Solution Sheet 13

Discussion 11.12.2024

Solution 1 - Displacement Current

a) The current I_c in the circuit due to charging of the capacitor is

$$I_c = C\frac{d\Phi_c}{dt} \tag{1}$$

with Φ_c is the voltage on the capacitor and the capacitance of the capacitor C is

$$C = \frac{\epsilon_0 A}{d} \tag{2}$$

Using Kirchoff's law, in this circuit we have

$$\begin{aligned} \Phi_0 &= R_1 I_c + \Phi_c \\ &= R_1 C\frac{d\Phi_c}{dt} + \Phi_c \end{aligned} \tag{3}$$

thus

$$\frac{dt}{\tau} = \frac{d\Phi_c}{\Phi_0 - \Phi_c} \tag{4}$$

with the time constant

$$\tau = R_1 C \tag{5}$$

Integrate the above equation from 0 to t on the left side and from 0 to Φ_c on the right side gives

$$\int_0^t \frac{dt}{\tau} = \int_0^{\Phi_c} \frac{d\Phi_c}{\Phi_0 - \Phi_c} \tag{6}$$

thus

$$\Phi_c(t) = \Phi_0(1 - e^{-t/\tau}) \tag{7}$$

thus the electric field E between the plates is

$$E(t) = \frac{\Phi_0}{d}(1 - e^{-t/\tau}) \tag{8}$$

Note that the method to derive the voltage on the capacitor was already discussed in another lecture and is only reproduced here for clarity.

b) To determine the magnetic field B in the solenoid, we use the Ampere's law on the central circular path of the solenoid of radius a, as

$$\oint \vec{B}\, d\vec{l} = \int\int \mu_0 \epsilon_0 \frac{d\vec{E}}{dt}\, d\vec{S} \tag{9}$$

$$B 2\pi a = \mu_0 \epsilon_0 \pi a^2 \frac{dE}{dt} \tag{10}$$

thus

$$B(t) = \frac{\mu_0 a \Phi_0}{2 R_1 A} e^{-t/\tau} \tag{11}$$

the last step uses $\tau = R_1 C$ and $C = \epsilon_0 A/d$.

c) In the solenoid, the current I_s and voltage Φ_s are

$$\begin{aligned} I_s &= \frac{dQ}{dt} \\ \Phi_s &= R_2 I_s \end{aligned} \tag{12}$$

thus charge Q which flows through the solenoid is

$$\begin{aligned} Q &= \int_0^\infty I_s\, dt \\ &= \int_0^\infty \frac{\Phi_s}{R_2} dt \\ &= \int_0^\infty \frac{-1}{R_2} \frac{d\varphi}{dt} dt \\ &= \int_0^\infty \frac{-1}{R_2} d\varphi \\ &= \frac{NS}{R_2} [B(0) - B(\infty)] \\ &= \frac{NS \mu_0 a \Phi_0}{2 A R_1 R_2} \end{aligned} \tag{13}$$

where we have used the Faraday law $\Phi_{\mathrm{em}} = -d\varphi/dt$, the total flux $\varphi = NSB(t)$ and the fact that $B(\infty) = 0$ as the field vanishes exponentially.

Solution 2 - AC magnetic field

Similar to the discussion in the previous exercise and with $C = \epsilon_0 A/d$, the E-field between the plates is

$$E(t) = \frac{Q(t)}{\epsilon_0 \pi a^2} \tag{14}$$

select a circular path of radius r between the two plates, whose plane is parallel to the plates, and use Ampere's law, as

$$\oint \vec{B}(t) d\vec{l} = \iint \mu_0 \epsilon_0 \frac{d\vec{E}(t)}{dt} d\vec{S} \tag{15}$$

when $r < a$,

$$B(t) \cdot 2\pi r = \mu_0 \epsilon_0 \frac{1}{\epsilon_0 \pi a^2} \frac{dQ(t)}{dt} \pi r^2 \tag{16}$$

thus

$$B(t) = \frac{r\omega Q_0 \mu_0}{2\pi a^2} \cos \omega t \tag{17}$$

when $r \geqslant a$,

$$B(t) \cdot 2\pi r = \mu_0 \epsilon_0 \frac{1}{\epsilon_0 \pi a^2} \frac{dQ(t)}{dt} \pi a^2 \tag{18}$$

thus

$$B(t) = \frac{\omega Q_0 \mu_0}{2\pi r} \cos \omega t \tag{19}$$

Solution 3 - Waves

a) Recall that $\sin\alpha \cos\beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$. Rewrite Ψ in the following way

$$\begin{aligned} \Psi &= \frac{A}{2}[\sin(Bx + Ct) + \sin(Bx - Ct)] + D \\ &= \frac{A}{2}[\sin(B(x + \frac{C}{B}t)) + \sin(B(x - \frac{C}{B}t))] + D \end{aligned} \tag{20}$$

Clearly we can replace $\sin(Bx) = f(x)$, and $f(x - vt)$, $f(x + vt)$ are the solution of wave equations.

b) Setting $x - Ct/B = \text{const}$ shows that the speed is $v = C/B$.

c) To verify that $\chi = (x - v_1 t)^2 + (x + v_2 t)^{-1/2}$ is indeed a solution of the wave equation $\frac{\partial^2 \chi}{\partial t^2} = v^2 \nabla^2 \chi$, we put χ into the wave equation, as

$$\begin{aligned}\nabla^2 \chi &= \frac{\partial^2 \chi}{\partial x^2} \\ &= \frac{\partial}{\partial x}[2(x - v_1 t) - \frac{1}{2}(x + v_2 t)^{-3/2}] \\ &= 2 + \frac{3}{4}(x + v_2 t)^{-5/2}\end{aligned} \tag{21}$$

$$\begin{aligned}\frac{\partial^2 \chi}{\partial t^2} &= \frac{\partial}{\partial t}[-2v_1(x - v_1 t) - \frac{v_2}{2}(x + v_2 t)^{-3/2}] \\ &= 2v_1^2 + \frac{3v_2^2}{4}(x + v_2 t)^{-5/2}\end{aligned} \tag{22}$$

To satisfy the wave equation it is required that $v_1 = v_2 = v$.

This result can also be obtained by considering that $h(x, t) = f(x - ct) + g(x + ct)$ is a general solution of the wave equation. However, in many less simple examples it will be necessary to go through the solution as described above.