EPFL

Angl. 1

Final exam
General Physics I, PHYS-101 (English)
January 21, 2022
Prof. Suliana Manley

Sciper :

Name :

Signature :

Instructions

1. Write clearly in ink (pen), explain your reasoning, and indicate clearly your result.
2. Only responses written inside of the boxes on the A4 sheets of the exam (front and back) will be taken into account.
3. Justify your calculations and responses with diagrams and by identifying the laws of physics you have used.
4. This exam should be taken in 3 hours.

Allowed:

- one page A4 (front and back) *handwritten by you*
- language dictionary

Not allowed:

- Any form of electronic device including a calculator
- Leaving the room without authorization

(Do not mark on this page below this line)

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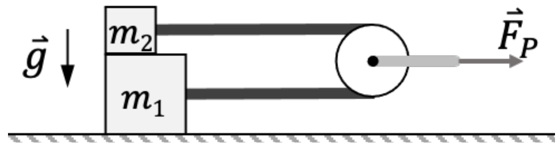
1. Accelerated Atwood machine.

Two blocks of masses m_1 and m_2 ($m_1 < m_2$) are stacked on top of each other on the surface of a frictionless table (see diagram). The blocks' centers of mass are connected by a string, which is wrapped around a pulley. The mass of the pulley and string are negligible.

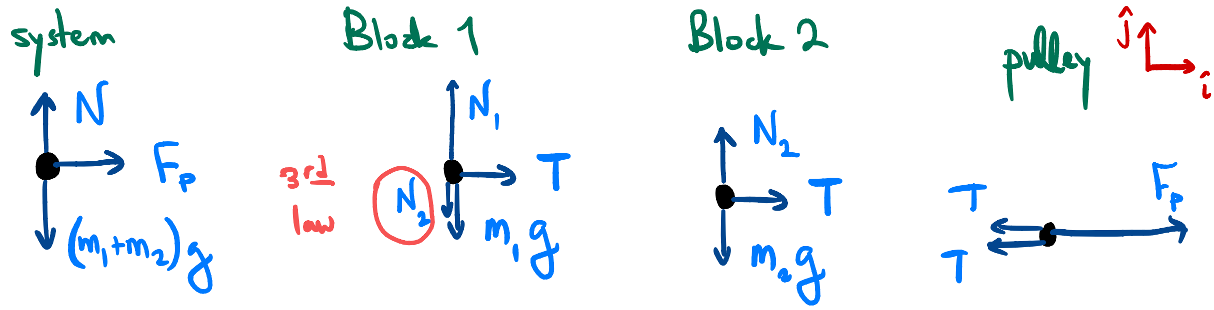


The system starts at rest, then a force $\vec{F}_P$ is applied horizontally to the pulley.

a) Assume there is no friction between the blocks or in the pulley. Determine the forces acting on the system and each object (Block 1, Block 2, pulley) separately. Draw free body diagrams.

Answer inside the box

system　　Block 1　　Block 2　　pulley



normal force from table to system　$\vec{N} = (m_1+m_2)g\hat{j}$

normal force from table to block 1　$\vec{N}_1 = (m_1 g + N_2)\hat{j} = (m_1+m_2)g\hat{j}$

normal force from block 2 to 1　$\vec{N}_2 = m_2 g\hat{j}$

tension in string　$\vec{T} = \frac{1}{2}F_p\hat{j}$

b) What are the accelerations, $\vec{a}_1$ and $\vec{a}_2$, of the blocks?

As the system FBD shows, Newton's 2nd law becomes:

$\hat{i}$: Block 1　$T = m_1 a_1$　　pulley: $F_p - 2T = \cancel{m_p}^{0} a_p$

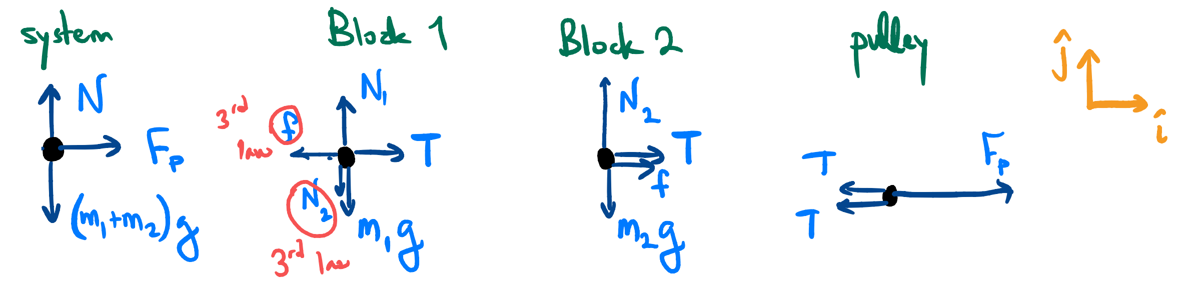
Block 2　$T = m_2 a_2$　　$\Rightarrow F_p = 2T$

$$\Rightarrow \vec{a}_1 = \frac{F_p}{2m_1}\hat{i} \qquad \vec{a}_2 = \frac{F_p}{2m_2}\hat{i}$$

Note $a_1 > a_2$.

1. Accelerated Atwood machine.

c) Now, assume there is a coefficient of friction between the blocks, μ (same for static and kinetic cases). Re-draw the free body diagrams for each block. What are the accelerations of each block, for "low" ($\vec{a}_{1L}$, $\vec{a}_{2L}$) and "high" ($\vec{a}_{1H}$, $\vec{a}_{2H}$) values of F_P?



"low" F_p : tension doesn't exceed static friction. Blocks do not slip.

$$\Rightarrow \quad \vec{a}_{1L} = \vec{a}_{2L} = \frac{F_p}{m_1+m_2}\hat{i}$$

"high" F_p : blocks slip. All forces balance in $\hat{j}$.

$\hat{i}$: Block 1 $\quad T - \mu m_2 g = m_1 a_1 \qquad$ pulley : $F_p - 2T = \cancelto{0}{m_p} a_p$

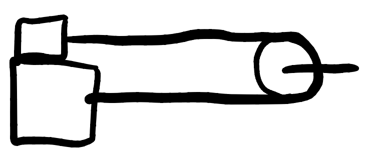
Block 2 $\quad T + \mu m_2 g = m_2 a_2 \qquad \Rightarrow F_p = 2T$

$$\Rightarrow \quad \vec{a}_{1H} = \left(\frac{F_p}{2m_1} - \mu\frac{m_2 g}{m_1}\right)\hat{i} = \frac{F_p - 2\mu m_2 g}{2m_1}\hat{i}$$

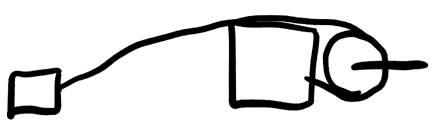
$$\vec{a}_{2H} = \left(\frac{F_p}{2m_2} + \mu g\right)\hat{i} = \frac{F_p + 2\mu m_2 g}{2m_2}\hat{i}$$

d) Sketch the configuration of the system (blocks and pulley) in the limit of long times, in each case described in (c) ("low" and "high").

Low:



High:



1. **Accelerated Atwood machine.**

e) What is the critical value of the force, F_P^*, which separates the two kinds of motion described in parts (c) and (d)?

As the force F_p decreases, $\vec{a}_1$ and $\vec{a}_2$ should become <u>equal</u>, at the point when they are no longer slipping.

$$\left(\frac{F_p^*}{2m_1} - \mu \frac{m_2 g}{m_1}\right) = \left(\frac{F_p^*}{2m_2} + \mu g\right)$$

multiply by $2m_1m_2$ $\Rightarrow$ $F_p^* m_2 - 2\mu m_2^2 g = F_p^* m_1 + 2\mu m_2 m_1 g$

$$\Rightarrow \quad F_p^*(m_2 - m_1) = 2\mu m_2 g (m_1 + m_2)$$

$$\Rightarrow \quad F_p^* = 2\mu m_2 g \frac{(m_1 + m_2)}{(m_1 - m_2)}$$

More detailed derivation for prob. 1

a)

From block 2, $\hat{j}$ component: $N_2 = m_2 g$

From block 1, $\hat{j}$ component: $N_1 = m_1 g + N_2 = (m_1+m_2) g$

From pulley, $\hat{i}$ component: $F_p = 2T \Rightarrow T = \frac{F_p}{2}$

c)

From block 1, $\hat{i}$ component

$$T - f = m_1 a_1$$

$$a_1 = \frac{F_p - 2f}{2m_1}$$

From block 2, $\hat{i}$ component

$$T + f = m_2 a_2$$

$$a_2 = \frac{F_p + 2f}{2m_2}$$

If blocks do not slip, we know that friction is static: $f < \mu N_2 = \mu m_2 g$

We also know that $a_1 = a_2 = a$

$$\frac{F_p - 2f}{\cancel{2}m_1} = \frac{F_p + 2f}{\cancel{2}m_2}$$

$$m_2 F_p - 2m_2 f = m_1 F_p + 2m_1 f$$

$$-2(m_1 + m_2) f = (m_1 - m_2) F_p$$

$$f = \frac{1}{2} \frac{m_2 - m_1}{m_2 + m_1} F_p$$ (this is positive as $m_2 > m_1$)

$$a = \frac{F_p}{2m_1} - \frac{1}{m_1}\left(\frac{1}{2} \frac{m_2 - m_1}{m_2 + m_1} F_p\right)$$

$$a = \frac{F_p}{2m_1}\left(\frac{\cancel{m_2} + m_1}{m_2 + m_1} - \frac{\cancel{m_2} - m_1}{m_2 + m_1}\right) = \frac{F_p}{\cancel{2m_1}} \frac{\cancel{2m_1}}{m_2 + m_1} = \boxed{\frac{F_p}{m_2 + m_1}}$$

Static friction holds as long as $f = \frac{1}{2} \frac{m_2 - m_1}{m_2 + m_1} F_p < \mu N_2 = \mu m_2 g$

$\Rightarrow$ kinetic friction if $\underline{F_p > 2\mu m_2 g \frac{m_2 + m_1}{m_2 - m_1}}$

If friction is kinetic, then $f = \mu N_2 = \mu m_2 g$ and $a_1 \neq a_2$

$$\boxed{a_1 = \frac{F_p - 2\mu m_2 g}{2m_1}} \qquad \boxed{a_2 = \frac{F_p + 2\mu m_2 g}{2m_2}}$$

e)

Which acceleration is larger? a_1 will be larger if

$$\frac{F_p - 2\mu m_2 g}{\cancel{2}m_1} > \frac{F_p + 2\mu m_2 g}{\cancel{2}m_2}$$

$$m_2 F_p - 2\mu m_2^2 g > m_1 F_p + 2\mu m_1 m_2 g$$

$$(m_2 - m_1) F_p > 2\mu m_2 g (m_1 + m_2)$$

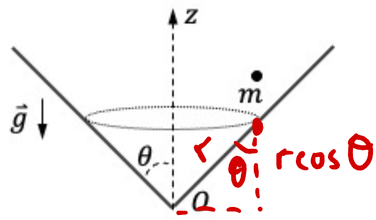
$F_p > 2\mu m_2 g \frac{m_2 + m_1}{m_2 - m_1}$ which is exactly the condition for kinetic friction

Thus, if the friction is kinetic, block 1 will accelerate faster than block 2

This is consistent with the limit $\mu \to 0$ (i.e. part b)

2. Funnel fun.

A point-like object of mass m slides on the inside of a frictionless, symmetrical cone, whose symmetry axis is vertical (parallel to gravity). The vertex half-angle is θ, as shown. Take the point of the cone as the origin of the coordinate system, O.



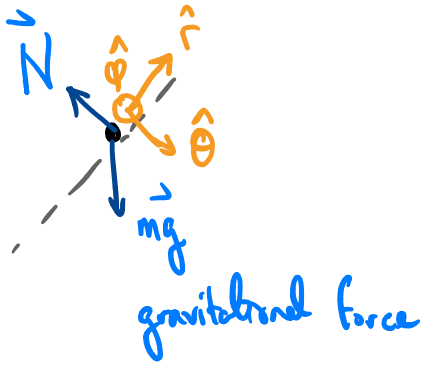
a) Determine the forces acting on the object. Draw a free body diagram.

Answer inside the box

Forces:

$\vec{N} = -N\hat{\theta}$ perpendicular to surface

$m\vec{g} = mg\underbrace{(-\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})}_{-\hat{k}}$



b) Write the differential equations of motion of the object in spherical coordinates, without solving them. Take into account all constraints.

Constraints: $\theta = \text{const.} \Rightarrow \dot{\theta} = 0, \ \ddot{\theta} = 0$

$\hat{r}$: $-mg\cos\theta = m(\ddot{r} - r\dot{\varphi}^2\sin^2\theta)$

$\hat{\theta}$: $mg\sin\theta - N = m(-r\dot{\varphi}^2\sin\theta\cos\theta)$

$\hat{\varphi}$: $0 = m(r\ddot{\varphi}\sin\theta + 2\dot{r}\dot{\varphi}\sin\theta)$

2. Funnel fun.

c) Suppose the object is placed at height H, and given an initial angular velocity of magnitude ω_0, that is in a horizontal plane and tangent to the surface of the cone. Find an expression for ω_0 such that the object will move in a horizontal, circular orbit.

In this case, constant $r = H/\cos\theta$, $\dot{r} = 0$, $\ddot{r} = 0$

$\hat{r}$: $-mg\cos\theta = m(\cancel{\ddot{r}} - r\underbrace{\dot{\varphi}^2}_{\dot{\varphi} = \omega}\sin^2\theta)$ $\Rightarrow$ $-m\omega_0^2 H\sin^2\theta/\cos\theta = -mg\cos\theta$

$\Rightarrow \omega_0 = \sqrt{\frac{g\cos^2\theta}{H\sin^2\theta}} = \sqrt{\frac{g}{H\tan^2\theta}}$

Alternative: motion will remain in a horizontal plane: $\rho = H\tan\theta$

$a_y = 0 \Rightarrow mg = N\sin\theta$ $\quad ma_x = -m\omega_0^2\rho = -N\cos\theta$

$\Rightarrow -m\omega_0^2 H\tan\theta = -\frac{mg\cos\theta}{\sin\theta} \Rightarrow \omega_0 = \sqrt{\frac{g}{H\tan^2\theta}}$

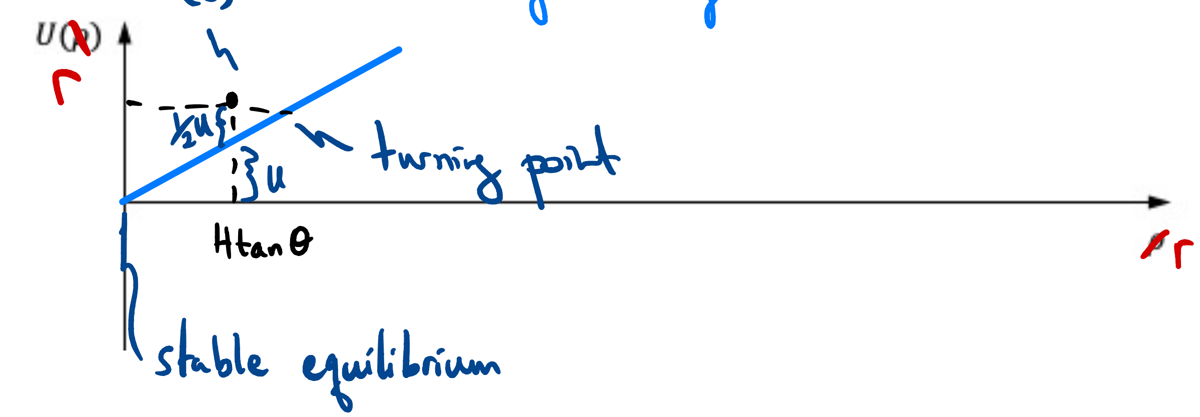
d) What is the relationship between the kinetic energy K and the potential energy U in the scenario described in (c)?

$v^2 = \omega_0^2\rho^2 = \frac{g}{H\tan^2\theta}\cdot H^2\tan^2\theta = gH$

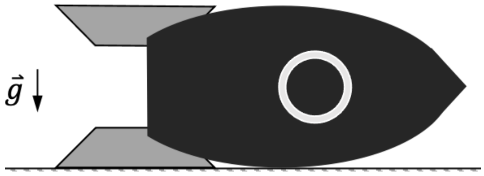
$\Rightarrow K = \frac{1}{2}mv^2 = \frac{1}{2}mgH = \frac{1}{2}U$

e) Sketch the potential energy function $U(\cancel{\rho}\, r)$ where $\cancel{\rho}\, r$ is the ~~horizontal~~ distance from the ~~z-axis,~~ origin and indicating the kinetic and potential energies for the scenario described in (c). Note any stable or unstable equilibria and turning points.

$U = mgh = mgr\cos\theta$



3. Tabletop rocket.



A rocket has mass m_0, including fuel of mass $m_0/2$. It starts at rest on a horizontal surface, whose coefficient of friction is μ (same for static and kinetic cases). At time $t = 0$, the fuel is ignited. Assume combusted fuel is ejected horizontally toward the left at a constant rate $\gamma = |dm/dt|$ with a constant velocity u relative to the rocket. Thus, the thrust force from the fuel burning is $\vec{F}_P = \gamma\vec{u}$. (You will <u>not</u> need to derive the "rocket equation.")

a) Draw momentum diagrams representing time t and time $t + \Delta t$ in the ground reference frame.

Answer inside the box

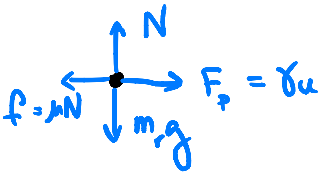
Time $t_i = t$:

Time $t_f = t + \Delta t$:

$m_r(t) = m_0 - \gamma t$

b) Draw a free body diagram. What is the condition on γ for the rocket to start moving? In the case that it does not start moving at time $t = 0$, at what time will it start?

<u>FBD</u>:



Condition for motion: $\sum F_x > 0 \Rightarrow \gamma u - \mu m_r g > 0$

or $\gamma > \frac{\mu m_r g}{u}$

The rocket's mass is $m_r(t) = m_0 - \gamma t$. So, if the above inequality isn't satisfied at $t = 0$, it will be at:

$$\gamma = \frac{\mu(m_0 - \gamma t) g}{u}$$

$\Rightarrow m_0 - \gamma t = \frac{\gamma u}{\mu g}$ or $t = \frac{m_0}{\gamma} - \frac{u}{\mu g}$

3. Tabletop rocket.

In the following parts (c-e), assume the condition in (b) is met at $t = 0$.

c) Determine the velocity as a function of time, $v(t)$ as a function of the given variables. What will be the maximum speed v_{max} reached by the rocket?

Newton's 2nd: $\sum F_x = \gamma u - \mu m_r g = ma = m_r \frac{dv}{dt}$

$$\Rightarrow \frac{dv}{dt} = \frac{-u}{m_r}\frac{dm_r}{dt} - \mu g$$

as γ = |dm$_r$/dt| and dm$_r$/dt<0, so γ = -dm$_r$/dt

separation of variables

$$\Rightarrow dv = -u\frac{dm_r}{m_r} - \mu g\, dt$$

integrate: $\int dv = -\int u \frac{dm_r}{m_r} - \int \mu g\, dt$

$$v(t) = \underbrace{C}_{\text{integrative const.}} - u\ln(m_r) - \mu g t$$

Boundary condition: $v(t=0) = 0 \Rightarrow C = u\ln(m_o)$

$$v(t) = -u\ln\left(\frac{m_r(t)}{m_o}\right) - \mu g t$$

Maximum speed is reached when all fuel is burned.

$$m_r(t) = m_o - \gamma t = \frac{m_o}{2} \Rightarrow t = \frac{m_o}{2\gamma}$$

$$v_{max} = u\ln 2 - \mu g \frac{m_o}{2\gamma}$$

3. Tabletop rocket.

d) What is the distance d traveled by the rocket after all of the fuel is burned? Express your answer in terms of the given variables and v_{max}.

After the fuel is burned, motion under constant (negative) acceleration.

Can be solved using equation of motion

$$\frac{m_0}{2} a = -\mu \frac{m_0}{2} g \Rightarrow a = -\mu g \qquad v = v_{max} - \mu g t$$

$$d = v_{max} t - \frac{1}{2} \mu g t^2$$

Condition: time at which velocity $\to 0$ $t^* = v_{max} / \mu g$

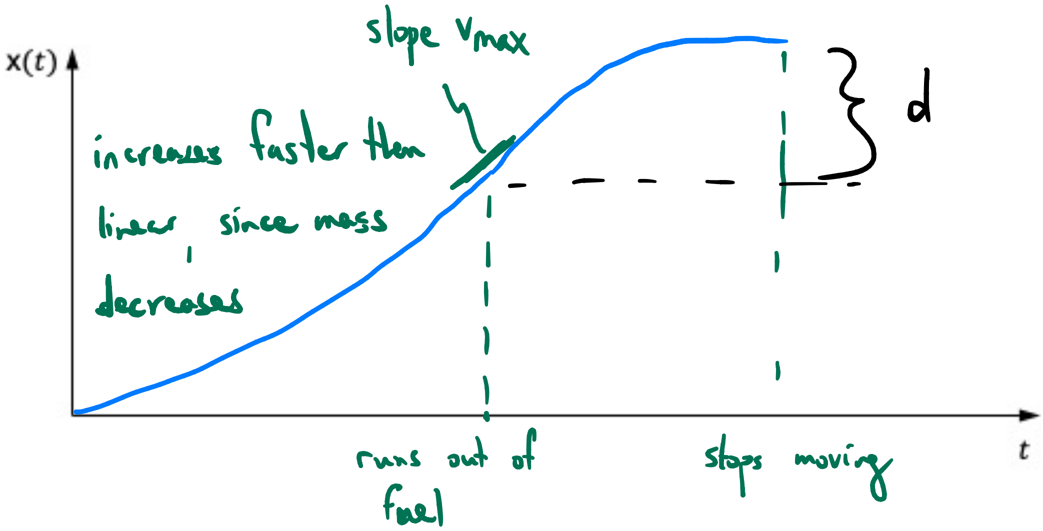
Substitute into d equation.

Alternatively, use energy (work-kinetic energy theorem).

$$\Delta K = \frac{1}{2} \frac{m_0}{2} v_{max}^2 - 0 = W = \mu \frac{m_0}{2} g d$$

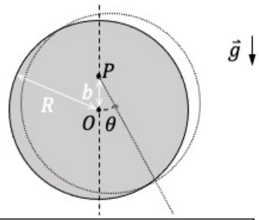
$$\Rightarrow d = \frac{v_{max}^2}{2 \mu g}$$

e) Sketch the position as a function of time of the rocket from time $t = 0$ until it is at rest. Be sure to note any important features.

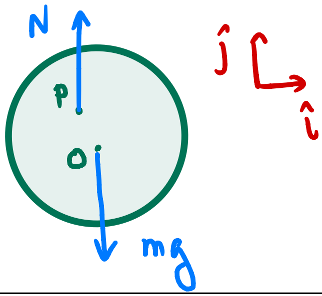


4. Swinging saucer.

A solid, uniform thin disk of mass m and radius R hangs vertically. The disk is attached to pivot about the point P, which is a distance b from the disk's center, O. All motion takes place in the plane of the disk. Neglect friction.



a) Determine the forces acting on the disk (draw a free body diagram).



Forces:
gravitational force, from center of mass
normal force, from pivot point.

b) The disk is rotated so that a line through P and O makes an angle θ_0 relative to gravity, and then released. Write an expression for the initial torque, τ, relative to point P.

$$\vec{\tau} = \vec{r}_p \times \vec{F} = b\left(\sin\theta\,\hat{i} - \cos\theta\,\hat{j}\right) \times mg(-\hat{j})$$

$$= -mgb\sin\theta\,\hat{k}$$

c) Calculate from general principles the moment of inertia, I_P, of the disk about the pivot point.

Using Steiner's theorem (parallel axis), we know

$$I_p = I_{cm} + mb^2$$

For a disk, divide into infinitesimal rings centered about O.

$$dm = \underbrace{\frac{m}{\pi R^2}}_{\text{density, } \sigma} \underbrace{2\pi r\,dr}_{\text{area of ring}} = \frac{2m}{R^2} r\,dr$$

$$I_{cm} = \int_{\text{disk}} r^2\,dm = \int_0^R \frac{2m}{R^2} r^3\,dr = \frac{2m}{R^2} \frac{r^4}{4}\Big|_0^R$$

$$= \frac{m}{2} R^2$$

$$\Rightarrow I_p = \frac{1}{2} mR^2 + mb^2$$

4. Swinging saucer.

d) Write the differential equation of motion of the disk in terms of the given variables.

Sum of torques about pivot point:

$$\sum \tau_p = I_p \alpha$$

$$-mgb\sin\theta = \left(\tfrac{1}{2}mR^2 + mb^2\right)\ddot{\theta}$$

Rewrite: $$\ddot{\theta} + \frac{bg}{\left(\frac{1}{2}R^2+b^2\right)}\sin\theta = 0$$

e) Assuming the initial angle θ_0 is small, what is the period ω_0 of the disk's oscillation? Show that your answer makes sense in the appropriate limit, when compared with a pendulum composed of a point mass suspended from a string.

Under the small angle approximation, $\sin\theta \approx \theta$

$\Rightarrow$ eq'n of motion $$\ddot{\theta} + \frac{bg}{\left(\frac{1}{2}R^2+b^2\right)}\theta = 0$$

This is simple harmonic motion with frequency

$$\omega_0 = \sqrt{\frac{bg}{\frac{1}{2}R^2+b^2}} \text{ rad/s}$$

and period $$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{\frac{1}{2}R^2+b^2}{bg}} \text{ s}$$

This is like a point mass in the limit $R \to 0$ and $b \to L$

$\Rightarrow$ $\omega_0 = \sqrt{\frac{g}{L}}$ just as for a point pendulum