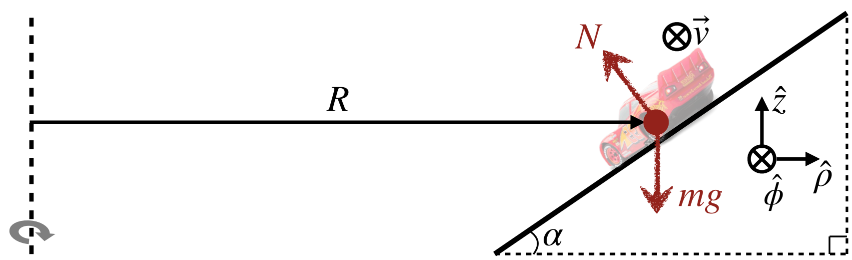
Solutions to Problem Set 4

Circular motion

PHYS-101(en)

1. Circular motion: banked turn

1. In this part, the static friction can be considered to be $F_s \approx 0$ because the coefficient of static friction is so small. As we are analyzing circular motion, we choose a cylindrical coordinate system as shown in the figure below, where the unit vector $\hat{\rho}$ points in the outward radial direction, $\hat{\phi}$ points into the page around the curve, and $\hat{z}$ points upwards. The free body diagram on the car is also shown.



Given the free body diagram and the form of the centripetal acceleration $\vec{a} = -(v_0^2/R)\hat{\rho}$ for circular motion, Newton's second law $\sum \vec{F} = m\vec{a}$ in the $\hat{\rho}$ direction is

$$-N \sin\alpha = -\frac{mv_0^2}{R}.$$

We can tell that the trigonometric function in this equation is sine rather than cosine by imagining the case that $\alpha = 0$. If $\alpha = 0$, the $\hat{\rho}$ component of $\vec{N}$ would be 0. Since $\sin(0) = 0$ and $\cos(0) = 1$, the $\hat{\rho}$ component of $\vec{N}$ should contain $\sin(\alpha)$. In the $\hat{z}$ direction Newton's second law is

$$N \cos\alpha - mg = ma_z.$$

Because the car is traveling in a circle and does not slide up or down, the acceleration in the $\hat{z}$ direction is zero $a_z = 0$. Thus, the components of Newton's second law become

$$N \sin\alpha = \frac{mv_0^2}{R}$$

and

$$N \cos\alpha = mg.$$

Dividing these equations to eliminate N yields

$$\tan\alpha = \frac{v_0^2}{Rg},$$

which we can solve for the speed v_0 that is necessary to maintain circular motion. We find

$$v_0 = \sqrt{Rg\tan\alpha}.$$

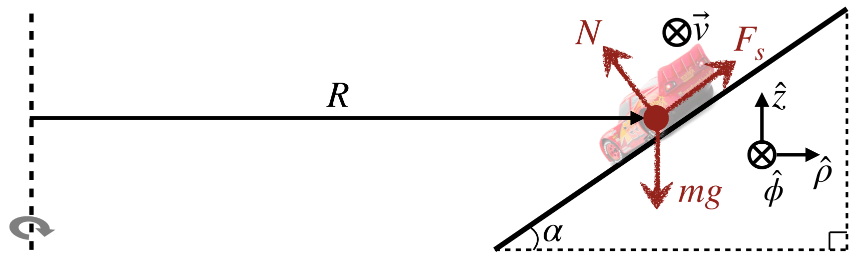
2. We now reconsider the problem with a non-zero coefficient of static friction μ_s. In this part, the speed of the car $v = v_{min}$ is so slow that it just barely doesn't slip down the bank. The static friction force must point up the incline as we know that it is preventing the car from slipping down. The free body diagram on the car is shown in the figure below, from which we see that Newton's second law is

$$-N\sin\alpha + F_s\cos\alpha = -\frac{mv^2}{R}$$

in the $\hat{\rho}$ direction and

$$N\cos\alpha + F_s\sin\alpha - mg = ma_z$$

in the $\hat{z}$ direction.



When $v = v_{min}$, the car still isn't slipping so the acceleration in the $\hat{z}$ direction $a_z = 0$ is still zero, but the static friction has its maximum magnitude of $F_s = \mu_s N$. Thus, Newton's second law becomes

$$-N\sin\alpha + \mu_s N\cos\alpha = -\frac{mv_{min}^2}{R}$$

and

$$N\cos\alpha + \mu_s N\sin\alpha = mg.$$

Dividing these equations to eliminate N yields

$$\frac{-\sin\alpha + \mu_s\cos\alpha}{\cos\alpha + \mu_s\sin\alpha} = -\frac{v_{min}^2}{Rg},$$

which can then be solved for the minimum speed necessary to avoid sliding down the embanked turn

$$v_{min} = \sqrt{Rg\frac{\sin\alpha - \mu_s\cos\alpha}{\cos\alpha + \mu_s\sin\alpha}} = \sqrt{Rg\frac{\tan\alpha - \mu_s}{1 + \mu_s\tan\alpha}}.$$

The limiting cases of this result can be checked. In the limit $\mu_s \to 0$, $v_{min} \to \sqrt{Rg\tan\alpha}$, which is consistent with the result for part 1. In the limit $\mu_s \to \tan\alpha$, $v_{min} \to 0$, which is the solution for the static case of a block sitting on an incline.

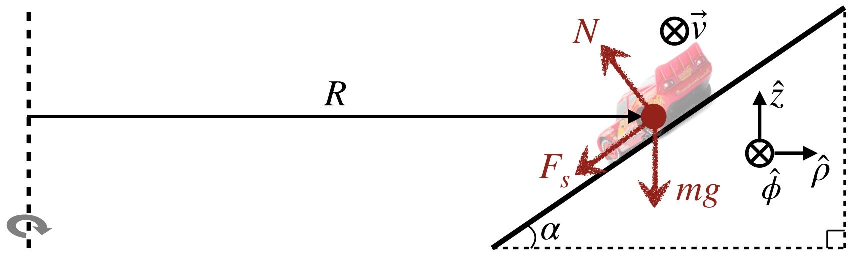
3. We now consider the case where the car is at the maximum speed $v = v_{max}$ such that it is almost slipping up the inclined plane. For this case, the direction of static friction now points down the incline plane and the free body diagram is shown below.

The analysis is identical to the previous case, except the static friction force changes sign. Thus Newton's second law become

$$-N\sin\alpha - F_s\cos\alpha = -\frac{mv^2}{R} \tag{1}$$

in the $\hat{\rho}$ direction and

$$N\cos\alpha - F_s\sin\alpha - mg = 0 \tag{2}$$



in the $\hat{z}$ direction. When $v = v_{max}$, the static friction has its maximum value of $F_s = \mu_s N$, so Newton's second law becomes

$$-N\sin\alpha - \mu_s N\cos\alpha = -\frac{mv_{max}^2}{R}$$

and

$$N\cos\alpha - \mu_s N\sin\alpha = mg.$$

Dividing these two equations to eliminate N yields

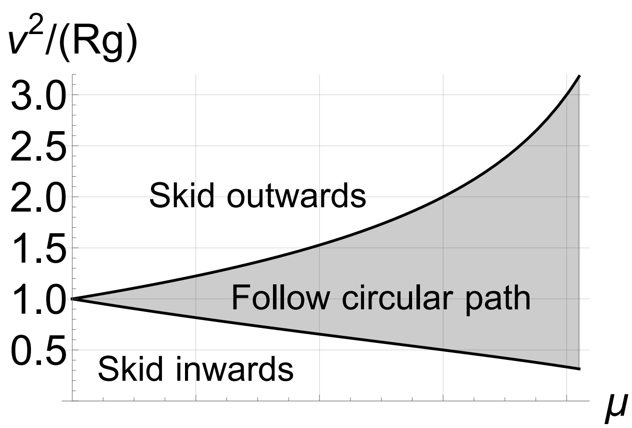
$$\frac{-\sin\alpha - \mu_s\cos\alpha}{\cos\alpha - \mu_s\sin\alpha} = -\frac{v_{max}^2}{Rg},$$

which can then be solved for the maximum speed v_{max} to avoid sliding up the embanked turn

$$v_{max} = \sqrt{Rg\frac{\sin\alpha + \mu_s\cos\alpha}{\cos\alpha - \mu_s\sin\alpha}} = \sqrt{Rg\frac{\tan\alpha + \mu_s}{1 - \mu_s\tan\alpha}}.$$

This solution is identical to that of part 2, except the sign in front of μ_s is opposite.

The figure below shows a plot of $v^2/(Rg)$ versus μ_s when $\alpha = 45°$. The shaded area represents the set of $(\mu_s, v^2/(Rg))$ points where the car remains in a circular path. Above the shaded region the car will slid up and out, while below the shaded region the car will slide down and in.



4. The analysis is the same as in part 3, but the magnitude of the static friction is less than its maximum value. However, now the problem statement gives us the velocity v as a known quantity. Hence, we can combine equations (1) and (2) to eliminate N and solve for F_s. To do so, we multiply (1) by $\cos\alpha$ and (2) by $\sin\alpha$ to find

$$-N\sin\alpha\cos\alpha - F_s\cos^2\alpha = -\frac{mv^2}{R}\cos\alpha$$

and

$$N\cos\alpha\sin\alpha - F_s\sin^2\alpha - mg\sin\alpha = 0.$$

Adding these two equations yields

$$-F_s\left(\cos^2\alpha + \sin^2\alpha\right) - mg\sin\alpha = -\frac{mv^2}{R}\cos\alpha.$$

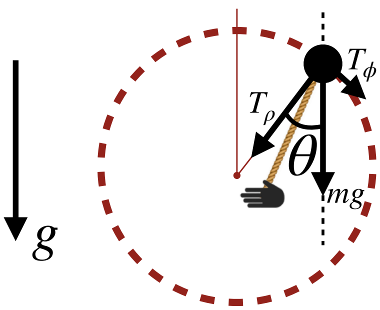
Using the identity $\cos^2\alpha + \sin^2\alpha = 1$ gives

$$F_s = m\left(\frac{v^2}{R}\cos\alpha - g\sin\alpha\right)$$

for the magnitude of the static friction force.

2. Swinging ball

1. The free body diagram for the ball is shown below. Note the $\hat{\phi}$ component of the tension arises because Sally's hand does not stay in the center of the circle, so the string does not pull precisely in the radial direction. From the free body diagram, we see that Newton's second law $\sum\vec{F} = m\vec{a}$ in the radial



direction is

$$-T_\rho - mg\cos\theta = -mR\omega^2,$$

where we have used the fact that the centripetal acceleration needed for circular motion is $\vec{a} = -R\omega^2\hat{\rho}$. Rearranging, we find the tension to be

$$T_\rho = mR\omega^2 - mg\cos\theta.$$

Because the angular frequency is related to the period through $\omega = 2\pi/t_0$, the magnitude of the radial component of the tension in the string is

$$T_\rho(\theta) = \frac{4\pi^2 mR}{t_0^2} - mg\cos\theta.$$

2. A string cannot push. Thus, the magnitude of the radial component of the tension in the string T_ρ must be greater than zero at all points in the circular motion. If $T_\rho < 0$ the ball will move inwards and the path will no longer be circular. This means that, using the final result from part 1, circular motion will not be maintained if

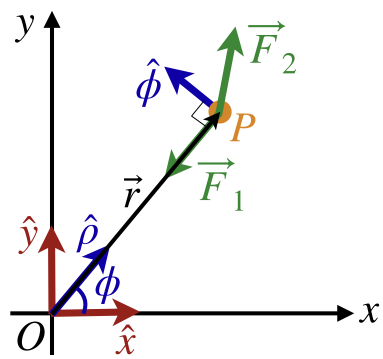
$$T_\rho(\theta) = \frac{4\pi^2 mR}{t_0^2} - mg\cos\theta < 0$$

at any point along the trajectory. From this equation we see that $\theta = 0$ (i.e. the top of the circle) is the most prone location to breakdown. This is because the gravitational term $-mg\cos\theta$ makes its largest negative contribution to the required radial force T_ρ (since $\cos(0) = 1$ is the maximum value of $\cos\theta$). Thus, taking $\theta = 0$ and solving the above equation for the period t_0 gives the condition for the breakdown of circular motion. We find that circular motion will **not** be maintained if

$$t_0 > 2\pi\sqrt{\frac{R}{g}}.$$

3. Spiral motion of a point mass

1. We start by representing the system in polar coordinates, as shown below. Note that, until we solve the equations of motion, we don't know the direction of $\vec{v}$ and, hence, $\vec{F}_2$, so we draw it with an arbitrary direction (and include both radial and azimuthal components to be as general as possible).



2. We are given that the forces acting on the mass are

$$\vec{F}_1 = -mk^2\vec{r}$$

$$\vec{F}_2 = -2m\lambda\vec{v} \quad \text{where} \quad 0 < \lambda < k.$$

To calculate motion from forces, we will apply Newton's second law

$$\sum \vec{F} = \vec{F}_1 + \vec{F}_2 = m\vec{a},$$

but we must first rewrite $\vec{F}_1$, $\vec{F}_2$, and $\vec{a}$ in polar coordinates. The position vector in polar coordinates is $\vec{r} = \rho\hat{\rho}$ (which does not include a $\hat{\phi}$ component because the direction of the radial unit vector $\hat{\rho}$ changes with time such that it always points at the point mass). Thus, the spring-like force can be written as

$$\vec{F}_1 = -mk^2\left(\rho\hat{\rho}\right).$$

The problem statement gives the form of the velocity in polar coordinates, which we can substitute to write the friction-type force as

$$\vec{F}_2 = -2m\lambda\left(\dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi}\right).$$

Lastly, the problem statement tells us that the acceleration vector is

$$\vec{a} = \left(\ddot{\rho} - \rho\dot{\phi}^2\right)\hat{\rho} + \left(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}\right)\hat{\phi}$$

in polar coordinates.

Substituting these three equations into Newton's second law gives the equations of motion

$$\ddot{\rho} + 2\lambda\dot{\rho} + \left(k^2 - \dot{\phi}^2\right)\rho = 0$$

in the $\hat{\rho}$ direction and

$$\rho\ddot{\phi} + 2\left(\dot{\rho} + \lambda\rho\right)\dot{\phi} = 0$$

in the $\hat{\phi}$ direction. Since $\dot{\phi} \neq 0$ and $\ddot{\phi} = 0$, the equation in the $\hat{\phi}$ direction simplifies to

$$\dot{\rho} = -\lambda\rho.$$

3. Given the solution form in the problem statement, we can immediately solve the $\hat{\phi}$ equation and use the initial condition $\rho(0) = \rho_0 = C$ to find

$$\rho\left(t\right) = \rho_0 e^{-\lambda t}.$$

We can then rearrange the $\hat{\rho}$ equation to isolate $\dot{\phi}$ according to

$$\dot{\phi}^2 = \frac{\ddot{\rho}}{\rho} + \frac{2\lambda\dot{\rho}}{\rho} + k^2.$$

Substituting our solution for $\rho(t)$, we find

$$\dot{\phi} = \pm\sqrt{k^2 - \lambda^2},$$

where we note that this is a real number as the problem statement tells us that $k > \lambda$. Integrating this equation with respect to time and using the initial condition that $\phi\left(0\right) = 0$ to determine the integration constant, we find

$$\phi\left(t\right) = \pm t\sqrt{k^2 - \lambda^2}.$$

Solving this for t and substituting it into our expression for $\rho(t)$ gives

$$\rho\left(\phi\right) = \rho_0 e^{\mp\frac{\lambda\phi}{\sqrt{k^2-\lambda^2}}}.$$

From the problem statement we know that the velocity in polar coordinates is

$$v = \dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi}.$$

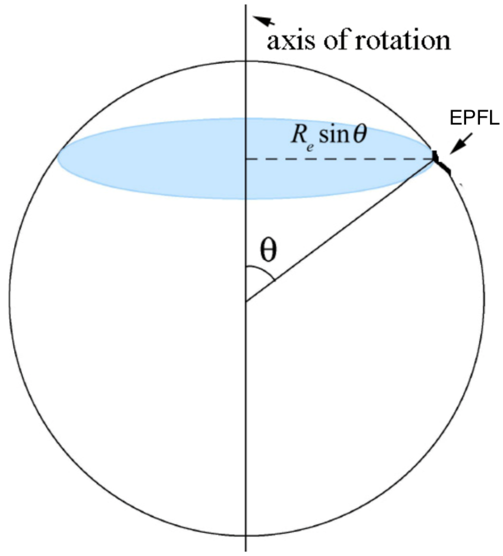
Substituting our solutions from above, we find

$$\vec{v}(t) = -\rho(t)\left(\lambda\hat{\rho} \mp \sqrt{k^2 - \lambda^2}\hat{\phi}\right).$$

The speed is just the norm of this, which simplifies to

$$v(t) = k\rho(t).$$

4. Circular motion of the earth



1. The rotational period of the earth is given by

$$T_e = (23 \text{ hr}) \left(60 \, \frac{\text{min}}{\text{hr}}\right) \left(60 \, \frac{\text{s}}{\text{min}}\right) + (56 \text{ min}) \left(60 \, \frac{\text{s}}{\text{min}}\right) + 4 \text{ s} = 86164 \text{ s},$$

which is less than 24 hr. Twenty-four hours is one solar day (i.e. noon to noon), while the above period is one “sidereal” day. “Sidereal” means with respect to the fixed stars and, if you think about things, you should be able to see why the two are different. A person at EPFL undergoes circular motion about the axis of the earth (as shown in the picture). The radius of the orbit is given by

$$R = R_e \sin\theta,$$

where θ is the angle between EPFL and the axis of rotation as shown in the figure. Since the latitude $\lambda = 46°31'\text{N} = (46 + 31/60)° = 46.52°$ is measured from the equator,

$$\theta = \frac{\pi}{2} - \lambda$$

and

$$\sin\theta = \sin\left(\frac{\pi}{2} - \lambda\right) = \cos\lambda$$

using trigonometric identities. The angle θ is sometimes called the “colatitude”. The radius of the orbit of a person at EPFL is

$$R = R_e \cos\lambda = \left(6.38 \times 10^6 \text{ m}\right) \cos(46.52°) = 4.39 \times 10^6 \text{ m}.$$

Because the circular motion is uniform, during one period of rotation T the person travels a distance

$$d = 2\pi R = vT$$

at a constant speed v, where $d = 2\pi R$ is the circumference. Solving for the speed gives

$$v = \frac{2\pi R}{T}.$$

Thus a person at EPFL has a velocity of

$$\vec{v} = \frac{2\pi R}{T}\hat{\phi} = \frac{2\pi \left(4.39 \times 10^6 \text{ m}\right)}{(86164 \text{ s})}\hat{\phi} = \left(320 \, \frac{\text{m}}{\text{s}}\right)\hat{\phi},$$

where $\hat{\phi}$ is the unit vector pointing east.

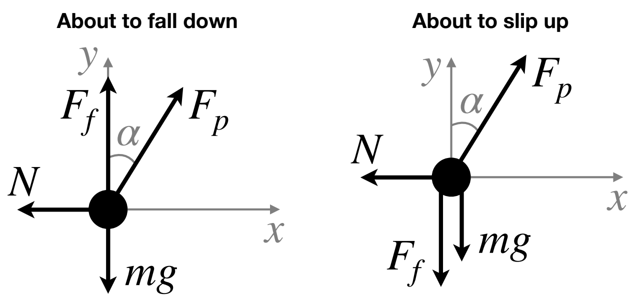
2. The centripetal acceleration is given by

$$\vec{a} = -\frac{v^2}{R}\hat{\rho} = -\frac{(320 \text{ m/s})^2}{(4.39 \times 10^6 \text{ m})}\hat{\rho} = -2.33 \times 10^{-2} \ \frac{\text{m}}{\text{s}^2}\hat{\rho},$$

where $-\hat{\rho}$ is the unit vector pointing towards the closest point on the axis of rotation (not towards the center of the earth).

5. Homework: Pushing a book against a wall

1. Let m, μ_s, and α be defined as in the problem and let $\vec{F}_f$ represent the friction force. Additionally, let $\vec{F}_p$ be the force of your push on the book. We will define our coordinate system such that $x > 0$ corresponds to the wall, so that you are pushing from the $x < 0$ side. The free body diagrams for both cases are given below. Note that the static friction force opposes the direction that the book is almost moving in.



2. We first consider the case when the book is almost falling down. The frictional force points up and has its maximum value, meaning it has a **norm** given by

$$F_f = \mu_s N.$$

Applying Newton's second law and requiring equilibrium (i.e. $\vec{a} = 0$) gives

$$N = F_p \sin\alpha$$

in the $\hat{x}$ direction and

$$\mu_s N + F_p \cos\alpha - mg = 0$$

in the $\hat{y}$ direction. Using these two equations to eliminate the normal force N and solve for F_p gives the solution of

$$F_p = \frac{mg}{\cos\alpha + \mu_s \sin\alpha}.$$

Next we consider the case when the book is about to slip up. The frictional force points down and has its maximum value, meaning it has a **norm** given by

$$F_f = \mu_s N.$$

Applying Newton's second law and requiring equilibrium (i.e. $\vec{a} = 0$) gives

$$N = F_p \sin\alpha$$

in the $\hat{x}$ direction and

$$-\mu_s N + F_p \cos\alpha - mg = 0$$

in the $\hat{y}$ direction. Using these two equations to eliminate the normal force N and solve for F_p gives the solution of

$$F_p = \frac{mg}{\cos\alpha - \mu_s \sin\alpha}.$$

3. To find the force for which the friction is zero, we can take the limit that $\mu_s \to 0$ in either of the solutions to part 2. This gives

$$F_p = \frac{mg}{\cos\alpha}.$$

Alternatively, one could draw the free body diagrams without the friction force and solve the resulting components of Newton's second law, which gives the same answer.

When $\alpha = 0$, $F_p = mg/\cos(0) = mg$ and when $\alpha = 90°$, $F_p = mg/\cos(90°) \to \infty$, which are both consistent with our intuition.