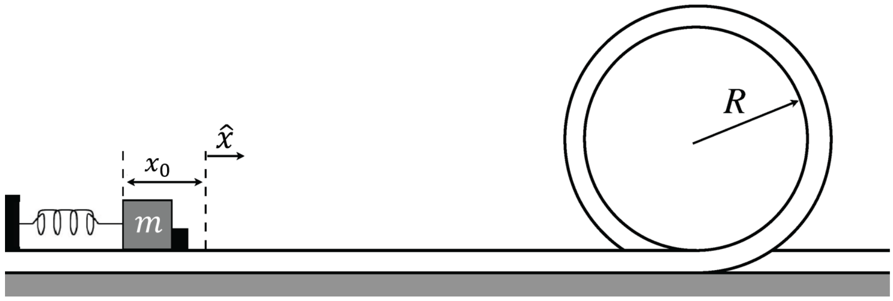
1. Spring and loop (8 points)



a. (**1.0 points**) *Assuming that the block is able to go around the loop, what is the speed of the block after exiting the loop? What is its velocity? Give the answer in terms of m, k, and x_0.*

Scoring: 0.5 points for applying conservation of energy. 0.25 points for correctly finding the speed of the block and 0.25 points for the velocity.

To find the speed of the block after exiting the loop, conservation of energy can be applied. Initially, the block is at rest and its kinetic energy is $K_0 = 0$. The block is also resting on the track at a height of $y_0 = 0$, so the gravitational potential energy is $U_{g,0} = 0$. Only the potential energy of the spring is relevant:

$$E_i = K_i + U_{s,i} + U_{g,i} = 0 + \frac{1}{2}k\,x_0^2 + 0 = \frac{1}{2}k\,x_0^2 \tag{1}$$

After exiting the loop, the block is again at a height of $y_f = 0$ and $U_{s,f} = 0$. Therefore, the only contributor to E_f is the kinetic energy of the block:

$$E_f = \frac{1}{2}m\,v_f^2 \tag{2}$$

Since the track is frictionless, $E_i = E_f$. Solving for the speed of the block after exiting the loop, v_f, we find:

$$v_f = x_0\sqrt{\frac{k}{m}} \tag{3}$$

In this case, the block is moving purely in the $\hat{x}$ direction so the velocity of the block is:

$$\vec{v}_f = x_0\sqrt{\frac{k}{m}}\,\hat{x} \tag{4}$$

b. (**2.0 points**) *If the force exerted by the loop on the block (i.e., the normal force) at the top of the loop is equal to the weight, find the speed of the block at the top of the loop in terms of g and R. What is the initial spring compression x_0 required for this to occur?*

Scoring: 0.5 points for equating the forces on the block in the radial direction to the centripetal force. 0.5 points for correctly solving for v_t. 0.5 points for applying conservation of energy to solve for x_0. 0.5 points for correctly solving for x_0.

First we consider the forces acting on the block at the top of the loop. These include the normal force N, where the problem statement tells us that $N = mg$ and the force of gravity, $F_g = mg$. Both of these forces are acting in the negative $\hat{y}$ direction. From Newton's second law and the properties of uniform circular motion, we know that the net force in the radial direction is equal to the centripetal force:

$$-mg - N = -\frac{mv_t^2}{R}, \tag{5}$$

Substituting in the value for N and solving for v_t:

$$v_t = \sqrt{2gR} \tag{6}$$

To find the compression of the spring required for this value of v_t, we return to conservation of energy. The energy of the block at the top of the loop is given by:

$$E_t = K_t + U_{g,t} = \frac{1}{2}m\,v_t^2 + m\,g\,y_t \tag{7}$$

where $y_t = 2R$. The initial energy of the block is only due to the compression of the spring, given by equation 1. Equating E_i and E_t, we find:

$$\frac{1}{2}k\,x_0^2 = \frac{1}{2}m\,v_t^2 + m\,g\,(2R) \tag{8}$$

Substituting in the value for v_t found in equation 6 and solving for x_0:

$$x_0 = \sqrt{\frac{6mgR}{k}} \tag{9}$$

c. (**1.0 points**) *What is the minimum value of x_0 required for the block to complete the loop without ever losing contact with the track?*

Scoring: As this problem is similar to part b., award 0.25 points for equating the forces on the block in the radial direction to the centripetal force and 0.25 points for correctly solving for v_t. Then, 0.25 points for applying conservation of energy to solve for x_0. 0.25 points for correctly solving for x_0.

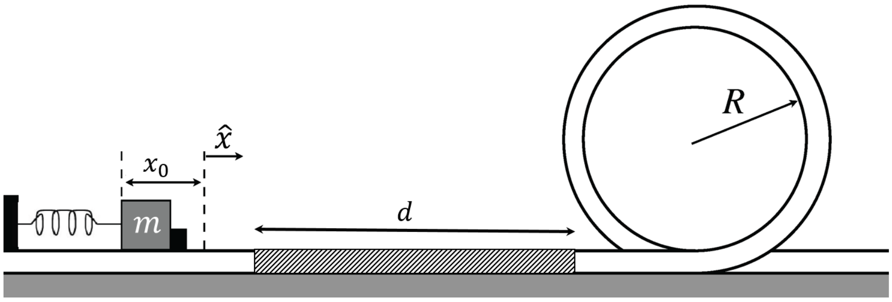
Now we consider the situation where the block just barely traverses the loop without losing contact with the track. Recognize that this corresponds to the situation where the normal force at the top of the loop goes to zero. We can then return to equation 5 where $N = 0$. Solving for v_t gives:

$$v_t = \sqrt{gR} \tag{10}$$

We can now solve for x_0 in the same way as part b., using conservation of energy with the new value for v_t:

$$\frac{1}{2}k\,x_0^2 = \frac{1}{2}m\,v_t^2 + 2m\,g\,R \tag{11}$$

$$x_0 = \sqrt{\frac{5mgR}{k}} \tag{12}$$



d. (**1.0 points**) *Now consider the situation in which the block has to move a distance d across a rough surface with a constant coefficient of kinetic friction μ_k before entering the loop. Compute the work done by friction on the block.*

Scoring: 0.5 points for attempting to use equation 13 and 0.5 points for correctly solving for W_{fr}.

To calculate the work done by friction on the block:

$$W_{fr} = \vec{F}_{fr} \cdot \vec{d} \tag{13}$$

Here, $\vec{F}_{fr}$ is the friction force $F_{fr} = -\mu_k N \hat{x}$ and the displacement is $\vec{d} = d \hat{x}$ over the rough part of the track. Since the sum of forces on the block in the $\hat{y}$ direction is 0, $N = mg$. Note that the friction force is acting opposite the motion of the block so the work done on the block is negative:

$$W_{fr} = -\mu_k m g d \tag{14}$$

e. (**1.0 points**) *What is the minimum value of x_0 required for the block to make it all the way across the rough surface?*

Scoring: 0.25 points for attempting to use the work-kinetic energy theorem. 0.25 points for correctly finding v_i. 0.25 points for applying conservation of energy and 0.25 points for correctly solving for x_0.

To find the compression required for the block to make it across d, we employ the work-kinetic energy theorem, i.e., $\Delta K = W_{\text{net}}$. In this case, only friction does work so $W_{\text{net}} = W_{fr}$ and:

$$\frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2 = W_{fr} \tag{15}$$

This allows us to solve for the velocity, v_i the block must have been traveling before traversing the rough surface to make it all the way across. In the case that the block just makes it across the surface, $v_f = 0$. Substituting the work done on the block from equation 14:

$$v_i = \sqrt{2\mu_k g d} \tag{16}$$

Now, conservation of energy can be used to find x_0. Here, we consider the situation at $t = 0$ where only the potential energy of the spring contributes and at the time t_i, immediately after the block is released from the spring, but prior to entering the rough stretch of track. In the former case, the energy of the block is given by equation 1. After being released from the spring, the block has only kinetic energy, with a speed of v_i. Equating E_0 and E_i and solving for x_0:

$$\frac{1}{2} k x_0^2 = \frac{1}{2} m v_i^2 \tag{17}$$

$$x_0 = \sqrt{\frac{2\mu_k\, mgd}{k}} \tag{18}$$

f. (**2.0 points**) *What is the minimum value of x_0 required for the block to complete the loop without ever losing contact with the track? This is different from part c. in that, now, the block has to move across the rough surface. What is the speed of the block after exiting the loop?*

Scoring: 0.5 points for attempting to use the work-kinetic energy theorem. Up to 0.5 points for substituting the correct values for the work-kinetic energy theorem, e.g., K_i, K_f, $U_{s,i}$, $U_{s,f}$, $U_{g,i}$, $U_{g,f}$ (subtract 0.1 points for each that is incorrect). 0.5 points for correctly solving for x_0 and 0.5 points for v_f.

As with part c., we are asked to find the minimum value of x_0 where the block completes the loop without losing contact with the track. Similarly, at the top of the track, $N = 0$ and $v_t = \sqrt{gR}$. However, now we must account for the work done by friction to calculate x_0. To do so, use the fact that, whenever non-conservative forces are present, the work-kinetic energy theorem leads to:

$$W_{\text{non}} = \Delta E_m = (K_f + U_{s,f} + U_{g,f}) - (K_i + U_{s,i} + U_{g,i}) \tag{19}$$

In this case, friction is the only non-conservative force that does work, so $W_{\text{non}} = W_{fr}$.

Thus, if we choose moment i to be the moment right before releasing the latch and f the moment in which the block gets to the upper part of the loop, we have $W_{fr} = K_f + U_{g,f} - U_{s,i}$. We can then plug in our values for W_{fr} (equation 14), K_f and $U_{g,f}$ (right-hand side of equation 11), and $U_{s,i}$ (equation 1). Solving for x_0:

$$-\mu_k\, m\, g\, d = \frac{1}{2} m\, g\, R + 2m\, g\, R - \frac{1}{2} k\, x_0^2 \tag{20}$$

$$x_0 = \sqrt{\frac{5mgR + 2\mu_k mgd}{k}} \tag{21}$$

Similarly, equation 19 can be applied to find the speed of the block after exiting the loop. Now, however, f refers to a time when the block has exited the loop. As a result, $U_{g,f} = 0$ and equation 19 becomes $W_{fr} = K_f - U_{s,i}$. Applying this to solve for v_f:

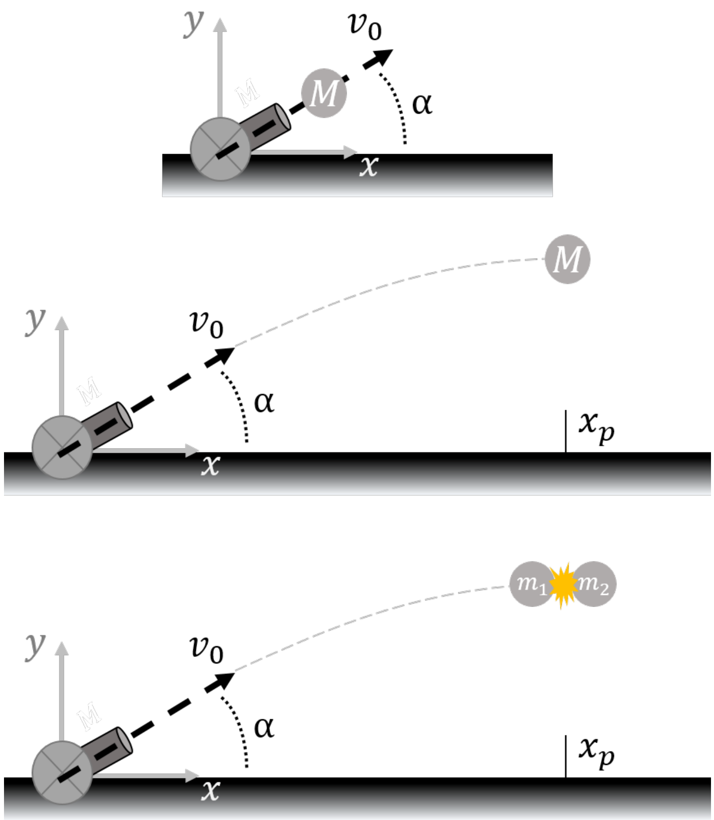
$$-\mu_k\, m\, g\, d = \frac{1}{2} m\, v_f^2 - \frac{1}{2} k\, x_0^2 \tag{22}$$

$$v_f = \sqrt{5gR} \tag{23}$$

2. Fragmenting projectile (8 points)

a. (**1.0 points**) *Make a sketch of the problem. Define clearly your reference frame.*

This is subjective, but the most natural reference frame would place the cannon at the origin in Cartesian coordinates, as shown below:



b. (**1.0 points**) *At what time does the explosion occur?*

Scoring: 0.5 points for finding equation 24 and 0.5 points for correctly solving for t_p.

The explosion occurs at the peak of the projectile's trajectory, corresponding to the location where $v_y = 0$. Since there is no air drag, the projectile is only subject to gravity and experiences a constant acceleration of $\vec{a} = -g\,\hat{y}$. Therefore, to find the time of the explosion, t_p:

$$v_y = v_{0,y} - g\,t \;\Rightarrow\; 0 = v_0 \sin(\alpha) - g\,t_p \tag{24}$$

$$t_p = \frac{v_0}{g}\sin(\alpha) \tag{25}$$

c. (**1.0 points**) *Determine the position at which the explosion occurs.*

Scoring: 0.25 points for correctly finding each of the four terms: $x(t)$, $y(t)$, x_p, and y_p (up to 1 point total). Subtract 0.2 points if the student does not consolidate x_p and y_p into a vector describing the position.

To find the location of the explosion, we can use the fact that the projectile experiences a constant acceleration. This dictates that the position of the projectile is given by:

$$y(t) = -\frac{1}{2} g\, t^2 + v_{0,y}\, t + y_0 \tag{26}$$

$$x(t) = v_{0,x}\, t + x_0 \tag{27}$$

In the reference frame shown in part a., $x_0 = 0$ and $v_{0,x} = v_0 \cos(\alpha)$. Plugging in the value for t_p found in part b. and using the identity $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$, the position in the x-direction where the explosion occurs is:

$$x_p = x(t_p) = \frac{v_0^2}{g} \sin(\alpha)\cos(\alpha) = \frac{v_0^2}{2g} \sin(2\alpha) \tag{28}$$

In the same reference frame, $y_0 = 0$ and $v_{0,y} = v_0 \sin(\alpha)$. The position in the y-direction is then:

$$y_p = y(t_p) = -\frac{1}{2} g\, t_p^2 + v_0 \sin(\alpha)\, t_p = -\frac{g}{2}\left(\frac{v_0}{g}\sin(\alpha)\right)^2 + \frac{v_0^2}{g}\sin^2(\alpha) = \frac{v_0^2}{2g}\sin^2(\alpha) \tag{29}$$

The position of the explosion is then the vector:

$$\vec{r}_p = x_p\, \hat{x} + y_p\, \hat{y} = \frac{v_0^2}{2g}\sin(2\alpha)\, \hat{x} + \frac{v_0^2}{2g}\sin^2(\alpha)\, \hat{y} \tag{30}$$

d. (**0.5 points**) *At what time do the fragments hit the ground? Why do they hit the ground at the same time?*

Scoring: 0.25 points for finding t_f, 0.25 points for an accurate explanation.

As specified in the problem statement, the fragments experience no change in the vertical component of their velocities following the explosion. Therefore, the fragments fall freely from a height of y_p, accelerating towards the ground under the influence of gravity. This takes the same amount of time that the projectile took to reach the apex of its trajectory, t_p. The fragments then hit the ground at time t_f, given by:

$$t_f = 2\, t_p = \frac{2v_0}{g}\sin(\alpha) \tag{31}$$

e. (**1.0 points**) *Determine the distance from the cannon at which the center of mass (CM) intercepts the ground.*

Scoring: Full points for finding the correct value of $x_{CM,f}$, 0.5 points if an accurate explanation is offered but the value found for $x_{CM,f}$ is incorrect.

The forces exerted on the fragments during the explosion are all internal to the system. The explosion then has no effect on the motion of the center of mass (CM), which continues to move like a point mass of mass M under the sole influence of gravity. This can be described by:

$$M\, \vec{a}_{CM} = \vec{F}_{\text{ext}} = \vec{F}_g = -M g\, \hat{y} \tag{32}$$

The center of mass will then reach the ground, i.e., $y = 0$, at twice the horizontal distance of the apex of the trajectory:

$$x_{CM,f} = 2\,x_p = \frac{v_0^2}{g}\sin(2\alpha) \tag{33}$$

Note that $x_{CM,f}$ is the distance to the cannon since the origin of the reference frame was fixed at the cannon.

f. (**1.5 points**) *Consider the case in which fragment m_1 reaches the cannon. At what distance from the cannon does m_2 land?*

Scoring: Full points for finding the correct value of $x_{2,f}$. If the correct value is not found, up to 1 point can be awarded based on the student's work, at the discretion of the grader.

There are at least two ways to solve this problem using conservation of momentum at the explosion or the center of mass. The latter is more straightforward but both are valid.

From the definition of the center of mass, after the explosion:

$$\vec{R}_{CM} = \frac{m_1\,\vec{r}_1 + m_2\,\vec{r}_2}{m_1 + m_2} \quad \Rightarrow \quad x_{CM} = \frac{m_1\,x_1 + m_2\,x_2}{m_1 + m_2} = \frac{m_1\,x_1 + m_2\,x_2}{M} \tag{34}$$

At time t_f:

$$x_{CM}(t_f) = x_{CM,f} = \frac{v_0^2}{g}\sin(2\alpha) = \frac{m_1\,x_{1,f} + m_2\,x_{2,f}}{M} \tag{35}$$

Now solving for when $x_{1,f} = 0$ (the case when the fragment m_1 reaches the cannon):

$$x_{2,f} = \frac{M}{m_2}x_{CM,f} = \frac{M}{M - m_1}\,\frac{v_0^2}{g}\sin(2\alpha) \tag{36}$$

g. (**0.5 points**) *What would the landing distance of m_2 be if, instead, m_1 landed at half the distance between the cannon and the horizontal location of the explosion?*

Using the same argument as part g., we can now solve for $x_{2,f}$ when $x_{1,f} = \frac{1}{2}x_p$:

$$x_{2,f} = \frac{M}{m_2}x_{CM,f} - \frac{m_1}{m_2}x_{1,f} = \frac{M}{m_2}(2x_p) - \frac{m_1}{m_2}(\frac{1}{2}x_p) \tag{37}$$

$$x_{2,f} = \left(\frac{4M - m_1}{M - m_1}\right)\frac{v_0^2}{4g}\sin(2\alpha) \tag{38}$$

h. (**1.5 points**) *Assuming that m_1 reaches the cannon, as in part f., what are the velocities of m_1 and m_2 immediately after the explosion?*

Scoring: Full points for finding the correct values of $\vec{v}_1$ and $\vec{v}_2$. If the correct value is not found, up to 1 point can be awarded based on the student's work: 0.5 points for recognizing that the impulse approximation should be used, 0.25 points for arriving at equation 39, and 0.25 points for using the correct value for v_1.

The system is subject to an external force (gravity), but we can use the impulse approximation to argue that momentum is conserved between the time immediately before and after the explosion.

Before the explosion: $\Sigma\vec{p}_i = Mv_{0,x}\,\hat{x} = Mv_0\cos(\alpha)\,\hat{x}$

After the explosion: $\Sigma\vec{p}_f = m_1\, v_1\, \hat{x} + m_2\, v_2\, \hat{x}$

Therefore:

$$M v_0 \cos(\alpha) = m_1\, v_1 + m_2\, v_2 \tag{39}$$

Since m_1 takes a time t_p to reach the cannon after the explosion (as specified by equation 31), it must be the case that $v_1 = -v_{0,x} = -v_0 \cos(\alpha)$. Then:

$$m_2\, v_2 = M v_0 \cos(\alpha) - m_1\, v_1 = M v_0 \cos(\alpha) - m_1(-v_0 \cos(\alpha)) = (M + m_1)\, v_0 \cos(\alpha) \tag{40}$$

$$v_2 = \frac{M + m_1}{m_2} v_0 \cos(\alpha) = \frac{M + m_1}{M - m_1} v_0 \cos(\alpha) \tag{41}$$

The answer is then:

$$\vec{v}_1 = -v_0 \cos(\alpha)\, \hat{x} \tag{42}$$

$$\vec{v}_2 = \frac{M + m_1}{M - m_1} v_0 \cos(\alpha)\, \hat{x} \tag{43}$$