Solution Sheet 3

Discussion 25.09.2024

Solution 1 - Airplane take-off speed

a) The airplane will take off when the force arising from the difference in pressure above and below the wings equals the gravitational force.
Using Bernoulli we can find the expression linking v_1, p_1 (below the wings) to v_2, p_2 (above the wings), and to the expression for the force:

$$F = 2\,A(p_1 - p_2) = 2A\frac{1}{2}\rho(v_2^2 - v_1^2) \tag{1}$$

The gravitational force is:

$$F = mg \tag{2}$$

We find the expression for v_1 (assimilated to the speed of the airplane):

$$v_2^2 - v_1^2 = \frac{mg}{A\rho} \tag{3}$$

With $v_2 = 1.2v_1$ we obtain

$$v_1 = \sqrt{\frac{mg}{0.44\,A\rho}} = 150 \text{ m/s} = 541 \text{ km/h} \tag{4}$$

b) The actual take-off speed is around 300 km/h. This difference is primarily caused by the fact that also the body is designed such as to create a lift force. Further corrections arise from the presence of turbulence and the fact that at these velocities air can no longer be considered incompressible.

Discussion 1 - Streamlines

The denser the streamlines, the higher the velocity. Importantly, streamlines can never cross, neither itself or another streamline. If they would cross this would imply two different velocities for the same point. In this case it should be made clear that the velocity on one side of the house is higher as on the other side and thus the pressure lower. This will cause a air flow in the house (streamlines...) that will cause the door to slam. Thus it is not the wind blowing through the house that is the reason.

Solution 2 - Streamlines in 2D

In the following we determine the expression $y(x)$ for the streamlines. We have: $v_x = (v_0/l)x$ and $v_y = -(v_0/l)y$. The streamlines are, in each point, tangential to the velocity vector, and thus verify the following equation:

$$\frac{dy}{dx} = \frac{v_y}{v_x} = \frac{-yv_0/l}{xv_0/l} = -\frac{y}{x}\ ,$$

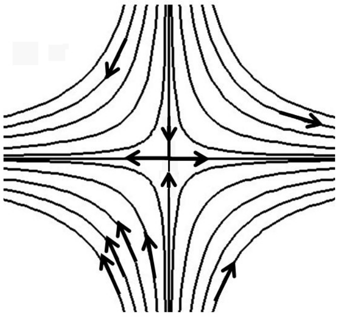
where we can separate the variables and obtain:

$$\int \frac{dx}{x} = -\int \frac{dy}{y} \ ,$$

i.e.:

$$-\ln y = \ln x + C \ , \quad \rightarrow \quad xy = C' \ .$$

So, each streamline is a hyperbola.



In general, the acceleration is :

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} = \frac{\partial \mathbf{v}}{\partial t} + v_x \frac{\partial \mathbf{v}}{\partial x} + v_y \frac{\partial \mathbf{v}}{\partial y} + v_z \frac{\partial \mathbf{v}}{\partial z} \ .$$

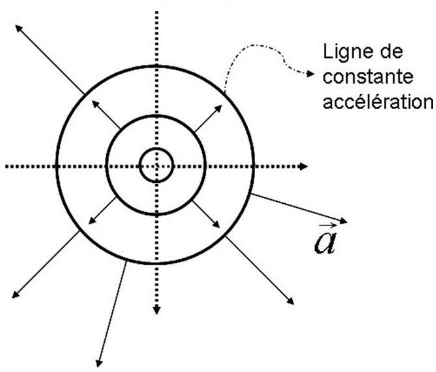
The flow is stationary so $\frac{\partial \mathbf{v}}{\partial t} = 0$. Moreover, the flow is bidimensional so $v_z = 0$ and $v_z \cdot \frac{\partial \mathbf{v}}{\partial z} = 0$. The acceleration then becomes :

$$\mathbf{a} = v_x \frac{\partial \mathbf{v}}{\partial x} + v_y \frac{\partial \mathbf{v}}{\partial y} = \left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \right) \hat{\mathbf{x}} + \left(v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} \right) \hat{\mathbf{y}} \ ,$$

and finally:

$$\mathbf{a} = x \frac{v_0^2}{l^2}\,\hat{\mathbf{x}} + y \frac{v_0^2}{l^2}\,\hat{\mathbf{y}} = \frac{v_0^2}{l^2}\, r\,\hat{\mathbf{r}} \ .$$

The acceleration is radial. Its amplitude is constant on the concentric circles around the origin.



Solution 3 - Viscosity, shear stress, and shear strain

Since the gradient of the velocity is constant, and since we only need to consider the force along the x-direction, we find that

$$S_{yx} = \eta \frac{dv_x}{dy} = \eta \frac{v_0}{d} \tag{5}$$

The expression for the shear rate is

$$\dot{e}_{yx} = \frac{de}{dt} = \frac{d}{dt}\frac{\Delta x}{d} = \frac{1}{h}\frac{d\Delta x}{dt} = \frac{v_0}{d} \tag{6}$$

The shear strain can also be expressed as

$$S_{yx} = \eta \dot{e}_{yx} \tag{7}$$

Solution 4 - Viscous drag

The lower disk will be set in motion by the shear stress exerted by the fluid.
By hypothesis:

$$\frac{\partial v}{\partial z} = \alpha(r,t) \quad \text{at any point at distance } r \text{ from the axis.} \tag{8}$$

We deduce:

$$v(z) = \alpha(r,t)z + \beta(r,t) \tag{9}$$

Boundary conditions (non-slip):

$v(z=0) = \omega r \implies \beta$ is independent of time and is $\beta = \omega r$;

$v(z=d) = \Omega r \implies \alpha = \dfrac{(\Omega - \omega)r}{d}$.

The shear stress at a distance r is given by:

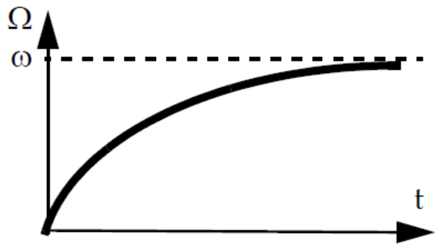
$$S_z = \eta \frac{\partial v}{\partial z} = \frac{\eta(\Omega - \omega)}{d} r \tag{10}$$

and is thus independent of z.

The moment of force exerted by the fluid on the upper disk ($z = 0$) is given by the sum of all the small moments of force exerted by the constraints on each surface element of the disk:

$$C = \int_A S_z r \, dA = \int_0^R S_z 2\pi r^2 \, dr = \frac{2\pi}{d}\eta(\Omega - \omega)\int_0^R r^3 \, dr = \frac{\pi}{2d}\eta(\Omega - \omega)R^4; \quad \vec{C} = C\vec{e_z} \tag{11}$$

The angular momentum of the lower disk is given by $I\Omega$. If the fluid exerts a moment of forces C on the upper disk, it must exert a moment $-C$ on the lower disk. Here we have assumed that the fluid is no longer accelerating (steady flow), or rather that the moment of inertia of the fluid is zero and thus directly follows the rotation of the upper disk. The dynamics of the lower disk is now given by the law of angular momentum: $\frac{d}{dt}(I\Omega) = -C$.



We thus obtain the differential equation allowing to determine the temporal dependence $\Omega(t)$:

$$I\dot{\Omega} + \frac{\pi}{2d}\eta R^4 \Omega = \frac{\pi}{2d}\eta R^4 \omega \tag{12}$$

$$\Omega(t) = \omega \left(1 - \exp\left(-\frac{\pi}{2d}\eta \frac{R^4}{I} t\right) \right) \tag{13}$$