Solution Sheet 14

Discussion 18.12.2024

Solution 1 - Refraction of Electromagnetic Waves

(a) From the lecture, the velocity in medium:

$$c_m = \frac{c}{\sqrt{\epsilon_r \mu_r}} \simeq \frac{c}{\sqrt{\epsilon_r}}$$

The frequency ν remains same in different materials.

$$\lambda_m = \frac{c_m}{\nu} = \frac{c_m}{c/\lambda_0} = \frac{c_m}{c}\lambda_0$$

In glass: $c_{glass} = \frac{c}{\sqrt{4.7}} \simeq 0.46c$, $\lambda_{glass} = \frac{c_{glass}}{c}\lambda_0 \simeq 0.46\lambda_0$;
in water: $c_{water} = \frac{c}{\sqrt{1.77}} \simeq 0.75c$, $\lambda_{water} = \frac{c_{water}}{c}\lambda_0 \simeq 0.75\lambda_0$.
where $\lambda_0 = 589.3nm$ and $c = 3 \times 10^8 m/s$

$$c_m = \begin{cases} 0.46c & 0 < h < 10\text{mm} \\ 0.75c & 10\text{mm} < h < 30\text{mm} \\ c & h > 30\text{mm} \end{cases}$$

$$\lambda_m = \begin{cases} 0.46\lambda_0 & 0 < h < 10\text{mm} \\ 0.75\lambda_0 & 10\text{mm} < h < 30\text{mm} \\ \lambda_0 & h > 30\text{mm} \end{cases}$$

where h is the vertical distance from bottom.

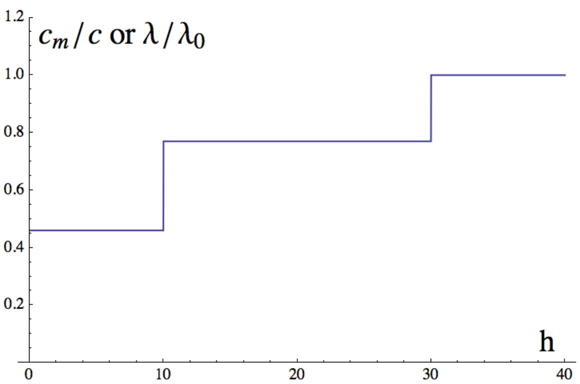

Figure 1: wavelength or velocity varying with distance

(b) From the refractive law:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

where $n_1 = \sqrt{\epsilon_1}$ and $n_2 = \sqrt{\epsilon_2}$ are refractive index in different materials.
From air to glass:

$$\sin\theta_2 = \frac{n_1 \sin\theta_1}{n_2} = \frac{1\sin 70°}{\sqrt{4.7}} \Rightarrow \theta_2 \simeq 26°$$

From glass to water:

$$\sin\theta_3 = \frac{n_2 \sin\theta_2}{n_3} = \frac{n_1 \sin\theta_1}{n_3} = \frac{1\sin 70°}{\sqrt{1.77}} \Rightarrow \theta_3 \simeq 45°$$

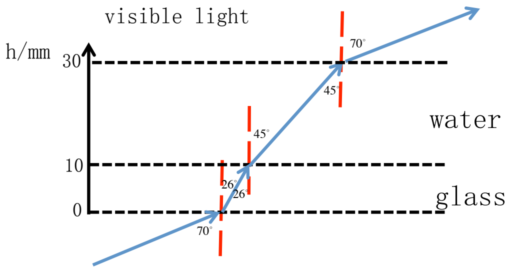

Figure 2: the trajectory of the laser (visible light)

(c) Similiar with **(a)** and **(b)**, for $\lambda_{micro} = 10\text{mm}$
we can get

$c'_{glass} = \frac{c}{\sqrt{3.8}} \simeq 0.51c$, $\lambda'_{glass} = \frac{c'_{glass}}{c}\lambda_{micro} \simeq 0.51\lambda_{micro}$

$c'_{water} = \frac{c}{\sqrt{33.6}} \simeq 0.17c$, $\lambda'_{water} = \frac{c'_{water}}{c}\lambda_{micro} \simeq 0.17\lambda_{micro}$

From air to glass:

$$\sin\theta'_2 = \frac{1\sin 70°}{\sqrt{3.8}} \Rightarrow \theta'_2 \simeq 29°$$

From glass to water:

$$\sin\theta'_3 = \frac{1\sin 70°}{\sqrt{33.5}} \Rightarrow \theta'_3 \simeq 9.3°$$

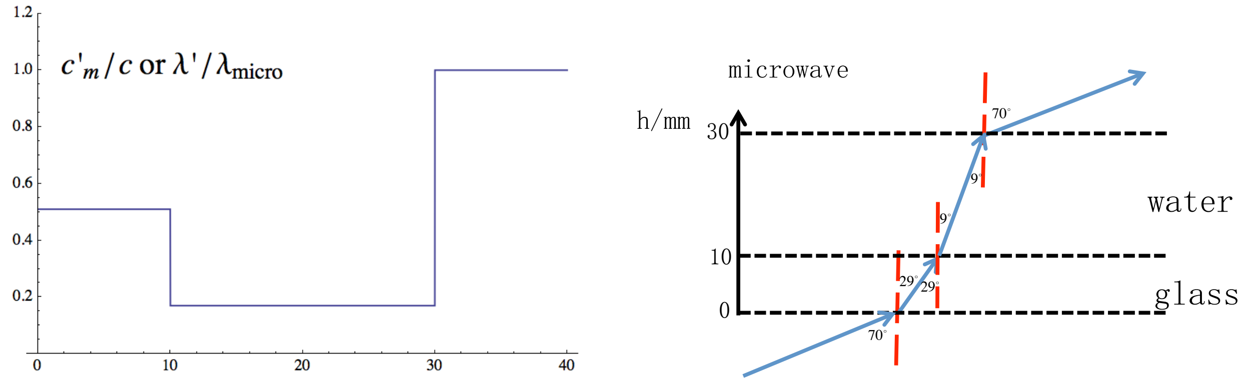

Figure 3: (left) the wavelenth and the velocity varying with distance, (right) the trajectory of the microwave

(d) Cherenkov radiation is emitted when a charged particle is moving at a speed greater than the velocity of light in a dielectric medium. Therefore the minimum velocity required for the electron to emit Cherenkov radiation is $v_{e,min} = c_{water} = c/\sqrt{1.77}$ from **(2a)** and the corresponding kinetic energy is

$$E_{e,min} = \frac{1}{2}m_e v_{e,min}^2 = \frac{1}{2}m_e c_{water}^2 = \frac{1}{2}m_e \frac{c^2}{1.77} \simeq 0.28 m_e c^2 \tag{1}$$

Solution 2 - Oscillating dipoles: antennas

(a) If we take the second derivate of the given $p(t)$

$$\begin{aligned} p(t) &= ql\sin(wt) \\ \dot{p}(t) &= ql\cos(wt)w \\ \ddot{p}(t) &= -qlw^2\sin(wt) \end{aligned}$$

Then, if we substitute $\ddot{p}$ to the given E field equation,

$$\bar{E}_{rad} = -\frac{qlw^2\sin(wt')}{4\pi\epsilon_0 c^2}\frac{\sin\theta}{r}\hat{\theta}$$

where $t' = t - \frac{r}{c}$. We can realize this oscillating dipole with an antenna by switching the positive and negative charges over a period of time based on a period T. Since $p(t) = ql\sin(wt)$, we have two separate voltages driving charges at 180° of phase. Then, the switch of the charges is realized as shown in Figure 4 where T is 2π, i.e., 360°.

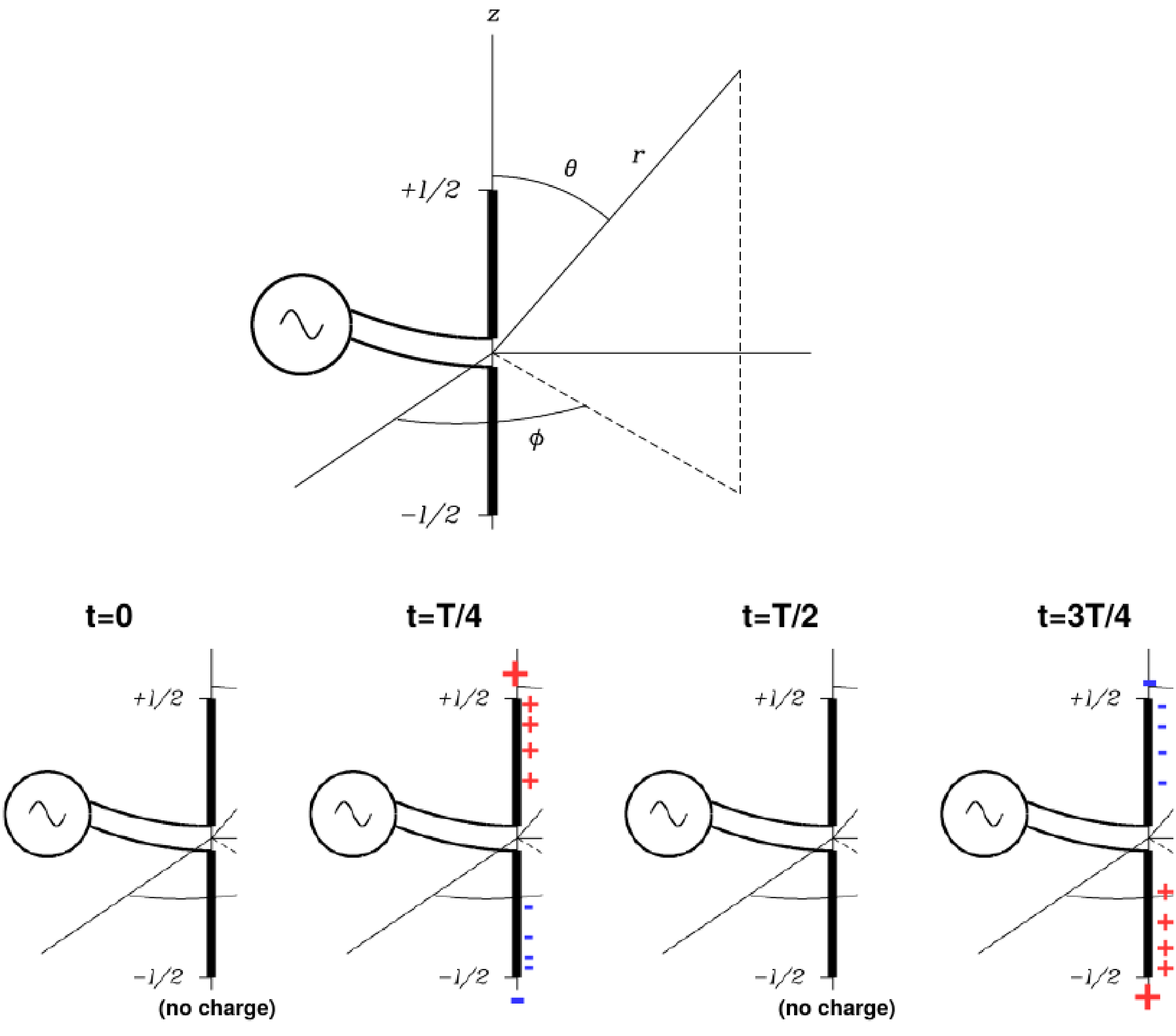

Figure 4: positive and negative charges in the antenna

(b) Since $\bar{B} = \frac{1}{c}\hat{r} \times \bar{E}$,

$$\begin{aligned}
\bar{B} &= \frac{1}{c}\hat{r} \times \bar{E} \\
&= -\frac{1}{c}\frac{qlw^2 \sin(wt')}{4\pi\epsilon_0 c^2}\frac{\sin\theta}{r}(\hat{r} \times \hat{\theta}) \\
&= \frac{1}{c}\frac{qlw^2 \sin(wt')}{4\pi\epsilon_0 c^2}\frac{\sin\theta}{r}\hat{\phi} \\
&= -\frac{\mu_0 qlw^2}{4\pi c}\frac{\sin\theta}{r}\sin(wt')\hat{\phi}
\end{aligned}$$

, where $\hat{r} \times \hat{\theta} = \hat{\phi}$ and $\mu_0\epsilon_0 = \frac{1}{c^2}$.

(c) The Poynting vector, $\bar{S}$, is

$$\begin{aligned}
\bar{S} &= \frac{1}{\mu_0}(\bar{E} \times \bar{B}) \\
&= \left(\frac{1}{\mu_0}\frac{-qlw^2 \sin(wt')}{4\pi\epsilon_0 c^2}\frac{\sin\theta}{r}\right)\left(\frac{-\mu_0 qlw^2}{4\pi c}\frac{\sin\theta}{r}\sin(wt')\right)\hat{\theta} \times \hat{\phi} \\
&= \frac{\mu_0}{c}\left(\frac{qlw^2 \sin\theta}{4\pi r}\sin(wt')\right)^2 \hat{r}
\end{aligned}$$

, where $\hat{\theta} \times \hat{\phi} = \hat{r}$.