frac{3}{2}\pi$) it becomes

$$\vec{E} = \frac{\sqrt{2}Q}{6\pi^2\epsilon_0 R^2}\hat{y}$$

It is also clear that for a closed ring the field becomes zero, in accordance with the result of part (a). For the open ring the E-field away from the xyplane of the ring becomes more difficult to calculate, but it is a good exercise to consider how one would approach this problem.

Solution 2 - Electric field and potential of a straight wire/rod

a) From symmetry considerations it follows that the E-field is along the horizontal direction of the rod towards point B. We will call this direction as x-axis and set the origin ($x = 0$) of the x-axis at the right end of the rod.

The contribution of a small part of the rod dx with line charge density λ to the E-field will be

$$dq = \lambda dx \tag{6}$$

$$dE_x = \frac{dq}{4\pi\epsilon_0 x^2} = \frac{\lambda dx}{4\pi\epsilon_0 x^2} \tag{7}$$

$$\begin{aligned} E &= \int_{x_0}^{L+x_0} \frac{\lambda dx}{4\pi\epsilon_0 x^2} \\ &= \frac{\lambda}{4\pi\epsilon_0}\int_{x_0}^{L+x_0} \frac{dx}{x^2} \\ &= \frac{\lambda}{4\pi\epsilon_0}(-\frac{1}{x}) \Big|_{x_0}^{L+x_0} \\ &= \frac{\lambda}{4\pi\epsilon_0}(\frac{1}{x_0} - \frac{1}{L+x_0}) \\ &= \frac{\lambda L}{4\pi\epsilon_0 x_0(L+x_0)} \end{aligned} \tag{8}$$

To find the potential we integrate from our reference where $V = 0$ (at $\pm\infty$) to the point we are interested in (x_0). To be able to integrate along the positive x-direction we start at $-\infty$. Thus the potential is

$$\begin{aligned} V &= -\int_{-\infty}^{x_0} E dx \\ &= -\int_{-\infty}^{x_0} \frac{\lambda}{4\pi\epsilon_0}(\frac{1}{x} - \frac{1}{L+x}) \\ &= -\frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{x_0}(\frac{1}{x} - \frac{1}{L+x})dx \\ &= -\frac{\lambda}{4\pi\epsilon_0}(\ln(x) - \ln(L+x)) \mid_{-\infty}^{x_0} \\ &= -\frac{\lambda}{4\pi\epsilon_0}\ln(\frac{x}{L+x}) \mid_{-\infty}^{x_0} \\ &= -\frac{\lambda}{4\pi\epsilon_0}\ln(\frac{x_0}{L+x_0}) \end{aligned} \tag{9}$$

The last step uses the formula

$$\mathrm{Lim}_{x\to-\infty}\ln(\frac{x}{x+L}) = \ln(1) = 0 \tag{10}$$

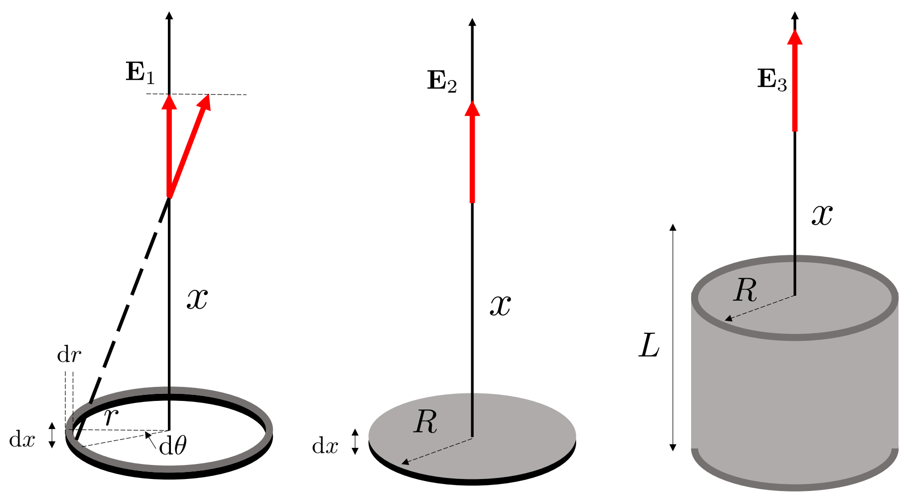


Figure 1: A circular ring, a circular disk and a cylinder.

b) The solution should be found by relating a circular ring, a circular disk and finally the cylinder. For a circular ring with radius r, width dr, thickness dx and distant x from point A, the E-field is E_1 in x-direction

$$dq = \rho r \cdot d\theta \cdot dr \cdot dx \tag{11}$$

$$\begin{aligned} E_1 &= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\frac{xdq}{(x^2+r^2)^{3/2}} \\ &= \frac{\rho r x dr dx}{4\pi\epsilon_0(x^2+r^2)^{3/2}}\int_0^{2\pi} d\theta \\ &= \frac{\rho r x dr dx}{2\epsilon_0(x^2+r^2)^{3/2}} \end{aligned} \tag{12}$$

For a circular disk with radius R, thickness dx and distant x from point A, the E-field is E_2 in x-direction

$$\begin{aligned} E_2 &= \int_0^R E_1 \\ &= \int_0^R \frac{\rho r x dr dx}{2\epsilon_0(x^2+r^2)^{3/2}} \end{aligned} \tag{13}$$

For a cylinder with radius R, length L and distant x_0 from point A, the E-field is E_3 in x-direction

$$\begin{aligned} E_3 &= \int_{x_0}^{x_0+L} E_2 \\ &= \int_{x_0}^{x_0+L}\int_0^R \frac{\rho r x dr dx}{2\epsilon_0(x^2+r^2)^{3/2}} \\ &= \frac{\rho}{2\epsilon_0}[L+\sqrt{R^2+x_0^2}-\sqrt{R^2+(x_0+L)^2}] \end{aligned} \tag{14}$$

Solution 3 - Drilling into a disk

Assume the surface charge density is σ. For a circular ring with radius r, width dr and distant z from point P, the E-field E_1 is

$$dq = \sigma r \cdot d\theta \cdot dr \tag{15}$$

$$\begin{aligned} E_1 &= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\frac{zdq}{(z^2+r^2)^{3/2}} \\ &= \frac{\sigma rdr}{4\pi\epsilon_0}\int_0^{2\pi}\frac{zd\theta}{(z^2+r^2)^{3/2}} \\ &= \frac{\sigma rzdr}{4\pi\epsilon_0(z^2+r^2)^{3/2}}\int_0^{2\pi}d\theta \\ &= \frac{\sigma rzdr}{2\epsilon_0(z^2+r^2)^{3/2}} \end{aligned} \tag{16}$$

For the circular disk with radius R, the E-field is E_2 in x-direction and is

$$\begin{aligned} E_2 &= \int_{R_2}^{R} E_1 \\ &= \int_{R_2}^{R}\frac{\sigma rzdr}{2\epsilon_0(z^2+r^2)^{3/2}} \\ &= \frac{\sigma z}{2\epsilon_0}\int_{R_2}^{R}\frac{rdr}{(z^2+r^2)^{3/2}} \\ &= \frac{\sigma z}{2\epsilon_0}\int_{z^2+R_2^2}^{z^2+R^2}\frac{dt}{2t^{3/2}} \\ &= \frac{\sigma z}{2\epsilon_0}\left(-\frac{1}{t^{1/2}}\right)\Big|_{z^2+R_2^2}^{z^2+R^2} \\ &= \frac{\sigma z}{2\epsilon_0}\left[\frac{1}{(z^2+R_2^2)^{1/2}}-\frac{1}{(z^2+R^2)^{1/2}}\right] \end{aligned} \tag{17}$$

It can also be solved in a simpler way using the superposition principle. The E-field of a charged disk of radius R is derived in the lectures as

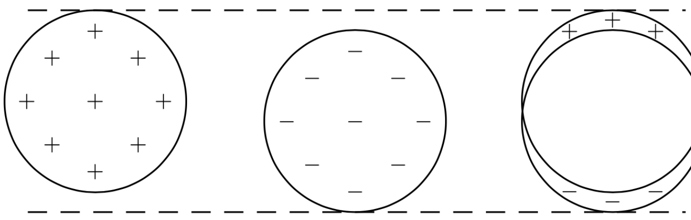
$$E(R) = \frac{\sigma z}{2\epsilon_0}\left[\frac{1}{z}-\frac{1}{(z^2+R^2)^{1/2}}\right] \tag{18}$$

Using the superposition principle, the drilled disk can be consider as a positively charged disk of radius R plus a negatively charged disk of radius R_2, therefore the total E-field is

$$\begin{aligned} E_2 &= E(R) - E(R_2) \\ &= \frac{\sigma z}{2\epsilon_0}\left[\frac{1}{(z^2+R_2^2)^{1/2}}-\frac{1}{(z^2+R^2)^{1/2}}\right] \end{aligned} \tag{19}$$

Solution 4 - Asymmetric charge distribution

The general idea is sketched in the figure. To obtain the $\sigma_0 \cos\theta$ charge distribution we consider two spheres that are slightly displaced by dz



The spheres are charged with a uniform positive and negative volume charge density of ρ and $-\rho$. Using Gauss's law it has been derived in the lecture that the potential of such a sphere is the same as for a point charge with the same total charge if we look outside the radius of the sphere a. In this case the total charge should be considered

$$Q = \frac{4}{3}\pi a^3 \rho$$

The potential from the positively charged sphere is

$$\Phi_+ = \frac{\frac{4}{3}\pi a^3 \rho}{4\pi\epsilon_0 r}$$

and from the negatively charged sphere $\Phi_- = -\Phi_+ - d\Phi_+$ whereby the last term is due to the small displacement dz. Thus the total potential is

$$\Phi = \Phi_+ + \Phi_- = -d\Phi = -\frac{\partial \left(\frac{\frac{4}{3}\pi a^3 \rho}{4\pi\epsilon_0 r}\right)}{\partial z} dz$$

Along the same lines as in the lecture we thus obtain

$$\Phi = \frac{\frac{4}{3}\pi a^3 \rho dz \cos\theta}{4\pi\epsilon_0 r^2}$$

For dz small, the charge in the parts of the spheres that don't overlap (c) can be considered the surface charge density σ_0 (volume charge density with one dimension removed becomes surface charge density):

$$\rho dz = \sigma_0$$

Thus the expression for the potential above can be rewritten as

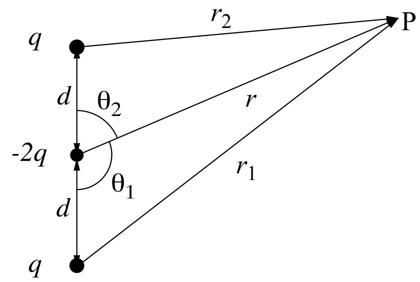
$$\Phi = \frac{\vec{p} \cdot \hat{r}}{4\pi\epsilon_0 r^2}$$

with

$$\vec{p} = \frac{4}{3}\pi a^3 \sigma_0 \hat{z} = \left(\frac{4}{3}\pi a^3 \rho dz \hat{z}\right)$$

The asymmetric charge distribution thus has the same potential as a dipole with this dipole moment. Note that the dipole moment can also be calculated by integrating the expression for σ with regard to the centre of the sphere.

Solution 5 - Potential of a quadrupole



We choose the origin on the central charge $-2q$. r_1 and r_2 are the distances of the lateral charges from the reference point P identified by $\vec{r}$.
From the hypothesis: r_1, r_2, $r \gg d$.

Using the superposition principle, the expression for the potential is:

$$\Phi(r) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_1} - \frac{2q}{r} + \frac{q}{r_2}\right) = \frac{q}{4\pi\epsilon_0\, r}\left(\frac{r}{r_1} + \frac{r}{r_2} - 2\right)$$

We can express r_1 and r_2:

$$r_1^2 = r^2 + d^2 - 2rd\cos\theta_1$$

$$r_2^2 = r^2 + d^2 - 2rd\cos\theta_2$$

Note that $\theta_1 = \pi - \theta_2$ such that $\cos\theta_1 = -\cos\theta_2$. We can then write:

$$r_1^2 = r^2 + d^2 - 2rd\cos\theta$$

and

$$r_2^2 = r^2 + d^2 + 2rd\cos\theta$$

where we have renamed $\theta = \theta_1$ and used the relation between θ_1 and θ_2.

Binomial expansion: we use $1/\sqrt{1+x} \approx 1 - x/2 + 3x^2/8$ valid for small x; terms of order higher than d^2/r^2 are neglected only after having carried out the expansion.

$$\frac{r}{r_1} = 1 - \frac{d}{r}\cos\theta + \frac{d^2}{r^2}\left(\frac{3\cos^2\theta - 1}{2}\right) + \ldots$$

and

$$\frac{r}{r_2} = 1 + \frac{d}{r}\cos\theta + \frac{d^2}{r^2}\left(\frac{3\cos^2\theta - 1}{2}\right) + \ldots$$

The expression for the potential is then

$$\Phi(r) = \frac{2qd^2}{4\pi\epsilon_0 r^3}\frac{3\cos^2\theta - 1}{2} = \frac{qd^2}{4\pi\epsilon_0 r^3}(3\cos^2\theta - 1)$$