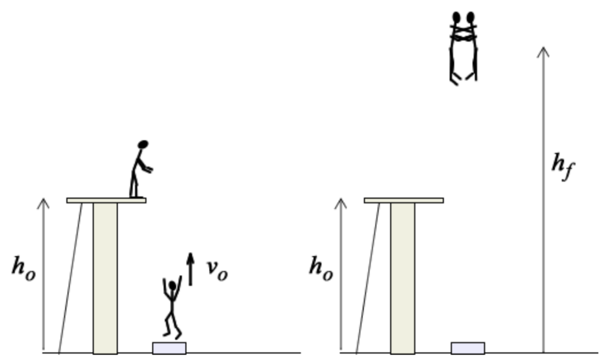
Solutions to Problem Set 7

Momentum and continuous mass transfer

PHYS-101(en)

1. Acrobat and clown



We start by defining our system to include both the acrobat and the clown. The first important observation is that there is a collision between the acrobat and the clown. This collision is completely "inelastic" in that the two bodies collide and stick together after the collision. The details of the collision are determined by the internal forces in the system. Since this is a one-dimensional motion, we will use a coordinate system with the origin at the trampoline and $\hat{y}$ defined to be upwards.

There are two important states to identify in this problem. State 1 is immediately before the collision, at a time that we'll call t_1. At this moment, acrobat has just arrived at the platform of the clown, so both are at the same position

$$y_A(t_1) = y_C(t_1) = h_0, \tag{1}$$

where $y_A(t)$ and $y_C(t)$ are the vertical positions of the acrobat and clown respectively. Additionally, just before grabbing the clown the acrobat has a velocity of $\vec{v}_A(t_1) = v_A(t_1)\hat{y}$.

The collision lasts a time Δt_{coll}. During this time interval, the acrobat grabs the clown.

State 2 is immediately after the collision, at a time t_2. After the collision, the two people rise together with the same velocity $\vec{v}_A(t) = \vec{v}_C(t) = v_{AC}(t)\hat{y}$. The key assumption is that the time over which the collision occurs is very short $\Delta t_{coll} = t_2 - t_1 \approx 0$.

Because the collision is so fast, the impulse delivered by the external gravitational force during the collision is negligibly small. This is the impulse approximation. Because of the impulse approximation, we can say that the total momentum of the system is constant during the collision. If the collision lasts a significant length of time, there would be some slowing down of the acrobat during the collision and we would have to calculate the effect of this. However, by assuming the collision is instantaneous, we can ignore this slowing down and consider the momentum of the system to be conserved.

Now let us analyze the time intervals separated by the two states. **Before state 1**, the acrobat is undergoing projectile motion. From one-dimensional kinematics, the vertical component of the position and velocity of the acrobat is given by

$$y_A(t) = -\frac{g}{2}t^2 + v_{A0}t + y_{A0} = -\frac{g}{2}t^2 + v_0 t \tag{2}$$

$$v_A(t) = -gt + v_{A0} = -gt + v_0 \tag{3}$$

respectively, where $y_{A0} = 0$ and $v_{A0} = v_0$ are the initial position and velocity of the acrobat as given in the problem. At $t = t_1$ the acrobat is at height $y(t_1) = h_0$, so we know

$$y_A(t_1) = h_0 = -\frac{g}{2}t_1^2 + v_0 t_1. \tag{4}$$

Applying the quadratic formula, we find

$$t_1 = \frac{-v_0 \pm \sqrt{v_0^2 - 4(-g/2)(-h_0)}}{2(-g/2)} = \frac{v_0 \mp \sqrt{v_0^2 - 2gh_0}}{g}. \tag{5}$$

Substituting this into equation (3) evaluated at $t = t_1$ allows us to find the velocity immediately before the collision

$$v_A(t_1) = \pm\sqrt{v_0^2 - 2gh_0} = \sqrt{v_0^2 - 2gh_0}, \tag{6}$$

where we have taken the plus sign as we know the physical velocity must be positive.

Between states 1 and 2, we know from the impulse approximation that momentum is conserved

$$\vec{p}_{sys}(t_1) = \vec{p}_{sys}(t_2). \tag{7}$$

Immediately before the collision, the momentum of the acrobat-clown system is only due to acrobat

$$\vec{p}_{sys}(t_1) = \vec{p}_A(t_1) + \vec{p}_C(t_1) = \vec{p}_A(t_1) = m_A v_A(t_1)\hat{y} = m_A\sqrt{v_0^2 - 2gh_0}\hat{y}, \tag{8}$$

where we have used equation (6). Immediately after the collision, the acrobat and clown are moving together, so the total momentum of the system is

$$\vec{p}_{sys}(t_2) = (m_A + m_C)v_{AC}(t_2)\hat{y}. \tag{9}$$

Enforcing conservation of momentum (i.e. substituting equations (8) and (9) into equation (7)) allows us to find the velocity of the acrobat and clown immediately after the collision

$$m_A\sqrt{v_0^2 - 2gh_0}\hat{y} = (m_A + m_C)v_{AC}(t_2)\hat{y} \quad \Rightarrow \quad v_{AC}(t_2) = \frac{m_A}{m_A + m_C}\sqrt{v_0^2 - 2gh_0}. \tag{10}$$

After state 2, the acrobat and clown experience projectile motion. Thus, their position and velocity is given by

$$y_{AC}(t) = -\frac{g}{2}(t - t_2)^2 + v_{AC0}(t - t_2) + y_{AC0} \tag{11}$$

$$v_{AC}(t) = -g(t - t_2) + v_{AC0} \tag{12}$$

respectively, where we have offset time such that $y_{AC0} = y_{AC}(t_2)$ and $v_{AC0} = v_{AC}(t_2)$ are the position and velocity of the acrobat at $t = t_2$. Just after the collision, the acrobat and clown are still at the level of the platform so $y_{AC0} = y_{AC}(t_2) = h_0$. Substituting this and equation (10) into equations (11) and (12) gives

$$y_{AC}(t) = -\frac{g}{2}(t - t_2)^2 + \frac{m_A}{m_A + m_C}\sqrt{v_0^2 - 2gh_0}(t - t_2) + h_0 \tag{13}$$

$$v_{AC}(t) = -g(t - t_2) + \frac{m_A}{m_A + m_C}\sqrt{v_0^2 - 2gh_0}. \tag{14}$$

To find the maximum height of their trajectory (at a time we'll call t_3), we first use equation (14) to find the elapsed time $t_3 - t_2$ at which the velocity is zero

$$v_{AC}(t_3) = 0 = -g(t_3 - t_2) + \frac{m_A}{m_A + m_C}\sqrt{v_0^2 - 2gh_0} \quad \Rightarrow \quad t_3 - t_2 = \frac{m_A}{m_A + m_C}\frac{1}{g}\sqrt{v_0^2 - 2gh_0}. \tag{15}$$

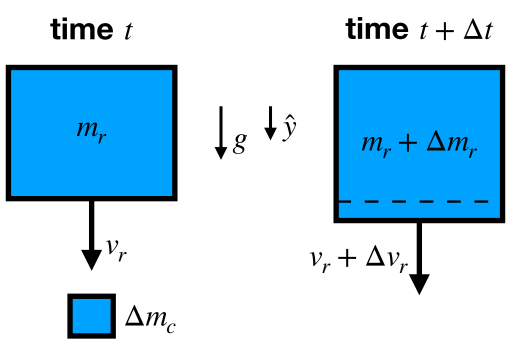
We can substitute this into equation (13) to find that the maximum height is

$$y_{AC}(t_3) = h_f = -\frac{g}{2}(t_3 - t_2)^2 + \frac{m_A}{m_A + m_C}\sqrt{v_0^2 - 2gh_0}(t_3 - t_2) + h_0 \tag{16}$$

$$h_f = \frac{1}{2g}\left(\frac{m_A}{m_A + m_C}\right)^2 (v_0^2 - 2gh_0) + h_0. \tag{17}$$

2. Falling raindrop

1. We start by choosing a coordinate system such that the $\hat{y}$ direction points downwards in the direction of the acceleration due to gravity. Note that the problem is one dimensional. Next, at an arbitrary time t, we consider a system that is composed of the raindrop of instantaneous mass m_r and a small differential mass element Δm_c from the cloud. The raindrop is moving downwards at an instantaneous velocity v_r, while the differential mass element is at rest (since the cloud is stationary). Thus, we can draw the momentum diagram shown below at time t. A very short time later at $t + \Delta t$, the differential mass element from the cloud has been incorporated into the raindrop. This slightly changes the mass of the raindrop to $m_r + \Delta m_r$ as well as the velocity to $v_r + \Delta v_r$. This is reflected in the momentum diagram shown below at time $t + \Delta t$.



From the momentum diagrams, we can use conservation of mass in the system to see that

$$m_r + \Delta m_c = m_r + \Delta m_r \quad \Rightarrow \quad \Delta m_c = \Delta m_r, \tag{1}$$

though this will not actually be needed to solve this problem. Additionally, we see that the total momentum of the system at time t is

$$\vec{p}_{sys}(t) = m_r v_r \hat{y} + \Delta m_c(0) = m_r v_r \hat{y}, \tag{2}$$

while at time $t + \Delta t$ it is

$$\vec{p}_{sys}(t + \Delta t) = (m_r + \Delta m_r)(v_r + \Delta v_r)\hat{y}. \tag{3}$$

We can now write down the generalized form of Newton's second law and use the limit form of the time derivative according to

$$\vec{F}_{net}^{ext} = \frac{d\vec{p}_{sys}}{dt} = \lim_{\Delta t \to 0}\frac{\Delta \vec{p}_{sys}}{\Delta t} = \lim_{\Delta t \to 0}\frac{\vec{p}_{sys}(t + \Delta t) - \vec{p}_{sys}(t)}{\Delta t}. \tag{4}$$

If we drew a free body diagram, we'd see that the only external force on the system is gravity $m_r g\hat{y}$. Using this and substituting equations (2) and (3) into equation (4), we find

$$m_r g = \lim_{\Delta t \to 0} \frac{(m_r + \Delta m_r)(v_r + \Delta v_r) - m_r v_r}{\Delta t} = \lim_{\Delta t \to 0} \frac{m_r \Delta v_r + \Delta m_r v_r + \Delta m_r \Delta v_r}{\Delta t} \tag{5}$$

in the $\hat{y}$ direction. We can neglect the final term $\Delta m_r \Delta v_r$ in this expression as it is product of two differential elements. Since the differential elements are infinitesimally small, a product of two differential elements will be much smaller than terms that include just one differential element (e.g. $\Delta m_r \Delta v_r \ll m_r \Delta v_r$). Thus, equation (5) becomes

$$m_r g = m_r \left(\lim_{\Delta t \to 0} \frac{\Delta v_r}{\Delta t} \right) + \left(\lim_{\Delta t \to 0} \frac{\Delta m_r}{\Delta t} \right) v_r. \tag{6}$$

Converting the limits back into derivatives, we find the differential equation

$$m_r g = m_r \frac{dv_r}{dt} + \frac{dm_r}{dt} v_r. \tag{7}$$

As an aside, note that this is equivalent to the standard generalized Newton's second law applied to the raindrop alone (i.e. $F_{net}^{ext} = dp_r/dt = d(m_r v_r)/dt = (dm_r/dt)v_r + m_r(dv_r/dt)$ using the product rule). This holds because the physical situation corresponds to a category 1 case (as defined in lecture 7a), in which the differential mass element carries no momentum. If the differential element did carry momentum (e.g. problem 4 below), the situation becomes more complicated and using the generalized Newton's second law for the raindrop alone would no longer be accurate.

To find an equation for $v_r(t)$, we can substitute the rate of mass gain from the problem statement into equation (7),

$$\frac{dm_r}{dt} = k m_r v_r, \tag{8}$$

which fortunately causes all the factors of m_r to cancel and gives

$$g = \frac{dv_r}{dt} + k v_r^2. \tag{9}$$

This is the differential equation the problem asked for.

2. To calculate the terminal velocity v_∞ of the raindrop, we could solve the differential equation and then take the limit at $t \to \infty$. However, there is a much easier way. We know that the raindrop will reach terminal velocity only when its acceleration $dv_r/dt = 0$ becomes zero. Otherwise the velocity would still be changing with time, so it wouldn't be "terminal". Thus, we can use this fact to write equation (9) as

$$g = 0 + k v_{r\infty}^2 \tag{10}$$

and find that

$$v_{r\infty} = \sqrt{\frac{g}{k}}. \tag{11}$$

3. Falling chain

This problem is challenging. We start by taking a coordinate system with $\hat{y}$ pointing downwards in the direction of the acceleration due to gravity and note that the problem is one dimensional. We will let $t = 0$ be the moment that the chain is dropped. Next, we can divide up the chain into many differential elements, each of length Δy. Then, we will consider the differential element that starts a distance D above the scale and define its initial position as the origin of our coordinate system. You can't push with a chain, so there is no force from the ground that is transmitted up the chain to the differential elements still in the air. In other words, each differential element of the chain is in free fall and governed by projectile motion until it makes contact with the scale. Thus, given our coordinate system and the fact that the chain starts at rest, the position and velocity of the differential element follows

$$\vec{r}(t) = y(t)\hat{y} = \frac{1}{2}gt^2\hat{y} \tag{1}$$

$$\vec{v}(t) = v_y(t)\hat{y} = gt\hat{y} \tag{2}$$

respectively. Using equation (1), we can calculate the time t just before the differential element of interest makes contact with the scale to be

$$y(t) = D = \frac{1}{2}gt^2 \quad \Rightarrow \quad t = \sqrt{\frac{2D}{g}}. \tag{3}$$

Substituting this into equation (2), we see that the element is traveling with a velocity of

$$\vec{v}(t) = v_y(t)\hat{y} = g\sqrt{\frac{2D}{g}}\hat{y} = \sqrt{2gD}\hat{y} \tag{4}$$

just before it impacts the scale.

A very short time later $t + \Delta t$, the differential element is at rest on the scale. We can calculate the force required to cause this through the change in momentum of the differential element. First, we need the mass Δm. We have choosen to divide up the rope into differential elements of length Δy. Thus, given that the linear mass density λ of the chain is uniform, we can find the mass according to

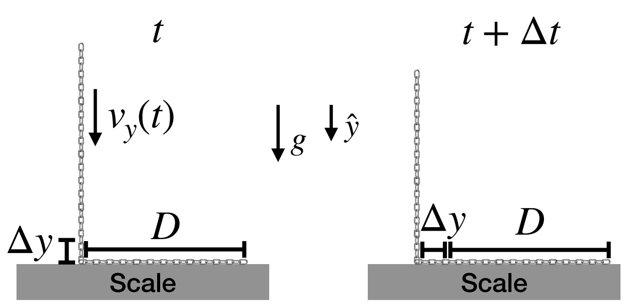
$$\lambda = \frac{M}{L} = \frac{\Delta m}{\Delta y} \quad \Rightarrow \quad \Delta m = \frac{M}{L}\Delta y. \tag{5}$$

We can now define a system composed of exclusively the differential element of interest and draw momentum diagrams at time t and $t + \Delta t$ (see below). Using equations (4) and (5), the momentum of such a system at time t is

$$\vec{p}_{sys}(t) = \Delta m\vec{v}(t) = \left(\frac{M}{L}\Delta y\right)\left(\sqrt{2gD}\hat{y}\right) = \sqrt{2gD}\frac{M}{L}\Delta y\hat{y}, \tag{6}$$

while at $t + \Delta t$ the total momentum is

$$\vec{p}_{sys}(t + \Delta t) = \Delta m(0) = 0. \tag{7}$$



We can then write down the generalized form of Newton's second law and use the limit form of the time derivative according to

$$\vec{F}^{ext}_{net} = \frac{d\vec{p}_{sys}}{dt} = \lim_{\Delta t\to 0}\frac{\Delta\vec{p}_{sys}}{\Delta t} = \lim_{\Delta t\to 0}\frac{\vec{p}_{sys}(t + \Delta t) - \vec{p}_{sys}(t)}{\Delta t}. \tag{8}$$

The external force will be only the normal force from the scale on the differential element

$$\vec{F}_{net}^{ext} = -F_{scale}\hat{y}, \tag{9}$$

which is what we are interested in calculating to determine the reading on the scale. This is because the gravitational force can be neglected through the impulse approximation, since the time interval Δt is so short.

Substituting equations (6), (7) and (9) into equation (8), we find

$$-F_{scale} = \lim_{\Delta t \to 0} \frac{0 - \sqrt{2gD}(M/L)\Delta y}{\Delta t} = -\sqrt{2gD}\frac{M}{L}\left(\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}\right) \tag{10}$$

in the $\hat{y}$ direction. Converting the limit back into a derivative gives

$$F_{scale} = \sqrt{2gD}\frac{M}{L}\frac{dy}{dt}. \tag{11}$$

Using the definition of velocity as the derivative of position $v_y = dy/dt$ and making use of equation (4) gives

$$F_{scale} = \sqrt{2gD}\frac{M}{L}v_y = \sqrt{2gD}\frac{M}{L}\left(\sqrt{2gD}\right) = 2gD\frac{M}{L}, \tag{12}$$

the magnitude of the upwards force exerted by the scale on the differential element of the chain.

If we drew a free body diagram for the scale at time $t + \Delta t$, we would see that there are two forces from the chain acting on it: the normal force from the differential element that just impacted the scale and the normal force from the part of the chain that is already sitting stationary on the scale. The length of chain already on the scale is D, so we can use the linear density to find its mass to be $m = \lambda D = D(M/L)$. Thus, the part of the chain sitting on the scale exerts a downward force with a magnitude of $mg = D(M/L)g$. Adding this to the action-reaction pair of equation (12) gives a total downwards force of the chain on the scale of

$$\text{Scale reading} = D\frac{M}{L}g + 2gD\frac{M}{L} = 3gD\frac{M}{L}. \tag{13}$$

This is our final answer for the reading on the scale.

Note that since D was an arbitrary location, we can use equation (3) to replace D in equation (13) with t. This allows us to see how the scale reading depends on time

$$\text{Scale reading} = \frac{3}{2}\frac{M}{L}g^2t^2. \tag{14}$$

Thus, the scale reading increases quadratically with time. The top end of the chain will land on the scale when $D = L$, at which time equation (13) becomes

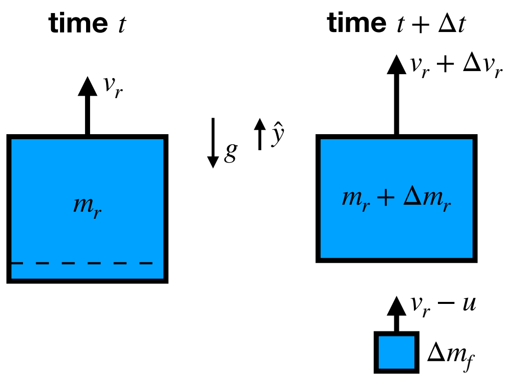
$$\text{Scale reading} = 3Mg. \tag{15}$$

In other words, the maximum reading of the scale is three times the mass of the rope. Of course, after the entire chain has come to rest, the reading on the scale will drop to the weight of the chain, i.e.

$$\text{Scale reading} = Mg. \tag{16}$$

4. Homework: Rocket with changing mass

We start by choosing a coordinate system such that $y = 0$ is the ground and the $\hat{y}$ direction points upwards in the opposite direction of the acceleration due to gravity. Note that the problem is one dimensional. Next, at an arbitrary time t, we consider a system that is composed of the rocket including *all* the fuel it currently contains, which we will denote as having a total instantaneous mass m_r. The instantaneous speed of the rocket is v_r, so we can draw the momentum diagram shown below at time t. A very short time later at $t + \Delta t$, the rocket has ejected a differential mass element Δm_f of fuel, which slightly alters the mass of the rocket to $m_r + \Delta m_r$. Note that it is important allow an arbitrary change in the rocket's mass by $+\Delta m_r$. This will help to prevent sign errors later in the derivation (e.g. accidentally accounting for the fact that the rocket's mass is decreasing twice) and accommodate more general calculations where the mass is changing due to several mechanisms. After ejecting the differential mass element, the velocity of the rocket is also slightly changed to be $v_r + \Delta v_r$. We must also include the momentum of the ejected fuel as it is still part of the system. It has a mass of Δm_f and a velocity of $-u\hat{y}$ *relative to the rocket*. This means that it has a velocity of $(v_r - u)\hat{y}$ in the inertial laboratory frame. We have drawn the momentum diagram at time $t + \Delta t$ below.



From the momentum diagram, we can use conservation of mass in the system to see that

$$m_r = m_r + \Delta m_r + \Delta m_f \quad \Rightarrow \quad \Delta m_f = -\Delta m_r. \tag{1}$$

Additionally, we see that the total momentum of the system at time t is

$$\vec{p}_{sys}(t) = m_r v_r \hat{y}, \tag{2}$$

while at time $t + \Delta t$ it is

$$\vec{p}_{sys}(t + \Delta t) = (m_r + \Delta m_r)(v_r + \Delta v_r)\hat{y} + \Delta m_f(v_r - u)\hat{y} = (m_r + \Delta m_r)(v_r + \Delta v_r)\hat{y} - \Delta m_r(v_r - u)\hat{y}, \tag{3}$$

making use of equation (1). We can now write down the generalized form of Newton's second law and use the limit form of the time derivative according to

$$\vec{F}^{ext}_{net} = \frac{d\vec{p}_{sys}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \vec{p}_{sys}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{p}_{sys}(t + \Delta t) - \vec{p}_{sys}(t)}{\Delta t}. \tag{4}$$

If we drew a free body diagram, we'd see that the only external force on the system is gravity $\vec{F}^{ext}_{net} = -m_r g \hat{y}$. Using this and substituting equations (2) and (3) into equation (4), we find

$$-m_r g = \lim_{\Delta t \to 0} \frac{(m_r + \Delta m_r)(v_r + \Delta v_r) - \Delta m_r(v_r - u) - m_r v_r}{\Delta t} = \lim_{\Delta t \to 0} \frac{m_r \Delta v_r + \Delta m_r \Delta v_r + \Delta m_r u}{\Delta t} \tag{5}$$

in the $\hat{y}$ direction. We can neglect the $\Delta m_r \Delta v_r$ term in this expression as it is product of two differential elements. Since the differential elements are infinitesimally small, a product of two differential elements will be much smaller than terms that include just one differential element (e.g. $\Delta m_r \Delta v_r \ll m_r \Delta v_r$). Thus, equation (5) becomes

$$-m_r g = m_r \lim_{\Delta t \to 0} \frac{\Delta v_r}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta m_r}{\Delta t} u. \tag{6}$$

Converting the limits back into derivatives, we find the differential equation

$$-m_r g = m_r \frac{dv_r}{dt} + \frac{dm_r}{dt} u. \tag{7}$$

Now, as in problem 2, we must determine dm_r/dt, the rate of change of the mass of the rocket (including the fuel it contains). We know that, at take-off when $t = 0$, the total mass of the rocket and fuel is M. Additionally, it ejects fuel at a constant rate of D. Thus, the total mass of the rocket as a function of time is

$$m_r(t) = M - Dt, \tag{8}$$

which gives

$$\frac{dm_r}{dt} = -D \tag{9}$$

after taking a derivative. Substituting this result into equation (7) gives the differential equation

$$-m_r g = m_r \frac{dv_r}{dt} - Du. \tag{10}$$

Rearranging and using equation (8) gives

$$\frac{dv_r}{dt} = -g + \frac{Du}{M - Dt}. \tag{11}$$

The problem statement ultimately asks us to find the speed and altitude of the rocket. Thus, we will integrate equation (11) to find the velocity

$$v_r(t) = -\int g dt + \int \frac{Du}{M - Dt} dt. \tag{12}$$

The first integral is straightforward, but to accomplish the second we must perform a change of variables. We know the integral of $1/x_1$ is the natural logarithm, so we will let

$$x_1 = M - Dt \tag{13}$$

and rewrite

$$v_r(t) = -\int g dt + Du \int \frac{1}{x_1} \frac{dt}{dx_1} dx_1. \tag{14}$$

Solving equation (13) for $t = M/D - x_1/D$ and taking a derivative gives $dt/dx_1 = -1/D$. Substituting this, taking the integrals, and using equation (13) gives

$$v_r(t) = -\int g dt + Du \int \frac{1}{x_1} \left(-\frac{1}{D} \right) dx_1 = -gt - u \ln(x_1) + C_1 = -gt - u \ln(M - Dt) + C_1, \tag{15}$$

where C_1 is a constant of integration. It can be determined by using the initial condition that the rocket is at rest at $t = 0$, i.e. $v_r(0) = 0$. This gives

$$v_r(0) = 0 = -u \ln(M) + C_1 \quad \Rightarrow \quad C_1 = u \ln(M). \tag{16}$$

Substituting this into equation (15) and using the property of logarithms that $\ln(A) - \ln(B) = \ln(A/B)$ gives

$$v_r(t) = -gt - u \ln(M - Dt) + u \ln(M) = -gt - u \ln \left(\frac{M - Dt}{M} \right). \tag{17}$$

To find the altitude of the rocket, we must integrate once more in time to find the position

$$y_r(t) = -\int gt dt - u \int \ln \left(\frac{M - Dt}{M} \right) dt. \tag{18}$$

Again the first integral is straightforward, but the second is challenging. We can use the second hint in the problem statement if we first perform a change of variables to

$$x_2 = \frac{M - Dt}{M}. \tag{19}$$

This allows us to write equation (18) as

$$y_r(t) = -\int g t dt - u \int \ln(x_2) \frac{dt}{dx_2} dx_2. \tag{20}$$

Solving equation (19) for $t = -M/D(x_2 - 1)$ and taking a derivative gives $dt/dx_2 = -M/D$. Substituting this, taking the integrals via hint 2, and using equation (19) gives

$$y_r(t) = -\int g t dt - u \int \ln(x_2) \left(-\frac{M}{D}\right) dx_2 = -\frac{g}{2}t^2 + \frac{Mu}{D} x_2 (\ln(x_2) - 1) + C_2 \tag{21}$$

$$= -\frac{g}{2}t^2 + \frac{Mu}{D}\frac{M - Dt}{M}\left(\ln\left(\frac{M - Dt}{M}\right) - 1\right) + C_2 = -\frac{g}{2}t^2 + u\frac{M - Dt}{D}\left(\ln\left(\frac{M - Dt}{M}\right) - 1\right) + C_2, \tag{22}$$

where C_2 is an integration constant. It can be determined by using the initial condition that the rocket starts on the ground at $t = 0$, i.e. $y_r(0) = 0$. This gives

$$y_r(0) = 0 = u\frac{M}{D}(\ln(1) - 1) + C_2 \quad \Rightarrow \quad C_2 = -u\frac{M}{D}(-1) = u\frac{M}{D}. \tag{23}$$

Substituting this into equation (22) gives

$$y_r(t) = -\frac{g}{2}t^2 + u\left[\frac{M - Dt}{D}\left(\ln\left(\frac{M - Dt}{M}\right) - 1\right) + \frac{M}{D}\right]. \tag{24}$$

Equations (17) and (24) are the speed and altitude as a function of time. To solve the problem, we are interested in their values at the time that the fuel is exhausted, which we'll denote by t_f. We know that the total amount of fuel is m_t and it is ejected at a rate of D. Thus, the fuel will be completely used up at a time $t_f = m_t/D$ after launch. Substituting this into equations (17) and (24) gives

$$v_r(t_f) = -g\frac{m_t}{D} - u\ln\left(\frac{M - m_t}{M}\right) \tag{25}$$

$$y_r(t_f) = -\frac{g}{2}\left(\frac{m_t}{D}\right)^2 + u\left[\frac{M - m_t}{D}\left(\ln\left(\frac{M - m_t}{M}\right) - 1\right) + \frac{M}{D}\right]. \tag{26}$$

Plugging in the numerical values from the problem statement (and noting that 1 ton = 1000 kg) gives $t_f = 160$ s and

$$v_r(t_f) = 3.2 \text{ km/s} \tag{27}$$

$$y_r(t_f) = 160 \text{ km}. \tag{28}$$