PHYS-200: Physique III
Part 1: Fluids

Lecture notes

Prof. Hugo Dil

École Polytechnique Fédérale de Lausanne (EPFL)

September 29, 2024

Contents

Introduction

Fluids are clearly part of our everyday life. Actually so much so that we rarely think about how weird their behaviour would appear for someone who has never encountered a real fluid and we certainly don't realise the complexities in describing the flow of fluids. One would be inclined to simplify the treatment by neglecting the viscosity, as will be done in the first chapters, but this will actually lead to behaviour which is completely alien to us. It would not even be possible to stir a cup of tea! Thus in the later chapters we will have to face the complexities of dissipative physics.

Let's start by trying to define what a fluid is. A working definition is that *a fluid fills a container, while maintaining its volume, under the influence of gravity.* This clearly applies to liquids, which is sometimes used synonymously to fluids, but also to gasses like our atmosphere, and to plasmas. In much of this lecture we will assume the fluid to be incompressible and the compressibility primarily starts to play a role for the transport of sound waves.

Actually, the compressibility is the only static deformation tensor that is non-zero in fluids, and this is what sets it clearly apart from solids. The Young modulus, which describes the change in length when a force is applied, and the shear stress, which describes the deformation upon an applied force, are both zero in the static case for fluids. Another important difference with solids is that in fluids the response will be isotropic i.e. it does not depend on the direction.

In the following we will often be using the *nabla* vector ($\vec{\nabla}$) which can be expressed as follows in Cartesian, cylindrical, and spherical coordinates respectively:

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \tag{1}$$

$$\vec{\nabla} = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{\partial}{\partial z} \right) \tag{2}$$

$$\vec{\nabla} = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \right) \tag{3}$$

In the rest of the lecture the vector sign will not be used and we will write ∇ instead of $\vec{\nabla}$.

Chapter 1

Hydrostatics

In this chapter we will look at the most important aspects of fluids at rest that will be relevant to our further discussion. It is assumed that the students are familiar with hydrostatics from other courses and high school physics.

Because of the absence of shear stress, the force $\vec{F}$ per unit area $\vec{A}$ will always be perpendicular to any surface we choose in the fluid and it will be the same for all orientations of this surface. The pressure p defined as

$$p = \frac{\vec{F}}{\vec{A}} \tag{1.1}$$

is thus also independent of which direction we look and **isotropic**. As a reminder, when we write a surface as a vector, this means that the modulus represents the area and the direction the surface normal. We will encounter this notation many times during this lecture.

Although the pressure is isotropic, this does not imply that its can't vary as a function of position. The clearest example of this dependency on position is when we consider a fluid column under the influence of gravity. If at a height z_0 the pressure is defined as p_0, the pressure will be $p_0 - \rho g h$ at a height h above and $p_0 + \rho g h$ at a height h below. Where ρ is the (mass) density. Thus we can say that

$$p + \rho g h = \text{const.} \tag{1.2}$$

In the following we will generalise this for three dimensions i.e. for $p(x, y, z)$

Let's consider a small cube with sides Δx, Δy, and Δz. At point x the pressure is p which is perpendicular to the surface as discussed above. Thus the force, pressure times area, on this face of the cube is $F_1 = p\Delta y\Delta z$. At point $x + \Delta x$ the pressure is slightly different, and, considering the basic definition of derivatives, it is $p + \frac{\partial p}{\partial x}\Delta x$. Because the surface normal has changed sign, the force here becomes $F_2 = -(p + \frac{\partial p}{\partial x}\Delta x)\Delta y \Delta z$. The total force along the x-direction is given by $F_x = F_1 - F_2 = -\frac{\partial p}{\partial x}\Delta x \Delta y \Delta z$. Similar considerations can be made for the y- and z-directions yielding $F_y = -\frac{\partial p}{\partial y}\Delta x \Delta y \Delta z$ and $F_z = -\frac{\partial p}{\partial z}\Delta x \Delta y \Delta z$. Thus the total force per unit volume $\vec{f}$ becomes

$$\frac{\vec{F}}{\Delta V} = \vec{f} = \left(-\frac{\partial p}{\partial x}, -\frac{\partial p}{\partial y}, -\frac{\partial p}{\partial z}\right) = -\nabla p \tag{1.3}$$

Where $\Delta V = \Delta x \Delta y \Delta z$ and ∇ is defined in Eq. 1. In words, the force is the gradient of the pressure.

Besides the force due to pressure, there might be other forces acting on the liquid, for example gravity. These can be summarised as a potential Φ per unit mass. For only gravity we would simply get $\Phi = gz$. The force per unit mass becomes $-\nabla\Phi$ (g) and the force per volume $-\rho\nabla\Phi$ (ρg). In equilibrium the pressure and potential forces should balance each other and we get

$$\vec{f} = -\nabla p - \rho\nabla\Phi = 0 \tag{1.4}$$

This is called the **equation of hydrostatics**. The first term can be considered as the internal forces and the second term as all external forces working on the fluid.

Whether Equation 1.4 has a solution depends on the properties of the density ρ, which breaks the symmetry of the expression. If ρ is constant, the solution is rather simple and becomes $p + \rho\Phi$ is constant, which is the same as Eq. 1.2 if we only consider gravity. If the density is only a function of the pressure ($\rho(p)$) a solutions exists, leading for example to the generally layered structure of our atmosphere. However, if the density also depends on other parameters ($\rho(p, x, y, z, t, T...)$) then no solution exists to Eq. 1.4 and one obtains phenomena like convection.

This ends our discussion of hydrostatics and we will now consider what happens if the fluid is in motion. This will rapidly become more complex, and certainly more interesting.

Chapter 2

Fluid dynamics without viscosity

The goal will be to describe the dynamic properties of the fluid at every point in space for some infinitely small volume. One can imagine we want to know how a small part of dust would move in the fluid. In order to be able to deal with the math in bits and pieces we will first ignore the internal friction in the fluid. Or in other words we will ignore the viscous forces. However, it should be noted that a fluid with zero viscosity is not a simple approximation of a real fluid, it is actually a very exotic state of matter comparable to what a superconductor is for electric transport. The only known example is superfluid Helium and this shows some incredible behaviour.

2.1 Equation of motion

As mentioned above, we want to describe all properties of the fluid for every point in space. This will clearly concern the pressure p and the velocity $\vec{v}$, but also the density ρ and the temperature T need to be described. One can go even further and also include what type of molecule is where, what its orientation is, and what quantum state it is in. These last points certainly go too far for this lecture, but even without them we will need to make some simplifications. The first is that we say that the temperature is not an independent property, but that it can be determined from the pressure and density.

The second assumption that we will often make is that the fluid is incompressible, and that thus the density is constant. This assumption is typically valid if the velocities involved are significantly lower as the speed of sound in the fluid. If we combine this with the conservation of matter in a continuity equation we obtain a useful expression. The mass flow can be expressed by $\rho\vec{v}$ and the total mass flow over some closed surface S must be equal to the change of mass M within this surface

$$\oiint_S (\rho\vec{v}) \cdot d\vec{S} = \frac{dM}{dt} \tag{2.1}$$

Using Gauss's theorem we can also express this locally with the divergence of the mass flow

$$\nabla \cdot (\rho\vec{v}) = -\frac{\partial\rho}{\partial t} \tag{2.2}$$

Now if ρ is constant then we obtain that the divergence of the velocity is zero

$$\nabla \cdot \vec{v} = 0 \qquad \text{(if incompressible)} \tag{2.3}$$

Which in principle states that there are no spontaneous sources of flow.

We are now ready to write Newton's law ($\vec{F} = m\vec{a}$) per unit volume whereby we use Eq. 1.4 for the force

$$\vec{f} = \rho\vec{a} = -\nabla p - \rho\nabla\Phi + \vec{f}_{visc} \tag{2.4}$$

Here $\vec{f}_{visc}$ represents the internal viscous forces which we set to zero in this chapter.

The next step will be to find an expression of the acceleration in terms of the velocity, which is our quantity of interest. One would be inclined to say that $\vec{a} = \frac{\partial\vec{v}}{\partial t}$ but this incorrect. This expression describes the acceleration at some fixed point in space, but not of the small unit volume, or dust particle, that we are following in the fluid. Instead, the acceleration is given by the change in velocity when going from point 1 to point 2. The velocity at point 1 is

$$\vec{v}_1 = \vec{v}(x, y, z, t)$$

and when going to point 2 it has changed by $\Delta\vec{v}$ and becomes

$$\vec{v}_2 = \vec{v}(x + \Delta x, y + \Delta y, z + \Delta z, t + \Delta t)$$

If we now consider that the change in a given direction is given by the velocity component along this direction multiplied by the time we can write $\Delta x = v_x\Delta t$, $\Delta y = v_y\Delta t$, and $\Delta z = v_z\Delta t$ and insert this to obtain

$$\begin{aligned}
\vec{v}_2 &= \vec{v}(x + v_x\Delta t, y + v_y\Delta t, z + v_z\Delta t, t + \Delta t) && (2.5)\\
&= \vec{v}(x, y, z, t) + \frac{\partial\vec{v}}{\partial x}v_x\Delta t + \frac{\partial\vec{v}}{\partial y}v_y\Delta t + \frac{\partial\vec{v}}{\partial z}v_z\Delta t + \frac{\partial\vec{v}}{\partial t}\Delta t && (2.6)
\end{aligned}$$

In the last step we went from a notation where the vector component are separate to a complete vectorial expression.

Now the acceleration becomes the full change in velocity ($\vec{v}_1 - \vec{v}_2$) divided by the time interval Δt

$$\vec{a} = \frac{\vec{v}_1 - \vec{v}_2}{\Delta t} = v_x \frac{\partial \vec{v}}{\partial x} + v_y \frac{\partial \vec{v}}{\partial y} + v_z \frac{\partial \vec{v}}{\partial z} + \frac{\partial \vec{v}}{\partial t} \tag{2.7}$$

As derived in the exercises this can be shortened into

$$\vec{a} = (\vec{v} \cdot \nabla)\vec{v} + \frac{\partial \vec{v}}{\partial t} \tag{2.8}$$

We can now insert Eq. 2.8 in Eq. 2.4 reshuffle the terms and leave out the viscosity to obtain

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{\nabla p}{\rho} - \nabla \Phi \tag{2.9}$$

This is our central equation of motion of a fluid in the absence of viscosity. Because we set out to determine our expression for every point in space, the pressure, density, and potential are scalar fields, and the velocity is a vector field.

The $(\vec{v} \cdot \nabla)\vec{v}$ term is not very intuitive, but it contains most of the information about the spatial dependency of the velocity field. Therefore it is instructive to rewrite this term and to introduce the **vorticity** $\vec{\Omega}$. This is a vector field, but we will drop the vector sign for simplicity. The first step is to rewrite the term using a vector identity

$$(\vec{v} \cdot \nabla)\vec{v} = (\nabla \times \vec{v}) \times \vec{v} + \frac{1}{2}\nabla(\vec{v} \cdot \vec{v}) \tag{2.10}$$

Now we define the vorticity as

$$\Omega = \nabla \times \vec{v} \tag{2.11}$$

and we simplify the scalar product in the last term to obtain

$$(\vec{v} \cdot \nabla)\vec{v} = \Omega \times \vec{v} + \frac{1}{2}\nabla v^2 \tag{2.12}$$

Inserting this in Eq. 2.9 we get

$$\frac{\partial \vec{v}}{\partial t} + \Omega \times \vec{v} + \frac{1}{2}\nabla v^2 = -\frac{\nabla p}{\rho} - \nabla \Phi \tag{2.13}$$

This will be the reference point for some of our discussion to come.

The vorticity represents the circulation of the velocity around any point and is more easily to consider as the $(\vec{v} \cdot \nabla)\vec{v}$ term above. If $\Omega = 0$ the flow is called **irrotational** and otherwise rotational or said to contain vorticity. However, we have to be careful. For example, a free vortex like a whirlpool or outflow of a basin is irrotational because the angular velocity decreases with one over the distance to the centre $\frac{1}{r}$. As a result, the curl of the velocity becomes zero at any point of the vortex except the centre. The piece of dust caught in this whirlpool will thus always face the same direction. In a forced vortex, where the rotation is induced by an external force, for example a rotating cylinder, there is vorticity. In this case the angular velocity is constant and the curl of the velocity, or Ω, takes a value of twice the local angular velocity. On the other hand, a steady flow along a single direction (parallel flow), but with a velocity gradient perpendicular to this direction, is rotational. $\Omega = 0$ only exactly at the centre line and takes finite opposite values on either side of this. Illustrations of the vorticity are provided in the lecture.

In a next step we can also eliminate the pressure from our equation of motion if the fluid is incompressible. To do this we take the curl $(\nabla\times)$ of both sides of Eq. 2.13 and use that the curl of any gradient is always zero. We then get

$$\frac{\partial \Omega}{\partial t} + \nabla \times (\Omega \times \vec{v}) = 0 \tag{2.14}$$

Together with Eq. 2.11 and Eq. 2.3 this now completely describes the velocity field. We can calculate the velocity from the vorticity, use this to calculate the change of vorticity (Eq. 2.14) and the change of velocity, and so on. We have, of course, sacrificed any knowledge on the pressure distribution using this method.

Looking at Eq. 2.14 we find a peculiarity. If the vorticity is zero for some time t then also $\frac{\partial \Omega}{\partial t} = 0$ and $\Omega = 0$ everywhere for all time. This means it is impossible to induce vorticity in an irrotational flow. This is counterintuitive as we know we can stir our tea, and one of the reasons why the exclusion of viscosity is far from realistic. We will come back to this in the next chapter.

2.2 Steady flow and Bernoulli

A **steady flow** is defined by the fact that the flow pattern does not change with time and thus that $\frac{\partial \vec{v}}{\partial t} = 0$. Remember that this is the velocity change at any given point and absolutely does not mean that everything is stationary. The velocity can still change as a function of position. This will now allow us to define the concept of **streamlines**. These are lines tangent to $\vec{v}$ and indicate the trajectory that a fluid (or dust) particle *would* take. They are similar to the field lines that we will encounter in electromagnetism. One can also define streamlines if the flow is *not* steady, but in this case they change with time and are not the trajectory of a fluid particle.

To derive **Bernoulli's theorem** we take the scalar product of the velocity ($\vec{v}\cdot$) and Equation 2.13 for steady flow ($\frac{\partial \vec{v}}{\partial t} = 0$) and realise that $\vec{v} \cdot (\Omega \times \vec{v}) = 0$ to obtain

$$\vec{v} \cdot \nabla(\frac{p}{\rho} + \Phi + \frac{1}{2}v^2) = 0 \tag{2.15}$$

If we now look along a streamline we know that by definition $\vec{v} \neq 0$ and thus the gradient term has to be zero. If the gradient is zero, it means that the object has to be constant and we thus get

$$\frac{p}{\rho} + \Phi + \frac{1}{2}v^2 = \text{const.} \tag{2.16}$$

whereby the value of the constant can depend on the streamline and we thus have to apply this **along a streamline**. This is Bernoulli's theorem, where often it is used that $\Phi = gz$.

If the flow is irrotational ($\Omega = 0$) and steady we directly obtain the result of Eq. 2.16 from Eq. 2.13 and the result is valid everywhere; i.e. the constant is the same in the whole flow.

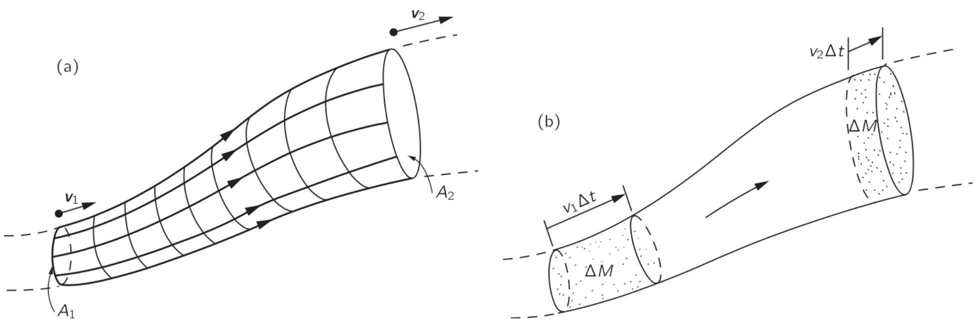

Figure 2.1: Bundle of streamlines and mass transport. (From Feynman lectures)

It is illustrative to also derive Bernoulli's theorem from conservation laws. To do this we consider a bundle of streamlines, or flow tube, as illustrated in Figure 2.1. By definition there is no velocity component, or flow, perpendicular to the streamlines and thus no fluid can leave the tube through the side surfaces. Conservation of mass now tells us that per unit time as much mass ΔM must enter the tube as that leaves it on the other side because nothing can leave the tube in any other way. This mass flow, or flux, is given by

$$\Delta M = \rho_1 A_1 v_1 \Delta t = \rho_2 A_2 v_2 \Delta t \tag{2.17}$$

Where A_1 and A_2 are the area at the entrance and exit of the tube, respectively. Per unit time we thus get that $\rho A v$ is constant and for an incompressible fluid

$$v \propto \frac{1}{A} \tag{2.18}$$

Thus if the area becomes smaller the flow has to be faster, which is rather intuitive.

As a next step we consider the work done by the pressure and equate this to the energy change. Work is force times distance, leading to

$$p_1 A_1 v_1 \Delta t - p_2 A_2 v_2 \Delta t = \Delta M (E_2 - E_1) \tag{2.19}$$

where E is the energy per unit mass which is composed of the kinetic ($\frac{1}{2}v^2$), potential (Φ), and internal/thermal energy (U). Thus we obtain

$$\frac{p_1 A_1 v_1 \Delta t}{\Delta M} - \frac{p_2 A_2 v_2 \Delta t}{\Delta M} = \frac{1}{2}v_2^2 + \Phi_2 + U_2 - \frac{1}{2}v_1^2 + \Phi_1 + U_1 \tag{2.20}$$

Using our expression for the mass flow Eq. 2.17 we see that the first term can be rewritten as $\frac{p}{\rho}$ which yields for a streamtube

$$\frac{p_1}{\rho_1} + \frac{1}{2}v_1^2 + \Phi_1 + U_1 = \frac{p_2}{\rho_2} + \frac{1}{2}v_2^2 + \Phi_2 + U_2 \tag{2.21}$$

For an incompressible fluid the density and internal energy are constant and we thus obtain the same expression as Bernoulli's theorem in Eq. 2.16.

2.3 Applications of Bernoulli

There are many applications of the Bernoulli theorem, whereby most are based on the relationship between velocity and pressure if the potential is constant. A higher velocity will lead to a lower pressure. This will lead to the shower curtain always being drawn inwards, to doors slamming in a draft, train windows rattling with a passing train, the possibility to let something fly that is heavier than air, and hurricanes to lift off roofs from houses. However, we will start with an example where the pressure is constant.

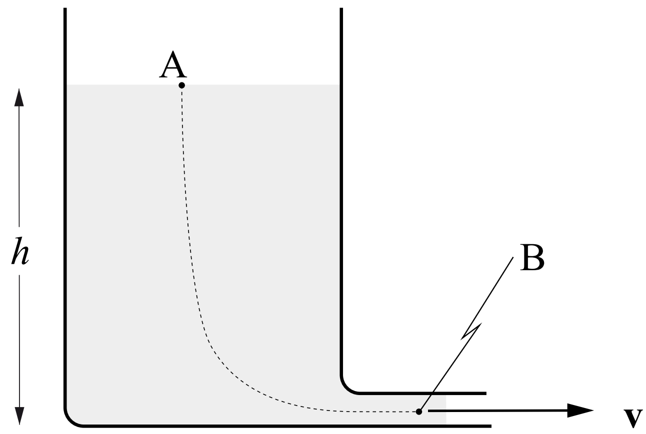

Figure 2.2: Torricelli flow

Consider the situation sketched in Figure 2.2 illustrating the flow from a large tank filled with liquid. For simplicity we consider the surface at the top of the tank to be much larger as the surface of the opening. This way we can ignore the flow speed at the top surface and set it to zero. We set the potential at this surface as a reference and set it to zero, thus the potential at the outflow is $-gh$. The pressure at the top surface and the outlet are both the external air pressure p_0 and putting this in Eq. 2.16 we obtain

$$p_0 = p_0 + \frac{1}{2}\rho v_{out}^2 - \rho g h \tag{2.22}$$

and thus

$$v_{out} = \sqrt{2gh} \tag{2.23}$$

This is the so-called **Torricelli flow**.

As a next example we consider the flow through a horizontal tube with cross section A_1which is at position 2 reduced to $A_2 < A_1$ and afterwards goes back to A_1. From Eq. 2.18 it directly follows that the velocity $v_2 > v_1$.

Using Eq. 2.16 with constant potential because of the same height for the middle streamline, we obtain

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$$

and directly see that $p_2 < p_1$. If at position 2 another small tube is placed perpendicular to the flow, we obtain suction through to tube due to the lower pressure. This pressure reduction forms the basis for a carburettor in a combustion engine, or a vaporiser. This is generally referred to as the **Venturi effect**. With a slight modification this can also be used to construct a pump without any moving parts, but where the rapid flow of water or oil creates an under pressure.

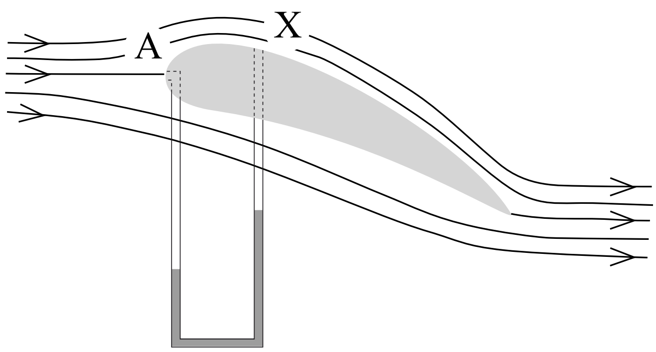

Figure 2.3: Pitot tube for speed measurements

For boats and especially for aircraft it is complicated, but essential, to measure the speed with regard to the water or air surrounding it. Also here Bernoulli provides a solution in the form of the **Pitot tube** illustrated in Fig. 2.3. Again, the potential at point A and X is the same. Furthermore, because point A is a stopping point for the flow, the velocity here $v_A = 0$. Now Eq. 2.16 simply becomes

$$p_A = p_X + \frac{1}{2}\rho v_X^2$$

and thus

$$v_X = \sqrt{\frac{2}{\rho}(p_A - p_X)}$$

where the pressure difference is measured by, for example, the height difference of a fluid in the tube and v_X is the speed of the object with respect to the surrounding.

The wings on an airplane or hydrofoil, or the rotor of a helicopter, are designed such that the density of the streamlines of the fluid is higher above as below. Thus the velocity of the fluid is also higher above as below and accordingly the pressure is lower above as below. This pressure difference creates an upward force (lift) that keeps an airplane in the air. The same effect is responsible for the fact that roofs are lifted up from buildings in storm.

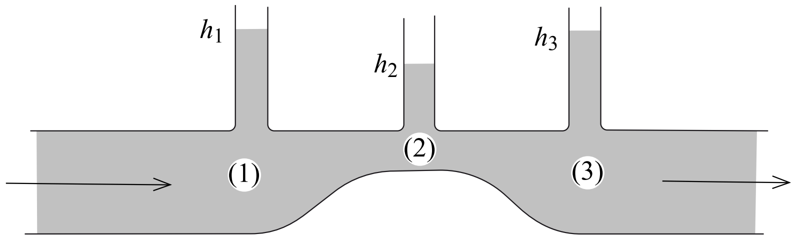

Figure 2.4: Venturi flow meter

Many other applications can be considered, but, here, as a last example we treat the **Venturi flow meter** shown in Figure 2.4 because it forms the transition to viscous flow. The cross sections A_1, A_2 and A_3 are known with $A_1 = 2A_2$ and we measure the heights h_1 and h_2 (and h_3). These heights are directly related to the pressure $p_1 = gh_1$ and $p_2 = gh_2$. In accordance with Eq. 2.18 we get that $v_2 = 2v_1$ and putting everything together in Eq. 2.16 we obtain

$$v_1 = \sqrt{\frac{2}{3}g(h_1 - h_2)}$$

and can thus determine the speed of flow.

If we look more carefully at h_3 we see that it is slightly lower as h_1 although the cross section is the same. This means that the conservation of energy that forms the basis of Bernoulli's theorem is not entirely valid. Somewhere energy has to be "lost" and this is due to internal friction in the fluid. This internal friction is called the **viscosity** and the inclusion of this in our equation of motion forms the basis of the next chapter.

Chapter 3

Flow in viscous fluids

There is a wide range of everyday observations that indicate that the treatment of the flow of fluids, as presented in the previous chapter, is incomplete. There is the pressure drop over a tube, the fact that one can induce vorticity, and other observations like turbulence and friction of a solid moving through a fluid. All these effects, and more, follow from the viscosity of the fluid. We will first derive the coefficient of viscosity and then change our equations of motion including this extra term and look at the consequences of this inclusion.

3.1 The coefficient of viscosity

We start with the observation that the velocity of a fluid is zero with respect to the surface of a solid. If the solid is stationary, a thin layer of the fluid will also stand still, and if the solid is moving, the fluid next to it will move with the same velocity. Now we consider two very large parallel plates with area A and distance d from each other, with a fluid in between. The bottom plate is not moving and the top plate moves with a velocity v_0. This will induce a velocity gradient in the fluid whereby the bottom part of the fluid is standing still and the top part also moves with v_0. Due to the friction in the fluid a force F needs to be applied to keep the plate moving with constant velocity. The **coefficient of viscosity** η with units [Ns/m^2] is now defined as:

$$\frac{F}{A} = \eta \frac{v_0}{d} \tag{3.1}$$

Here it should be realised that the force is applied parallel to the surface and thus represents a stress (and not a pressure).

In order to describe this stress accurately it is helpful to consider the deformation of a solid as illustrated in Figure 3.1. This so-called shear strain can be described by the displacement along the x-direction u_x as a function of y and vice versa (u_y). If the deformation is small, we can approximate $\tan\theta \approx \theta$ and thus $u_x = \theta y$ and $u_y = \theta x$. Using tensor notation one could now (wrongly) think that it is a good idea to describe the strain $\underline{e}$ of the deformation in x along the y-direction as

$$e_{xy} = \frac{\partial u_x}{\partial y} = \theta$$

and similar for

$$e_{yx} = \frac{\partial u_y}{\partial x} = \theta$$

. However, this causes a problem if we assume a negative deformation $-u_y$. In this case we would just have a trivial rotation of θ and no deformation at all, whereas our strain tensor would be non-zero.

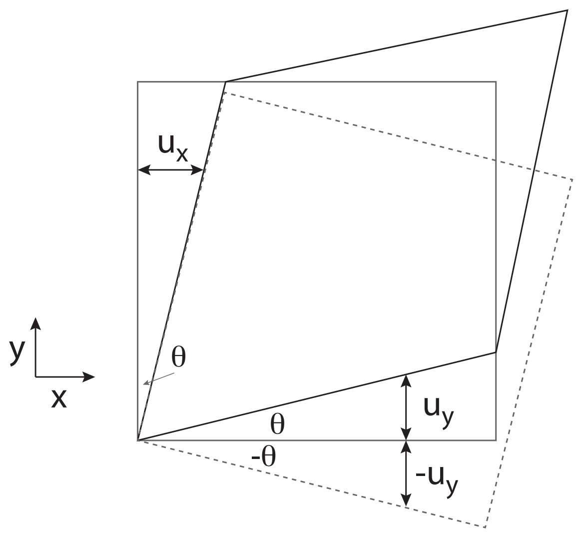

Figure 3.1: Deformation of a square (solid line) as compared to its rotation (dashed line)

To resolve this problem, and thus to avoid that we describe rotation as deformation, we need to take a combination of the strain along the x and y directions. The simplest is a linear combination

$$e_{xy} = e_{yx} = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) \tag{3.2}$$

Where we directly see that for a rotation we obtain zero. This of course means that the two tensor elements e_{xy} and e_{yx} are the same, and the strain tensor is thus symmetric. It should be noted that this does not indicate that the deformations along the two directions are always identical. One could have a finite u_x and $u_y = 0$ or similar. It just means that in our description the two are mixed in a single number in the strain tensor.

Now we can apply this to the stress, which is the rate of change ($\frac{\partial}{\partial t}$) of the strain, and sometimes referred to as the shear rate. With $\frac{\partial v_x}{\partial y} = \frac{\partial}{\partial t}\frac{\partial u_x}{\partial y}$ we can now write out the stress that needs to be applied to have this gradient of velocity. Using the notation of the symmetric rank 2 stress tensor $\underline{S}$ we can write

$$S_{xy} = S_{yx} = \eta \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right) \tag{3.3}$$

and similar expressions for S_{yz} and S_{zx}. Here the viscosity η is the coefficient proportionality. In tensor notation S_{xy} indicates the stress at the surface defined by $\hat{x}$ along the $\hat{y}$ direction (or at the surface $-\hat{x}$ along $-\hat{y}$).

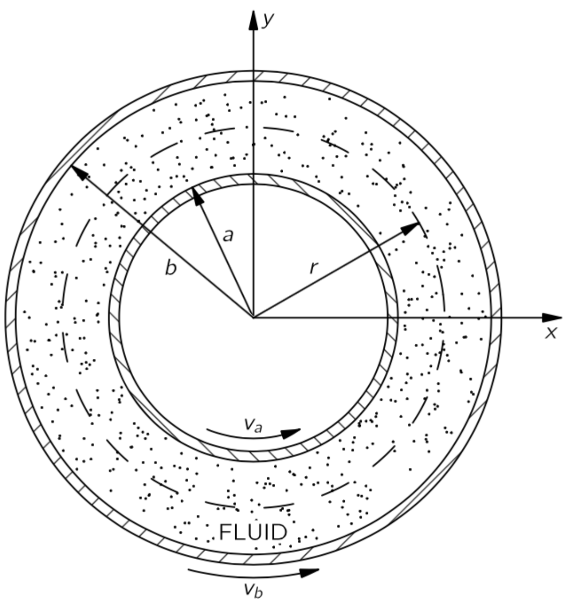

Figure 3.2: Measuring viscosity with two concentric rotating cylinders. (From Feynman lectures)

To illustrate the idea we will now consider a method to measure the viscosity of a fluid. It is based on two concentric rotating cylinders with a fluid in between as illustrated in Figure 3.2. The inner (outer) cylinder has radius a (b) and rotates with a velocity v_a (v_b), whereby both rotate in the same direction. We wait until a steady flow is established and in this case we can assume from symmetry that the velocity is only a function of r. For the angular velocity we can write $\omega = \frac{v}{r}$ and for the x and y coordinate we obtain: $x = r\cos\omega t$ and $y = r\sin\omega t$. For the velocities we thus get

$$v_x = \frac{\partial x}{\partial t} = -r\omega\sin\omega t = -\omega y \tag{3.4}$$

$$v_y = \frac{\partial y}{\partial t} = r\omega\cos\omega t = \omega x \tag{3.5}$$

The stress tensor from Eq. 3.3 thus becomes

$$\underline{S}_{xy} = \eta\left(\frac{\partial(\omega x)}{\partial x} - \frac{\partial(\omega y)}{\partial y}\right) = \eta\left(x\frac{\partial\omega}{\partial x} - y\frac{\partial\omega}{\partial y}\right) \tag{3.6}$$

If we now consider that for $y = 0$ then $x = r$ and take the rotational symmetry into account i.e. our choice of the axes is arbitrary. We thus obtain

$$\underline{S}_{xy} = \eta r\frac{\partial\omega}{\partial r} \tag{3.7}$$

Next we calculate the torque τ of the stress on the fluid, whereby the torque is given by the moment (arm) times the area (the cylinders have length l) times the stress given above

$$\tau = r2\pi rl\underline{S}_{xy} = 2\pi l\eta r^3\frac{\partial\omega}{\partial r} \tag{3.8}$$

Because the flow is steady, the torque is independent of r. Thus $r^3\frac{\partial\omega}{\partial r}$ is constant (C_1) and we get that $\frac{\partial\omega}{\partial r} = \frac{C_1}{r^3}$ and thus $\omega = -\frac{C_1}{2\pi r^2} + C_2$. The two constants can be determined from the boundary conditions that $\omega = \omega_a = \frac{v_a}{a}$ at $r = a$ and similar for $r = b$. We are here only interested in C_1 so we can insert it in Eq. 3.8:

$$C_1 = \frac{2a^2b^2}{b^2 - a^2}(\omega_b - \omega_a)$$

Thus we obtain for the torque

$$\tau = 2\pi l\eta C_1 = \frac{4\pi l\eta a^2b^2}{b^2 - a^2}(\omega_b - \omega_a) \tag{3.9}$$

We can now rotate one of the two cylinders and measure the torque on the other one to obtain η with all the other parameters known.

The coefficient of viscosity is typically strongly dependent on temperature, and for non-Newtonian fluids its also depends on the applied force. Often the so-called **kinematic viscosity** is used which is the coefficient of viscosity divided by the density: $\frac{\eta}{\rho}$. The values of the kinematic viscosity for different fluids are often closer together.

3.2 Viscous flow

We will now include the viscous force in our equation of motion as given in Eq. 2.4 and 2.9, thus aiming to complete the expression

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{\nabla p}{\rho} - \nabla\Phi + \frac{f_{visc}}{\rho} \tag{3.10}$$

To do this, we will need to turn the stress tensor into a force per volume.

Let's start with the stress tensor as defined in Eq. 3.3, but now also including forces that occur during compression of the fluid. If we later consider an incompressible fluid we can set that part to zero again.

$$\underline{S}_{ij} = \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) + \eta^\star \delta_{ij}(\nabla \cdot \vec{v}) \tag{3.11}$$

where $\eta^\star$ is the second coefficient of viscosity, δ_{ij} the Kronecker delta which is equal to 1 if $i = j$ and zero otherwise, and $x_i = x, y, z$. Considering that force is stress times surface and with a bit of algebra (to be included when time available) we obtain for the viscous force per unit volume

$$(f_{visc})_i = \sum_{j=1}^{3} \frac{\partial \underline{S}_{ij}}{\partial x_j} \tag{3.12}$$

and with Eq. 3.11 this becomes

$$(f_{visc})_i = \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \left(\eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \right) + \frac{\partial}{\partial x_j} \left(\eta^\star \delta_{ij}(\nabla \cdot \vec{v}) \right) \tag{3.13}$$

Considering that both coefficients of viscosity are homogeneous and thus $\frac{\partial \eta}{\partial x_i} = \frac{\partial \eta^\star}{\partial x_i} = 0$ we can rewrite this using the nabla operator

$$f_{visc} = \eta \nabla^2 \vec{v} + (\eta + \eta^\star)\nabla(\nabla \cdot \vec{v}) \tag{3.14}$$

It is a good student exercise to write this out and check that one obtains Eq. 3.14. The Laplacian operating on a vector field should be interpreted as

$$\nabla^2 \vec{v} = (\nabla^2 v_x, \nabla^2 v_y, \nabla^2 v_z)$$

and on a scalar field

$$\nabla^2 p = \sum_{j=1}^{3} \frac{\partial^2 p}{\partial x_i^2}.$$

If we now insert Eq. 3.14 in Eq. 3.10 we obtain the **Navier-Stokes** equation for a viscous fluid:

$$\rho \left(\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} \right) = -\nabla p - \rho \nabla \Phi + \eta \nabla^2 \vec{v} + (\eta + \eta^\star)\nabla(\nabla \cdot \vec{v}) \qquad (3.15)$$

And in terms of the vorticity defined in Eq. 2.11 this becomes

$$\rho \left(\frac{\partial \vec{v}}{\partial t} + \Omega \times \vec{v} + \frac{1}{2}\nabla v^2 \right) = -\nabla p - \rho \nabla \Phi + \eta \nabla^2 \vec{v} + (\eta + \eta^\star)\nabla(\nabla \cdot \vec{v}) \qquad (3.16)$$

For an incompressible fluid $\nabla \cdot \vec{v} = 0$ and the expression becomes much simpler. Now we again eliminate the pressure in a similar way as for the derivation of Eq. 2.14. Taking the curl of Eq. 3.16, remembering that ρ is constant, and simplifying yields

$$\frac{\partial \Omega}{\partial t} + \nabla \times (\Omega \times \vec{v}) = \frac{\eta}{\rho} \nabla^2 \Omega \qquad (3.17)$$

where we directly see the use of the dynamic viscosity.

Equation 3.17 represents a diffusion equation of the vorticity Ω whereby the term on the right is like a damping term. In the following we will use this equation to try to understand what can happen with the flow of a real fluid.

3.3 Helmholtz theorems

The Helmholtz theorems concern the vorticity Ω and allow to have a closer look at this vector field. We especially consider the **vortex lines** which are like the streamlines but for the vorticity. They are of course intricately connected with the streamlines through Eq. 2.11 and they can be considered to encircle each other. An example of this is illustrated in Fig. 3.3.

From the definition of the vorticity $\Omega = \nabla \times \vec{v}$ it directly follows that the divergence of the vorticity is always zero

$$\nabla \cdot \Omega \equiv 0 \tag{3.18}$$

This means that *vortex lines have no source or drain*, and always close on themselves. We will see something similar for the magnetic field lines later in this lecture.

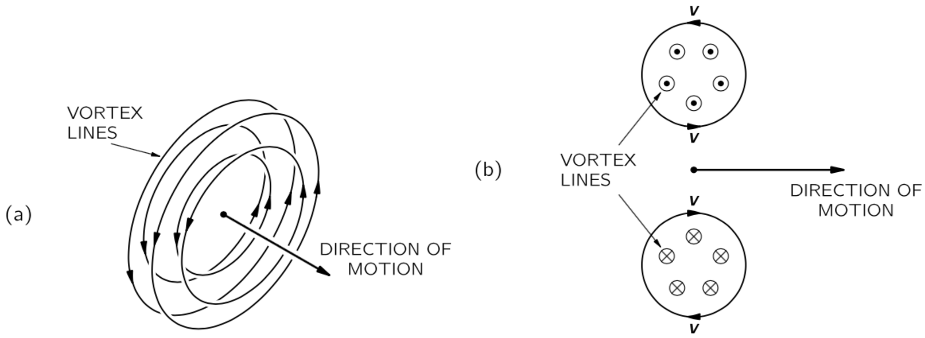

Figure 3.3: Torus of vortex lines and the connection between vortex lines and streamlines. (From Feynman lectures)

If we now consider fluids with very low viscosity ($\eta \approx 0$) the second theorem is that *vortex lines move with the fluid*. Thus the product of vorticity and area A is constant:

$$\Omega_1 A_1 = \Omega_2 A_2 \tag{3.19}$$

It is thus always the same particles that make up a vortex, and if the fluid is deformed by pressure forces the the vortex lines deform along. This is crucially different from, for example, a wave. The second theorem can be derived from the conservation of angular momentum in the absence of friction (viscosity). For this we consider a bundle of vortex lines at time t_0 and some time later when the fluid has moved or changed shape. Taking a cylindrical shape for simplicity, the moment of inertia of this bundle is $I = MR^2$ with R the radius of the cylinder and M its mass. The angular velocity of the fluid particles around the vortex lines is by definition proportional to Ω and the angular momentum becomes $MR^2\Omega$. Conservation of angular momentum means

$$M_1 R_1^2 \Omega_1 = M_2 R_2^2 \Omega_2 \tag{3.20}$$

For an incompressible fluid $M_1 = M_2$ and further $A = \pi R^2$ and we thus obtain Eq. 3.19. This is nicely illustrated by the vortex cannon shown during the lecture which produces a torus of vortex lines like in Fig. 3.3.

The third theorem we have already encountered in the previous chapter, and thus also only applies for $\eta \approx 0$. It states that *if for some time t the vorticity is zero, then it will be zero for all t.*

The vortex lines also help to understand what happens if we do include viscosity. Namely, the vorticity spreads over the fluid and moves through it. In the vortex cannon we saw this by the smoke ring becoming thicker. More importantly, if $\eta \neq 0$ vorticity can be (spontaneously) created in the fluid, leading to turbulence and drag forces.

3.4 Magnus effect

We consider a cylinder in a flow to the right, or equivalently, a cylinder moving to the left through a stationary fluid as illustrated in Figure 3.4. The velocity of the fluid relative to the cylinder is v_∞ far away from the object If the viscosity is zero, then there is symmetry along the horizontal and vertical direction and there is no net force on the cylinder.

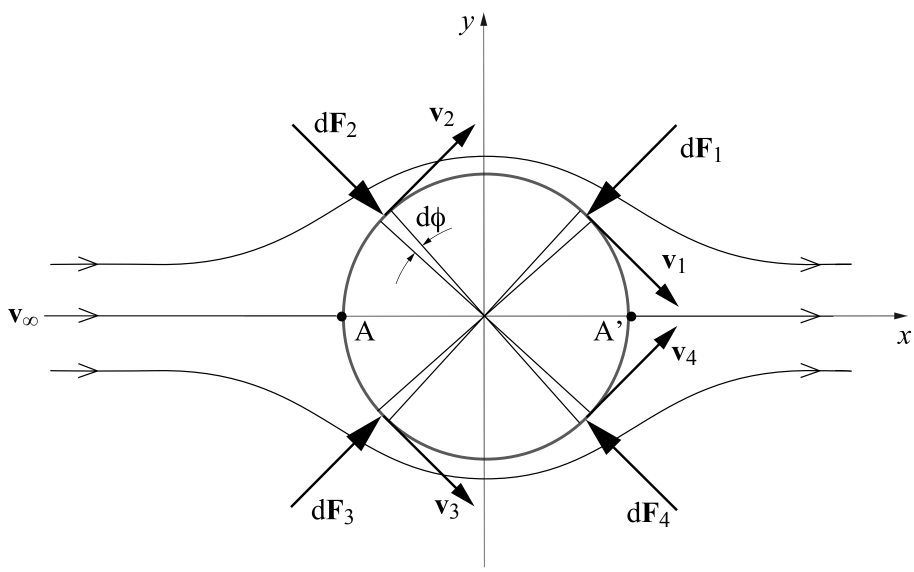

Figure 3.4: Cylinder in flow

If the fluid is viscous ($\eta \neq 0$) we still have symmetry along the x axis, but the symmetry with respect to the y axis is broken because due to the friction. As a result $dF_3 = dF_2 \neq dF_1$ and there is a net force along the $\hat{x}$ direction against the movement of the object or dragging it along in the flow. This is the so-called **drag force**. It has been empirically determined for various objects and here only the **Stokes flow** around a sphere of radius R is given:

$$F_x = 6\pi\eta R v_\infty \tag{3.21}$$

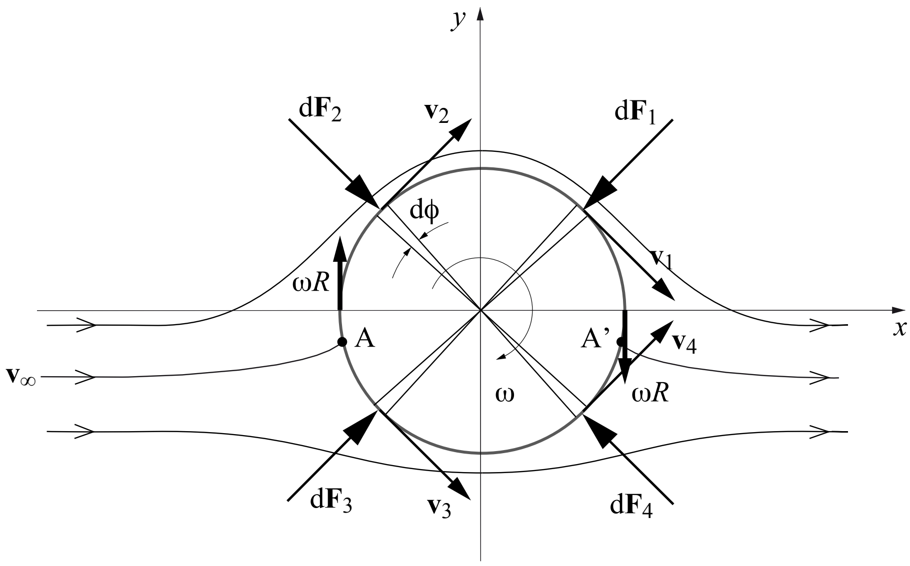

Figure 3.5: Rotating cylinder in flow

Now let's return to our cylinder, but consider that it is rotating with angular velocity ω in the clockwise direction i.e. $\vec{\omega} = -\omega_z$ as illustrated in Figure 3.5. Because at the surface the relative velocity between cylinder and fluid is zero, the rotation drags the fluid flow along. On the upper part ($y > 0$) the original flow velocity and the additional velocity from the rotation add up to increase the total velocity, and for $y < 0$ the total velocity is decreased: $\vec{v} = v_\infty \hat{x} + \vec{\omega} \times \vec{R}$. Thus $v_2 > v_3$ and $v_1 > v_4$ and if we assume that Bernoulli is still valid we obtain that $p_2 < p_3$ and $p_1 < p_4$. This means that the pressure forces on the upper part are lower as on the lower part and there is thus a net force along the positive $\hat{y}$-direction. If the direction of rotation is reversed then also the direction of the force is reversed.

It is left as a student exercise, but under the assumption that Bernoulli can be applied, we obtain that

$$F_y = 2\pi\omega R^2 \rho L v_\infty \tag{3.22}$$

where L is the length of the cylinder. We can generalise this for any object or flow by considering the circulation Γ, which describes the flow around an object. For the rotating cylinder we obtain

$$\Gamma = \oint \vec{v} \cdot d\vec{r} = 2\pi\omega R^2 \tag{3.23}$$

This means we can rewrite Eq. 3.22 as

$$F_y = \Gamma \rho L v_\infty \tag{3.24}$$

This expression is valid for any object, where L plays the role of the dimension perpendicular to the plane we are considering for the force. It also applies if the object is not rotating. The circulation can still be obtained by integrating the velocity around an object, or if we rewrite this using Stoke's theorem:

$$\Gamma = \iint_A (\nabla \times \vec{v}) \cdot d\vec{A} = \iint_A \Omega \cdot d\vec{A} \tag{3.25}$$

where A is a surface encompassing the object of interest. Equation 3.24 is the **Kutta-Joukowski** formula and is routinely used in engineering problems to calculate the (lift) force on objects. It again shows the usefulness of the vorticity field in fluid dynamics.

The force on a rotating object in a flow, and its subsequent movement, is referred to as the **Magnus effect**. It forms the basis of spin and slice in tennis and related sports, or the curving of a football. In the more general formulation of Eq. 3.24 it is also applied for the lift on airplane wings and the changes, or breakdown, of lift if the flow becomes turbulent.

3.5 Poiseuille flow

We will discuss here one more important example of flow before moving to the topic of turbulence. Let's consider a cylinder (or pipe) with length L and radius R, oriented along the x-direction. A pressure gradient $\frac{dp}{dx}$ induces the flow of an incompressible fluid along the x-axis. Conservation of mass means that $\frac{\partial v_x}{\partial x} = 0$ (just as much fluid has to go in as out) and we consider a steady flow and thus also $\frac{\partial v_x}{\partial t} = 0$. We explicitly want to find a solution for a **laminar flow** meaning that $v_r = v_\theta = 0$. The cylindrical symmetry imposes a further condition namely that there should be no dependency on θ and thus $\frac{\partial v_x}{\partial \theta} = 0$. Putting all this together we have that $\vec{v} = v_x(r)$

If we put this in the Navier-Stokes equation (Eq. 3.15) and realise that $(\vec{v} \cdot \nabla)\vec{v} = 0$ because only $\frac{\partial v}{\partial r} \neq 0$ we obtain

$$\frac{\partial p}{\partial x} = \eta \nabla^2 \vec{v} = \eta \left(\frac{\partial^2 v_x}{\partial r^2} + \frac{1}{r}\frac{\partial v_x}{\partial r} \right) \tag{3.26}$$

Here $\frac{\partial p}{\partial x}$ is just the pressure gradient as parameter and the solution is rapidly found to be

$$v_x(r) = \frac{r^2}{4\eta}\left(\frac{dp}{dx}\right) + C_1 \ln(r) + C_2 \tag{3.27}$$

We know that for $r = 0$ the velocity has to be finite and thus $C_1 = 0$. To find the other integration constant we consider the boundary condition at $r = R$ where the velocity has to be zero $v_x(R) = 0$ which leads to $C_2 = -\frac{R^2}{4\eta}\left(\frac{dp}{dx}\right)$ and thus for the velocity profile

$$v_x(r) = \frac{1}{4\eta}\left(\frac{dp}{dx}\right)(r^2 - R^2) \tag{3.28}$$

which is the parabolic profile sketched in Figure 3.6. Note that in the example $\frac{dp}{dx} < 0$ as the flow goes to positive x direction.

The maximum velocity is achieved for $r = 0$ and is thus

$$v_{max} = \frac{1}{4\eta}\left(\frac{dp}{dx}\right)$$

and the volume flow can be obtained by integrating the velocity over the area of the cylinder:

$$Q_{vol} = \iint_A \vec{v} \cdot d\vec{A} = \int_0^R v_x(r) 2\pi r dr \tag{3.29}$$

$$= \frac{\pi}{2\eta}\frac{dp}{dx}\int_0^R (R^2 r - r^3) dr = \frac{\pi R^4}{8\eta}\frac{dp}{dx} \tag{3.30}$$

The pressure gradient can be considered as the driving force and thus the factor $\frac{\pi R^4}{8\eta}$ is the resistance of the pipe per unit length.

It should be noted that Eq. 3.27 can also be used to calculate the flow profile in other situations. For example for concentric pipes, or if there is

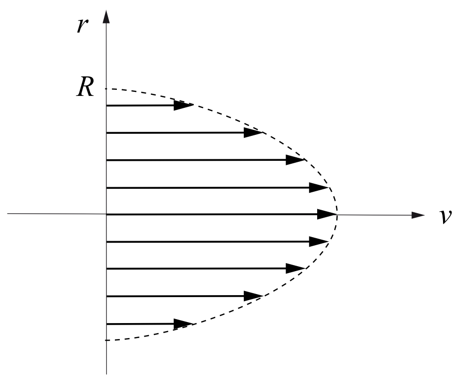

Figure 3.6: Velocity profile for Poiseuille flow

no pressure gradient but the pipe is moving. In these cases the boundary conditions, and thus the final expression for $v_x(r)$, are different.

To obtain a feeling for the nature of the vorticity, it is illustrative to calculate the vorticity for the Poiseuille flow described above. In calculating the curl of $\vec{v}$ only the $\frac{\partial v_x}{\partial r}$ term is unequal to zero and this appears in the Ω_θ component. We thus get that

$$\Omega_\theta = \left(\frac{\partial v_r}{\partial x} - \frac{\partial v_x}{\partial r}\right) = -\frac{r}{4\eta}\left(\frac{dp}{dx}\right) \tag{3.31}$$

The vortex lines thus circle around the axis of the cylinder as illustrated in Figure 3.7. The magnitude of the vorticity, and thus the density of the vortex lines, increase with r. Surfaces of constant vorticity form cylinders with fixed r.

3.6 Reynolds number

The Navier-Stokes equation including viscosity (Eq.3.15) is not only extremely difficult to solve once we move beyond steady and laminar flow, it is not even clear whether the obtained solution is unique. Furthermore, it shows very chaotic behaviour, where the solution strongly depends on the initial boundary conditions. However, solving this equation is extremely important for a wide range of applications, ranging from weather predictions to vehicle design. Using powerful supercomputers and iterative algorithms

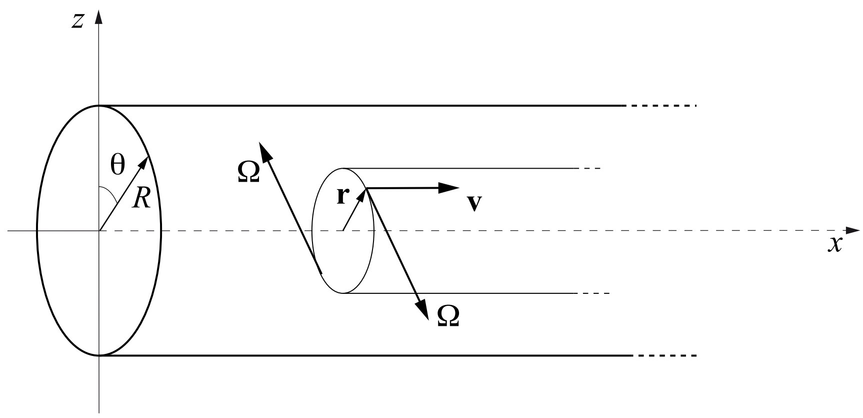

Figure 3.7: Vorticity in Poiseuille flow

the solutions have become better and better, but often rely on wind tunnel tests. One very important aspect of the Navier-Stokes equation is that it is scalable. If a solution is found for a given object and flow, it can be applied to larger or smaller objects or flows with different velocity or viscosity, as long as a special ratio between these factors stays constant. This is the **Reynolds number** $\mathcal{R}e$ which will be derived here.

We will again consider the flow of a fluid with viscosity η and density ρ around a cylinder with diameter $D = 2R$ with its axis along the z-direction. The fluid velocity far away from the cylinder is along the x-direction and $\vec{v} = v_x = v_0$. Our velocity field is defined by Eq. 3.17 and Eq. 2.11 with the boundary conditions that $\vec{v} = 0$ for $x^2 + y^2 = \frac{D^2}{4}$. The solutions clearly depend on D, v_0, η, ρ and our first step is to rewrite everything in units of the first two. We get that

$$x = x'D \quad y = y'D \quad z = z'D \qquad v = v'v_0 \tag{3.32}$$

Here the variables with a prime are our new variables. For example we now get that $v' = 1$ for $x' \gg 1$.

This change in units also leads to a new time

$$t = t'\frac{D}{v_0} \tag{3.33}$$

Furthermore, we will need to write our derivatives

$$\frac{\partial}{\partial t} \rightarrow \frac{v_0}{D}\frac{\partial}{\partial t} \tag{3.34}$$

$$\frac{\partial}{\partial x} \rightarrow \frac{1}{D}\frac{\partial}{\partial x} \tag{3.35}$$

whereby the last one is representative for all spatial derivatives, summarised in a new ∇'. Lastly, also the vorticity has to be expressed in these new units

$$\Omega = \nabla \times \vec{v} = \frac{v_0}{D}\nabla' \times \vec{v}' = \frac{v_0}{D}\Omega' \tag{3.36}$$

Now we have all the ingredients to rewrite the vorticity diffusion equation Eq. 3.17 in the units of D and v_0.

$$\frac{\partial \Omega'}{\partial t'} + \nabla' \times (\Omega' \times \vec{v}') = \frac{\eta}{\rho v_0 D}\nabla'^2\Omega' \; (= \frac{1}{\mathcal{R}e}\nabla'^2\Omega') \tag{3.37}$$

With the Reynolds number thus defined as

$$\mathcal{R}e = \frac{\rho}{\eta} v_0 D \qquad \text{(cylinder)} \tag{3.38}$$

In general the Reynolds number can be defined considering any characteristic dimension L perpendicular to the flow.

$$\mathcal{R}e = \frac{\rho}{\eta} v L \tag{3.39}$$

Exactly what dimension needs to be taken depends on the object and this is well tabulated for a wide variety of cases.

The consequence of Eq.3.37 is that the flow is the same, but scaled, if the Reynolds number is the same. For example, if we have determined the flow in a wind tunnel for a 1:10 scale model, we know that the found flow will be the same for the real object for 10 times lower flow velocity. Of course, one can simultaneously change the viscosity and density as well, just as long as the Reynolds number stays the same.

3.7 Turbulent flow

One of the most important applications of the Reynolds number is to determine when the transition between laminar and turbulent flow happens. In turbulent flow much energy is lost in the creation of vortices and the flow resistance typically increases. However, there are also regimes where the generation of a turbulent layer around the object actually reduces the resistance. Most of this is rather empirical and tables can be found on the internet (or in the library of STI).

As an example, the derivation of the Poiseuille flow required that the flow was laminar. This is only valid if $\frac{L}{R} > \frac{\mathcal{R}e}{48}$ where the Reynolds number for a pipe is $\mathcal{R}e = \frac{\rho \bar{v} 2R}{\eta}$ with L the length of the pipe and R its radius. If

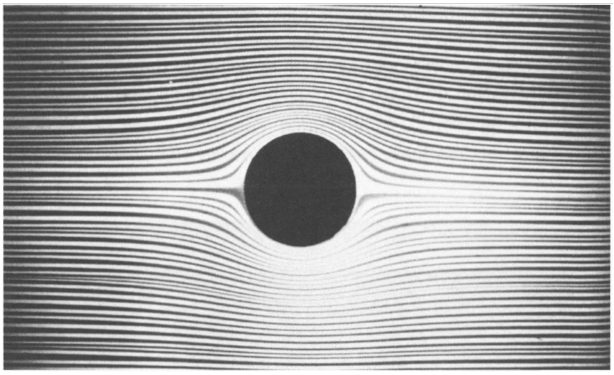

Figure 3.8: Laminar flow around cylinder for $\mathcal{R}e \approx 0$

we are outside this regime, the nice parabolic solution we obtained will no longer be valid.

In this section we will just consider some examples of turbulent flow around a cylinder for different Reynolds numbers, but similar flow patterns exist for different objects.

For a negligible Reynolds number the flow is nicely laminar and symmetric as shown in Figure 3.8. If we increase the flow velocity for the same cylinder, the Reynolds number will increase and around $\mathcal{R}e \approx 13$ we start seeing some vorticity behind the cylinder. For larger $\mathcal{R}e \approx 26$ this has developed into two large symmetric vortices as shown in Figure 3.9.

At a Reynolds number of $\mathcal{R}e \approx 40$ the character of the flow completely changes again and steady solutions no longer exist. The vortices behind the cylinder become unstable and get dragged along by the flow. This always happens in alternating fashion and creates the so-called **Karman vortex street** shown in Figure 3.10. The impressive consequence of the reduced dimensions in the Reynolds number is that such vortex streets also occur in cloud formations behind lone mountain tops, but then on the scale of hundreds of kilometres.

When the Reynolds number increases even further, the flow becomes increasingly noisy as shown in Figure 3.11. In this regime the turbulence can form a boundary layer that actually reduces the drag force and the resistance.

It should be noted that from Eq. 3.37 and Eq. 3.39 it can *not* be concluded that the flow for $\mathcal{R}e \to \infty$ is the same as for $\eta \to 0$. The distinction is clear from Figure 3.11 and the type of results we obtained when ignoring the viscosity. The reason for this difference is the second derivative in Eq. 3.37. This will allow for very rapid variations in the flow to compensate

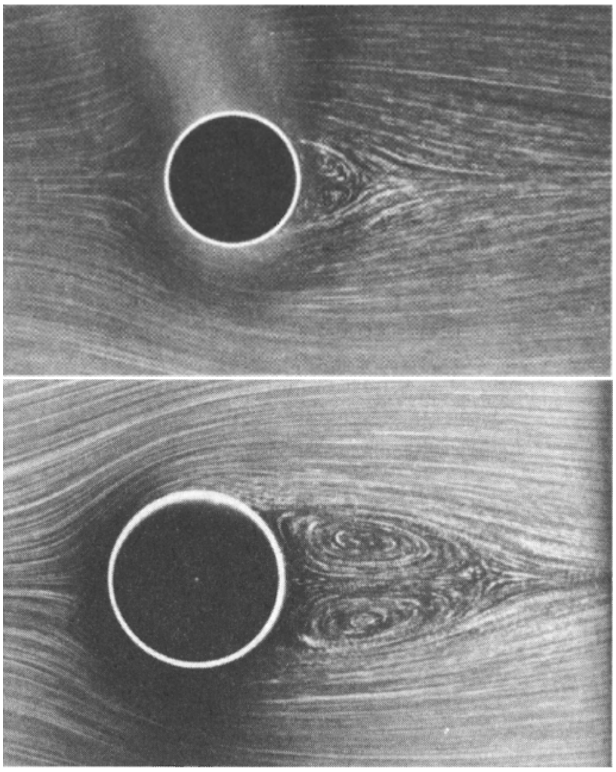

Figure 3.9: Onset of turbulent flow around cylinder for $\mathcal{R}e \approx 13$ and increased turbulence behind cylinder for $\mathcal{R}e \approx 26$

for the increase in $\mathcal{R}e$ and thus the product will not go to zero.

As a last remark, in the derivation of the Reynolds number we ignored the second coefficient of viscosity and the compressibility of the fluid because we used Eq. 3.17 as a starting point. This means that the use of $\mathcal{R}e$ is only valid for velocities well below the speed of sound in the fluid. If we want to go to higher velocities, we have to use the **Mach number**. In this case the flow is the same, but scaled, if both the Reynolds and Mach number are the same.

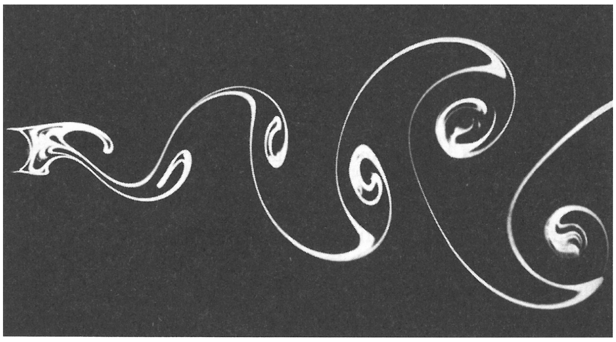

Figure 3.10: Karman vortex street behind a cylinder for $\mathcal{R}e \approx 140$

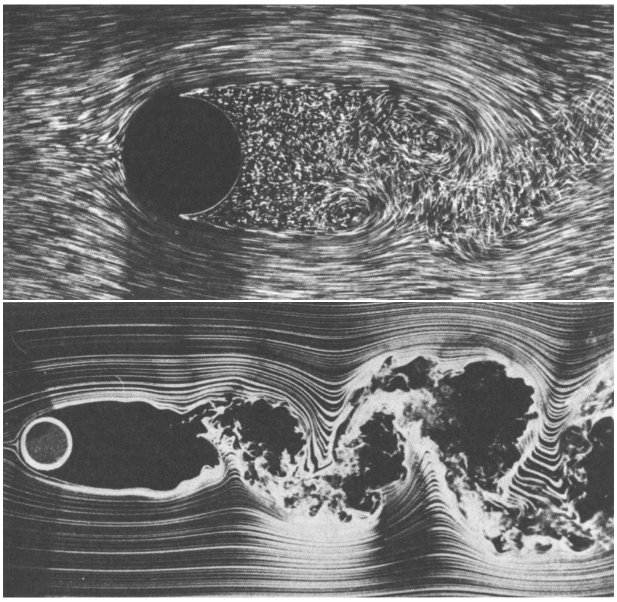

Figure 3.11: Turbulent flow around cylinder for $\mathcal{R}e \approx 2000$ and for $\mathcal{R}e \approx 10000$

Chapter 4

Various end stuff

This script has been written under substantial time pressure and any reader might recognise this. Although care has been taken to avoid any mistakes, I can't guarantee the absence of typos and other errors. Any comments in this respect are more than welcome.

It should be realised that this script is not meant to replace a proper text book and I have no claim with regard to completeness. Many topics would deserve a more in-depth treatment and there are excellent texts that do exactly this. My only aim is to provide an overview of my lecture as given at this level and covering the knowledge that I require from my students.

Lastly, many figures in this script have been taken from a variety of sources and I have not been as strict as I should have been listing these sources. This script is only meant for distribution with the relevant students at the EPFL and not for any further distribution or commercial gain. However, if anyone feels that their copyright has been inflicted I kindly ask them to contact me and the issue will be fixed immediately.