Final exam solutions

1 Escape through a window (10 points)

Breakdown of points

a) 4 points

b) 1 point

c) 2 points

d) 3 points

Solution

a) **What are the forces on the person** (M_P) **?**

- The force $\vec{T}$ from the rope, vertical and upward.

- The weight $\vec{F}_{gP} = M_P\vec{g}$ of the person, vertical and downward.

1 point for the two forces, in the correct directions, in words or a free body diagram

A

What are the forces on the table (M_T) **?**

- The force $\vec{T}'$ from the rope, parallel to the floor and rightward. This has the same magnitude as the tension force $\vec{T}$ acting on the person.

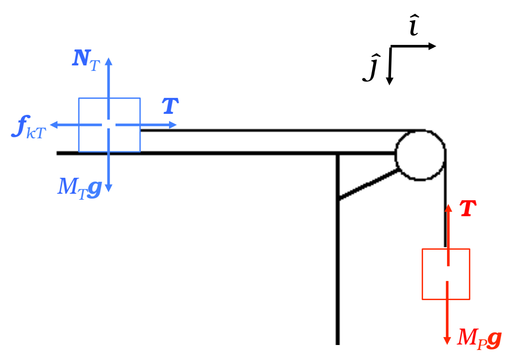
- The force of kinetic friction $\vec{f_{kT}}$, parallel and opposite in direction to the motion.

- The weight $\vec{F}_{gT} = M_T\vec{g}$ of the table, vertical and downward.

- The contact force $\vec{N}_T$, vertical and upward, balancing $\vec{F}_{gT}$.

1 point for the four forces, in the correct directions, in words or a free body diagram

B



Apply Newton's 2nd law $\Sigma\vec{F} = m\vec{a}$ **to each mass**

For the person :

$$M_P\vec{a}_P = M_P\vec{g} + \vec{T} = (M_Pg - T)\hat{\jmath} \tag{1}$$

For the table :

$$M_T\vec{a}_T = M_T\vec{g} + \vec{T}' + \vec{N}_T + \vec{f}_T = (M_Tg - N_T)\hat{\jmath} + (T' - \mu N_T)\hat{\imath} \tag{2}$$

1 point : $T = T'$ (0.5), condition for slipping $a_P = a_T > 0$ (or, $a_P = a_T = 0$) (0.5)

C

What is the maximum mass of the table ?

From (1) with the condition for slipping, $T = M_P g$. From (2), the table is in equilibrium in $\hat{j}$, so $N_T = M_T g$. Thus (2) in $\hat{\imath}$ becomes :

$$M_T = T/\mu_s g = M_P/\mu_s \quad \boxed{\text{1 point}}_{\text{D}} \tag{3}$$

b) **What is the minimum mass of the vase ?** The mass of the vase is added to the mass of the table, and the two can be considered as a single object since they don't move with respect to each other. (3) becomes :

$$M_V = M_P/\mu_s - M_T \quad \boxed{\text{1 point}}_{\text{E}} \tag{4}$$

c) Since the vase and the table move together, the two can be considered as a single object with the mass of the vase added to the mass of the table. Furthermore, all three masses(flatmate, vase and table) have the same magnitude of acceleration.
Apply Newton's 2nd law $\Sigma\vec{F} = m\vec{a}$ to each mass
For the flatmate, the equation is the same as in part a) with the mass M_F instead of M_P :

$$M_F\vec{a} = M_F\vec{g} + \vec{T} = (M_F g - T)\hat{j} \tag{5}$$

For the vase+table :

$$(M_T + M_V)\vec{a} = ((M_T + M_V)g - N_{TV})\hat{j} + (T - \mu N_{TV})\hat{\imath} \quad \boxed{\text{1 point}}_{\text{F}} \tag{6}$$

Solving for the acceleration,

$$a = \frac{(M_F - \mu_k(M_T + M_V))g}{M_T + M_V + M_F} \quad \boxed{\text{1 point}}_{\text{G}} \tag{7}$$

d) **What is the maximum value of M_F such that the vase doesn't slide with respect to the table ?**
The application of Newton's 2nd law to the person is given by (5).
Newton's 2nd for the table+vase is given by (6).
The only difference is that now we have to consider the vase independently from the table.

$$M_V\vec{a}_V = (M_V g - F_{VT})\hat{j} + (F_s)\hat{\imath} \tag{8}$$

where F_{VT} and $F_s < \beta_s F_{VT} = \beta_s M_V g$ are 3rd law action-reaction pairs, with the gravity/contact force and static friction respectively. $\boxed{\text{1 point for } F_{VT}}_{\text{H}}$ $\boxed{\text{1 point for } F_s}_{\text{I}}$
The condition for the vase not sliding on the table is that all of the masses in the system have the same magnitude of the acceleration, a. As before, the tension in the rope is the same everywhere. Solving the $\hat{\imath}$-component of (8) for a_x, the constraint on the magnitude of the acceleration becomes $a < \beta_s g$.
Since the acceleration of the vase has to be the same as the system vase+table which is defined by equation (7), it is possible to determine the maximum value of M_F by imposing :

$$\begin{aligned} a = \frac{(M_F - \mu_k(M_T + M_V))g}{M_T + M_V + M_F} &< \beta_s g \\ \frac{(M_F - \mu_k(M_T + M_V))}{M_T + M_V + M_F} &< \beta_s \\ \beta_s(M_T + M_V + M_F) &> M_F - \mu_k M_T - \mu_k M_V \\ -M_F &> -\mu_k(M_T + M_V) - \beta_s(M_T + M_V + M_F) \\ & F\boxed{\text{1 point}}_{\text{J}} \end{aligned} \tag{9}$$

2 Oscillating sled (11 points)

Breakdown of points

a) 3 points
b) 4 points
c) 2 points
d) 2 points

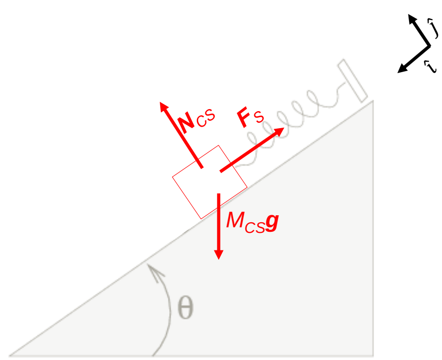
Solution

a) **What are the forces on the system child M_C + sled M_S ?**

- The force $\vec{F}_s$ from the spring, parallel to the plane. Its direction depends on the length of the spring relative to its rest length (l_0). At the equilibrium position, it points up the plane.

- The weight $\vec{F}_{gCS} = (M_C + M_S)\vec{g}$, vertical and downward.

- The contact force $\vec{N}$, perpendicular to the plane and balancing the projection of $\vec{F}_{gCS}$ onto the same axis. 1 point for the three forces, in the correct directions, in words or a free body diagram A



Apply Newton's 2nd law $\Sigma\vec{F} = m\vec{a}$ to the total mass $m = M_{CS} = M_C + M_S$

$$M_{CS}\vec{a}_{CS} = M_{CS}\vec{g} + \vec{N} + \vec{F_S} = (M_{CS}g\sin\theta - F_S)\hat{\imath} + (N - M_{CS}g\cos\theta)\hat{\jmath} \tag{10}$$

Since the system is in equilibrium,

$$\begin{aligned} M_{CS}g\sin\theta - F_S &= 0 \\ N - M_{CS}g\cos\theta &= 0 \end{aligned} \tag{11}$$

1 point for the equilibrium condition B

If we define l_0 as the rest length of the spring, then

$$\begin{aligned} F_S &= k(l_2 - l_0) \\ k(l_2 - l_0) &= M_{CS}g\sin\theta \\ l_2 &= l_0 + \frac{M_{CS}g\sin\theta}{k} \end{aligned} \tag{12}$$

Since we do not know l_0 we have to compute it using l_1 which is the length of the spring when only the sled is attached to the spring,

$$\begin{aligned}
F_S &= k(l_1 - l_0) \\
k(l_1 - l_0) &= M_S g \sin\theta \\
l_1 &= l_0 + \frac{M_S g \sin\theta}{k} \\
l_0 &= l_1 - \frac{M_S g \sin\theta}{k} \\
l_2 = l_1 - \frac{M_S g \sin\theta}{k} &+ \frac{M_{CS} g \sin\theta}{k} \\
l_2 &= l_1 + \frac{M_C g \sin\theta}{k}
\end{aligned} \tag{13}$$

$\boxed{\text{1 point for the expression of } l_2 \text{ as a function of the given variables}}$ C

b) **What is the speed of the sled when it returns for the first time to position l_1 ?** The velocity of the sled is equal to zero when it returns for the first time to the position l_1 since the system child + sled is at rest ($v = 0$) at the position l_1 at the time of release. This is due to conservation of energy. $\boxed{\text{1 point (0.5) } v = 0, \text{ (0.5) conservation of energy}}$ D

What is the maximum speed v of the sled, and what is the length of the spring when it occurs ? Since the motion is simple harmonic (oscillation), the maximum speed occurs at position l_2 $\boxed{\text{1 point}}$ E.

Strategy : apply the conservation of mechanical energy to the system child M_C + sled M_S between positions l_1 and l_2.

Choose as the origin the position l_2.

In the position l_1, the system child + sled has a height

$$h_1 = (l_2 - l_1)\sin\theta \tag{14}$$

The spring is elongated by $l_1 - l_0$ with respect to the rest position l_0. The system is at rest, so the mechanical energy is given by only the sum of the potential gravitational energy and the elastic energy.

$$\begin{aligned}
E_{mec,1} = U_{grav} + U_{spring} &= (M_C + M_S) g h_1 + \frac{1}{2}k(l_1 - l_0)^2 \\
E_{mec,1} &= (M_C + M_S)(l_2 - l_1) g \sin\theta + \frac{1}{2}k(l_1 - l_0)^2
\end{aligned} \tag{15}$$

In the position l_2, the total mechanical energy is given by the sum of the elastic energy and the kinetic energy

$$E_{mec,2} = \frac{1}{2}k(l_2 - l_0)^2 + \frac{1}{2}(M_C + M_S)v^2 \tag{16}$$

$\boxed{\text{1 point (0.5) for mechanical energy of position 1, (0.5) for position 2}}$ F

Using the equation (13), (15) and (16), we obtain,

$$\begin{aligned}
E_{mec,1} &= E_{mec,2} \\
(M_C + M_S) g \left(\frac{M_C g \sin^{2}\theta}{k}\right) + \frac{1}{2}k\left(\frac{M_S g \sin\theta}{k}\right)^2 &= \frac{1}{2}k\left(\frac{M_C + M_S}{k} g \sin\theta\right)^2 + \frac{1}{2}(M_C + M_S)v^2 \\
v &= g\sin\theta\sqrt{\frac{(M_C)^2}{k(M_C + M_S)}} \quad \boxed{\text{1 point}} \text{ G}
\end{aligned} \tag{17}$$

When the system reaches position l_2, some of the gravitational potential energy that the system had in the position l_1 has been converted into kinetic and elastic energy. When the system passes l_2, the potential energy continues to decrease and the kinetic energy starts to decrease as it is converted into elastic energy. When the system reaches its lowest point (l_3) the kinetic and gravitational potential energy are both zero, having been converted into elastic energy.

c) **Apply Newton's 2nd law $\Sigma\vec{F} = m\vec{a}$ to the total mass $m = M_C + M_S$** along the direction of motion $\hat{\imath}$,

$$\begin{aligned}(M_C + M_S)a &= (M_C + M_S)g\sin\theta - k(l - l_0)\\ (M_C + M_S)\ddot{x} &= (M_C + M_S)g\sin\theta - k(x - l_0)\\ \ddot{x} + \frac{k}{M_C + M_S}x &= g\sin\theta + \frac{kl_0}{M_C + M_S}\\ \ddot{x} + \omega^2 x &= b \quad \boxed{\text{1 point}}_{\text{H}}\end{aligned} \tag{18}$$

What is the angular frequency ω of the oscillation of the system?

$$\omega = \sqrt{\frac{k}{M_C + M_S}} \quad \boxed{\text{1 point}}_{\text{I}} \tag{19}$$

d) **What is the motion of the system child + sled when it is released from rest?**
A general solution to the equation of motion is

$$\begin{aligned}x(t) &= A\cos(\omega t + \phi) + x_0\\ \dot{x} &= -A\omega\sin(\omega t + \phi)\end{aligned} \tag{20}$$

where x_0 represents the position of the center of the oscillation (l_2) with respect to the center of the reference system.
Choose as the origin position l_2, so $x_0 = 0$.
Find the particular solution for the situation described in (b). At $t = 0$, we have

$$\begin{aligned}x(t=0) = l_2 - l_1 &= M_C g\frac{\sin\theta}{k}\\ \dot{x}(t=0) &= 0\end{aligned} \tag{21}$$

Then,

$$\begin{aligned}\dot{x}(t=0) &= -A\omega\sin(\phi) = 0 \rightarrow \phi = 0\\ x(t=0) = A\cos(\phi) + x_0 = A\cos(0) + 0 &= M_C g\frac{\sin\theta}{k} \rightarrow A = M_C g\frac{\sin\theta}{k}\end{aligned} \tag{22}$$

$\boxed{\text{1 point applying correct boundary conditions}}_{\text{J}}$

Use it to find the velocity at l_2. When the system is passing through the position l_2 the phase of the oscillation has to be equal to $\frac{\pi}{2}$ then, $\rightarrow \sin\left(\omega t = \frac{\pi}{2}\right) = 1$

$$\begin{aligned}\dot{x}(t) = -A\omega\sin(\omega t = \frac{\pi}{2}) = -A\omega &= -M_C g\frac{\sin\theta}{k}\sqrt{\frac{k}{M_C + M_S}}\\ |v| &= g\sin\theta\sqrt{\frac{(M_C)^2}{k(M_C + M_S)}} \quad \boxed{\text{1 point}}_{\text{K}}\end{aligned} \tag{23}$$

As found in part b!

<u>Alternative solutions :</u>

i) Choose as the origin position l_3.

$$\begin{aligned} x_0 = l_2 - l_1 = M_C g\frac{\sin\theta}{k} \\ x(t=0) = l_3 - l_1 = 2M_C g\frac{\sin\theta}{k} \\ \dot{x}(t=0) = 0 \end{aligned} \tag{24}$$

then,

$$\begin{aligned} \dot{x}(t=0) = -A\omega\sin(\phi) = 0 \to \phi = 0, 2\pi \\ x(t=0) = A\cos(\phi) + x_0 = A\cos(0) + M_C g\frac{\sin\theta}{k} = 2M_C g\frac{\sin\theta}{k} \to A = M_C g\frac{\sin\theta}{k} \end{aligned} \tag{25}$$

ii) Choose as the origin position l_1,

$$\begin{aligned} x_0 = -(l_2 - l_1) = M_C g\frac{\sin\theta}{k} \\ x(t=0) = 0 \\ \dot{x}(t=0) = 0 \end{aligned} \tag{26}$$

then,

$$\begin{aligned} \dot{x}(t=0) = -A\omega\sin(\phi) = 0 \to \phi = 0, 2\pi \\ x(t=0) = A\cos(\phi) + x_0 = A\cos(0) - M_C g\frac{\sin\theta}{k} = 0 \to A = M_C g\frac{\sin\theta}{k} \end{aligned} \tag{27}$$

Final exam solutions

3 Horizontal collision with a barbell (10 points)

Breakdown of points

a) 2 points
b) 1 points
c) 3 points
d) 4 points

Solution

a) **What is the total linear momentum of the system projectile + barbell ?**
Before the collision, in the reference frame where the barbell is at rest, the only contribution to the total momentum is given by the projectile

$$\vec{p_{tot}} = M_p \vec{v} \quad \boxed{\text{1 point}}_{\text{A}} \tag{28}$$

Is the linear momentum conserved in the collision ?
During the collision the linear momentum is conserved since there are no external forces that are doing work $\boxed{\text{1 point}}_{\text{B}}$

b) **What is the position l_{CM} of the center of mass ?**
The center of mass of a pair of particles with masses m_1 and m_2 and positions x_1 and x_2 is defined $\frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$. Thus,

$$l_{CM} = \frac{Ml}{M + M_p + m} \tag{29}$$

Show that $\lim_{M\to\infty} l_{CM} = l$.

$$\lim_{M\to\infty} l_{CM} = \lim_{M\to\infty} \frac{l}{1 + M_p/M + m/M} = l \quad \boxed{\text{1 point}}_{\text{C}} \tag{30}$$

c) **What is the angular momentum $\vec{L}$ after the collision ?**
What is the direction of the angular momentum ?
Since the rotation is in the x, y plane, the direction of the angular velocity will be along the z axis and coming out of the plane.

$$\vec{\omega} = \omega \hat{k} \tag{31}$$

If I is the total moment of inertia, the angular momentum is defined by :

$$\vec{L} = I\vec{\omega} \tag{32}$$

so, the angular momentum is in the same direction as $\vec{\omega}$. $\boxed{\text{1 point}}_{\text{D}}$
What is the moment of inertia I after the collision ? Note that this needs to be calculated with respect to the center of mass, so it is defined :

$$I = \sum_{i=1}^{2} m_i r_i^2 \tag{33}$$

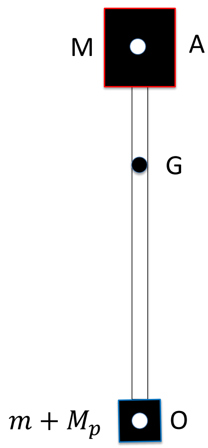
where the r_i are the distance of each mass from the center of mass calculated in (b). Thus,

$$I = M\left[l - l\left(\frac{M}{M_p + m + M}\right)\right]^2 + (M_p + m)l^2\left(\frac{M}{M_p + m + M}\right)^2 \quad \boxed{\text{1 point}}_{\text{E}} \tag{34}$$

Then, the modulus of the angular momentum becomes :

$$L = \frac{M(M_p + m)}{(M_p + m + M)} l^2 \omega \quad \boxed{\text{1 point}}_{\text{F}} \tag{35}$$

<u>Alternative solution :</u> Define the positions A, G and O as shown in the picture



$$\begin{aligned} |\vec{L}| &= |\sum_{i=1} \vec{R_i} \times m_i \vec{v_i}| \\ L &= AGMv_M + GO(M_p + m)v_{M_pm} \\ L &= \left[Ml\left(\frac{M_p + m}{M_p + m + M}\right)v_M + (M_p + m)l\left(\frac{M}{M_p + m + M}\right)v_{M_pm}\right] \end{aligned} \tag{36}$$

Since,

$$\begin{aligned} v_M &= \omega AG \\ v_{M_pm} &= \omega GO \end{aligned} \tag{37}$$

The equation (36) becomes,

$$\begin{aligned} L &= \left[Ml^2\left(\frac{M_p + m}{M_p + m + M}\right)^2 + (M_p + m)l^2\left(\frac{M}{M_p + m + M}\right)^2\right]\omega \\ L &= \frac{M(M_p + m)}{(M_p + m + M)^2} l^2 (M_p + m + M)\omega \\ L &= \frac{M(M_p + m)}{(M_p + m + M)} l^2 \omega \end{aligned} \tag{38}$$

d) **What is the angular frequency ω?**

The variation of the angular momentum is defined as :

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{tot} \tag{39}$$

The angular momentum is conserved because there are no external torques.
Let's define the angular momentum after the collision L_f, as found in part (c).
Just before the collision the angular momentum is given by :

$$L_i = |\vec{l}_{CM} \times M_p\vec{v}| = \frac{M}{(M_p + m + M)} l M_p v \quad \boxed{\text{1 point}}_{\text{G}} \tag{40}$$

Thus, by conservation of the angular momentum,

$$\begin{aligned} \frac{M}{(M_p + m + M)} l M_p v &= \frac{M(M_p + m)}{(M_p + m + M)} l^2 \omega \\ \omega &= \frac{M_p}{M_p + m} \frac{v}{l} \quad \boxed{\text{1 point}}_{\text{H}} \end{aligned} \tag{41}$$

When $M_p >> m$

$$\lim_{m/M_p \to 0} \omega = \lim_{m/M_p \to 0} \frac{1}{1 + m/M_p} \frac{v}{l} = \frac{v}{l} \tag{42}$$

The velocity of the projectile now completely defines the angular velocity of the system after the collision, and it is independent of any of the masses.

$\boxed{\text{1 point (0.5) for explanation and (0.5) for limit}}_{\text{I}}$

When $M_p << m$

$$\lim_{M_p \to 0} \omega \to 0 \tag{43}$$

The angular velocity will be around zero since the momentum transferred by the projectile will be very small.

$\boxed{\text{1 point (0.5) for explanation and (0.5) for limit}}_{\text{J}}$