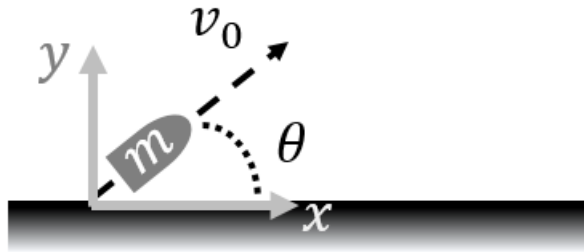


1. Projectile motion (5 points)

- a. (3.0 points) What are the velocity and position of the airborne projectile as a function of time? Give the answers in terms of v_0, θ, g and any initial position consistent with your reference frame.

First, choose a reference frame and show it clearly (1 point):

For example, you can draw a sketch of the figure in the problem statement and draw the reference frame on it. A convenient choice is the one shown below because it makes the initial position (at $t = 0$) be $\vec{r}_0 = 0$ (which means that the initial position along x is $x_0 = 0$ and the one along y is $y_0 = 0$). Furthermore, it makes the acceleration due to gravity have the simple form $\vec{g} = -g \hat{y}$.



Other choices of reference frames are nevertheless acceptable. What is important is that they be clearly stated and that they be correctly used in what follows.

Determine the velocity of the projectile (1 point):

We know that $\vec{g}$ is constant and only acts in the y direction. Therefore, the velocity along x and along y fulfill

$$\begin{aligned} v_x(t) &= v_{0,x} \\ v_y(t) &= v_{0,y} - gt \end{aligned} \tag{1}$$

where $v_{0,x}$ and $v_{0,y}$ are two constants corresponding to the projection along x and y (respectively) of the velocity at $t = 0$.

From the above figure, you can see that the velocity at $t = 0$ has magnitude v_0 . From trigonometry, it can be concluded that $v_{0,x} = v_0 \cos(\theta)$ and $v_{0,y} = v_0 \sin(\theta)$.

Replacing these values back into equation 1, and noticing that you are being asked to give the *velocity* (a **vector**), you get the answer of this part

$$\vec{v}(t) = v_0 \cos(\theta) \hat{x} + (v_0 \sin(\theta) - gt) \hat{y} \tag{2}$$

Failure to give the answer as a vector is penalized with -0.5 points.

Determinate the position of the projectile (1 point):

Since for the choice of reference frame made above the initial position is *zero*, we have

$$\begin{aligned} x(t) &= x_0 + v_{0,x} t = v_0 \cos(\theta) t \\ y(t) &= y_0 + v_{0,y} t - \frac{g}{2} t^2 = v_0 \sin(\theta) t - \frac{g}{2} t^2 \end{aligned} \quad (3)$$

Position is a **vector** whose components are $x(t)$ and $y(t)$. The right answer is then

$$\vec{r}(t) = (v_0 \cos(\theta) t) \hat{x} + (v_0 \sin(\theta) t - \frac{g}{2} t^2) \hat{y} \quad (4)$$

Failure to give the answer as a vector is penalized with -0.5 points.

- b. **(1.0 points)** Find the total duration of the flight of the projectile in terms of v_0, θ and g .

There are two approaches to this problem, both equally valid (both can get the full 1 point).

The first one involves considering the velocity along y . Since the ground is completely horizontal and flat, the time it takes the projectile to reach the top of the trajectory must be half of the time that it takes to come down and hit the ground.

The top of the trajectory is reached when the vertical velocity is *zero*. If we call the duration of the flight T , you can use equation 1 and do

$$\begin{aligned} 0 &= v_y\left(\frac{T}{2}\right) = v_0 \sin(\theta) - g \frac{T}{2} \\ \implies T &= \frac{2v_0}{g} \sin(\theta) \end{aligned} \quad (5)$$

This is the desired result.

Alternatively one can consider the position along y . When the projectile hits the ground, it has vertical position 0. From equation 3, this means that

$$0 = y(T) = v_0 \sin(\theta) T - \frac{g}{2} T^2$$

Upon factorizing T in the expression above you obtain

$$0 = T \cdot \left(v_0 \sin(\theta) - \frac{g}{2} T \right)$$

This can be satisfied either with $T = 0$ or with $\sin(\theta) T - \frac{g}{2} T = 0$. The former case corresponds to the moment of the launch. The latter corresponds to the impact and is identical to the condition found in equation 5 for which we concluded that

$$T = \frac{2v_0}{g} \sin(\theta) \quad (6)$$

- c. **(1.0 points)** Find the range of the projectile. What is its total displacement?

Since in the selected reference frame the launch site is at the origin, the range of the projectile is just the value of x_f shown in the figure.



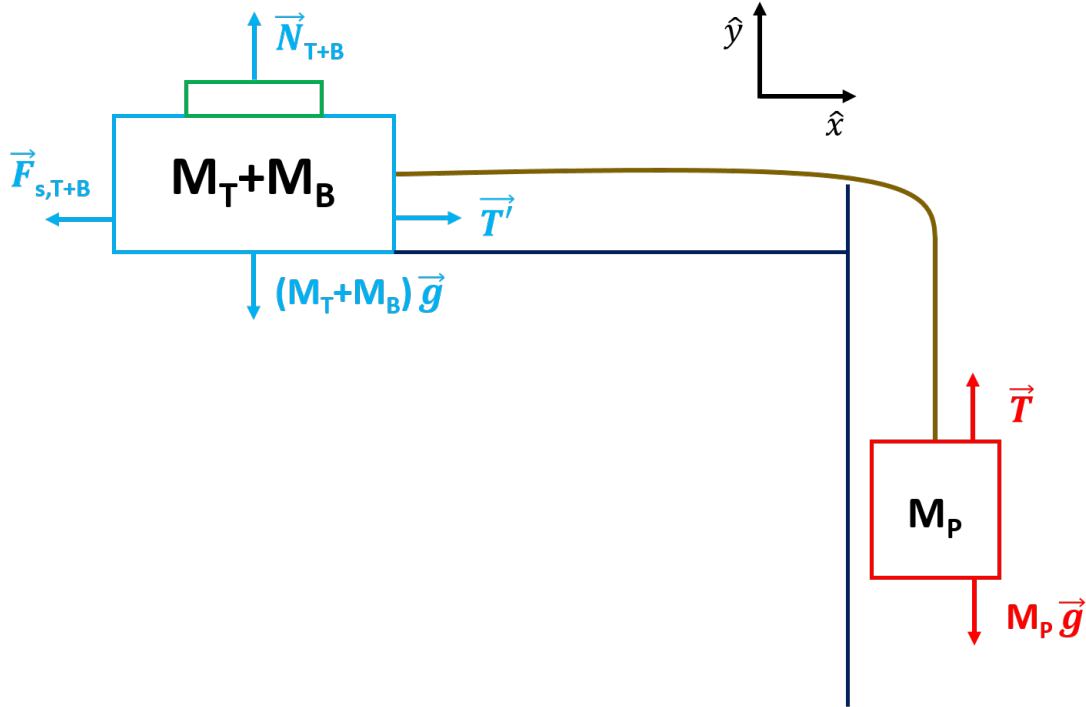
To find x_f you just need to find the position along x at the end of the flight, i.e. at the time T computed above. This is straightforwardly done using the expressions in equation 3:

$$\begin{aligned} x_f = x(T) &= v_0 \cos(\theta) T = v_0 \cos(\theta) \cdot \frac{2 v_0}{g} \sin(\theta) \\ &= \frac{2 v_0^2}{g} \sin(\theta) \cos(\theta) \end{aligned} \quad (7)$$

The total displacement is a **vector** computed as the final position *minus* the initial position. The initial position for our choice of reference frame is $\vec{r}_0 = 0$. The final position is $\vec{r}_f = x_f \hat{x} + 0 \hat{y} = x_f \hat{x}$. The answer is then

$$\Delta \vec{r} = \vec{r}_f - \vec{r}_0 = \frac{2 v_0^2}{g} \sin(\theta) \cos(\theta) \hat{x}$$

2. Escape through a window (10 points)



- a. (4.0 points) What is the minimum mass of the table to prevent it from sliding when you are suspended from the rope?

First, determine the forces acting on the person (1 point):

- The force $\vec{T}$ from the rope, vertical and upward.
- The weight $\vec{F}_{g,P} = M_P \vec{g}$ of the person, vertical and downward.

Next, find the forces acting on the table + book (1 point):

- The force $\vec{T}'$ from the rope, parallel to the floor and rightward. This has the same magnitude as the tension force $\vec{T}$ acting on the person.
- The force of static friction $\vec{F}_{s,T+B}$, parallel to the floor and opposite to the direction of $\vec{T}'$.
- The weight $\vec{F}_{g,T+B} = (M_T + M_B) \vec{g}$ of the table, vertical and downward.
- The normal force $\vec{N}_{T+B}$, vertical and upward, balancing $\vec{F}_{g,T+B}$.

Since the condition in the problem statement implies that the system is in a static state, recognize that $\Sigma F = 0$ in all directions.

- For the person, there are only forces in the $\hat{y}$ direction (0.5 points):

$$\Sigma F_{y,P} = 0 = T - M_P g \quad (8)$$

- For the table + book, there are forces in the $\hat{x}$ and $\hat{y}$ directions (0.5 points):

$$\Sigma F_{x,T+B} = 0 = T' - F_{s,T+B} \quad (9)$$

$$\Sigma F_{y,T+B} = 0 = N_{T+B} - (M_T + M_B) g \quad (10)$$

Where $F_{s,T+B} \leq \mu_s N_{T+B}$. The problem asks us to find the minimum mass of $M_T + M_B$ required to keep the table + book from sliding. Realize that this corresponds to the case where $F_{s,T+B} = \mu_s N_{T+B}$, as a larger mass would still fulfill $F_{s,T+B} \leq \mu_s N_{T+B}$. By then equating the forces in equations 8–10, find that (0.5 points):

$$T = M_P g \quad (11)$$

$$T' = \mu_s N_{T+B} \quad (12)$$

$$N_{T+B} = (M_T + M_B) g \quad (13)$$

Now recognize that $T = T'$ and solve for $M_T + M_B$ (0.5 points):

$$M_T + M_B = T/(\mu_s g) = M_P/\mu_s \quad (14)$$

b. **(3.0 points)** *After the book is removed from the table, what is your acceleration?*

To begin this problem, note that the forces acting on you are the same as part a). Now that the book has been removed, the forces acting on the table are similar except that the force of static friction (F_s) is replaced with the force of kinetic friction (F_k) acting in the negative $\hat{x}$ direction. The mass of the table + book system is also replaced with just M_T .

The primary difference with part a) is that the system is now accelerating so $\Sigma F \neq 0$. To solve for the acceleration, we then apply Newton's second law to each of the masses. Rewriting equations 8–10 (1 point):

– For the person:

$$\Sigma F_{y,P} = M_P a_{y,P} = T - M_P g \quad (15)$$

– For the table:

$$\Sigma F_{x,T} = M_T a_{x,T} = T' - F_{k,T} \quad (16)$$

$$\Sigma F_{y,T} = M_T a_{y,T} = N_T - M_T g \quad (17)$$

The key to solving this problem is recognizing that, because the rope is inextensible, you (mass M_P) and the table (mass M_T) have the same magnitude of acceleration. However, in the coordinate system where $\hat{y}$ is upwards, the direction of the acceleration is opposite for the table (in $\hat{x}$) and for you (in $\hat{y}$) (1 point):

$$a = a_{x,T} = -a_{y,P} \quad (18)$$

As with part a), the table is not accelerating in the $\hat{y}$ direction, so $a_{y,T} = 0$. This allows for N_T to be established from equation 17. From this, the expression for $F_{k,T}$ can also be found, as $F_{k,T} = \mu_k N_T$.

We can now use the condition in equation 18 and the fact that $T = T'$, to solve for a (1 point):

$$a_{y,P} = \frac{T - M_P g}{M_P} = -a \quad (19)$$

$$a_{x,T} = \frac{T' - \mu_k M_T g}{M_T} = a \quad (20)$$

After solving for T' in equation 20, we can substitute the resulting expression into equation 19. Following some algebra, we find:

$$a_{y,P} = -a = -\frac{g(M_P - \mu_k M_T)}{M_P + M_T} \quad (21)$$

This makes intuitive sense as the acceleration is proportional to $M_P - \mu_k M_T$, where the person is causing the acceleration of the system and the friction caused by the table is resisting it.

- c. **(3.0 points)** *What is the coefficient of static friction between the book and table (β_s) required such that the book doesn't slide with respect to the table?*

There are two approaches that can be used to solve this problem. For both, it is important to recognize that the no-slip condition in the problem statement mandates that the acceleration of the book (a_B) and the table (a_T) must be the same.

The first approach makes use of the fact that, since the table and book have the same acceleration, the acceleration can be found by treating the two as a single entity, as in part a). The difference is that now $\Sigma F \neq 0$ and Newton's second law must be used, as in part b). Doing so puts equations 8–10 in the form of equations 15–17:

- For the person (0.5 points):

$$\Sigma F_{y,P} = M_P a_{y,P} = T - M_P g \quad (22)$$

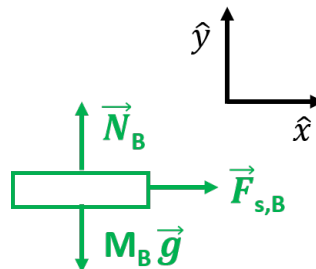
- For the table (0.5 points):

$$\Sigma F_{x,T+B} = (M_T + M_B) a_{x,T+B} = T' - F_{k,T+B} \quad (23)$$

$$\Sigma F_{y,T+B} = (M_T + M_B) a_{y,T+B} = N_{T+B} - (M_T + M_B) g \quad (24)$$

Using the same steps as part b), we can solve for the acceleration of the table + book system. The answer is in the same form as equation 21 with $M_T + M_B$ substituted for M_T (0.5 points).

$$a = \frac{g(M_P - \mu_k(M_T + M_B))}{(M_T + M_B) + M_P} \quad (25)$$



To find the coefficient of the static friction required between the book and the table, we now must consider the forces acting on the book. Referring to the free-body diagram, it becomes apparent that the only force acting on the book in the $\hat{x}$ direction is the force of static friction between the table and the book (0.5 points):

$$\Sigma F_x = M_B a_{x,B} = F_{s,B} \quad (26)$$

$$\Sigma F_y = M_B a_{y,B} = N_B - M_B g \quad (27)$$

Therefore, $a_{x,B}$ must be equal to the acceleration of the table+book system found in equation 25. Using the fact that $a_{y,B} = 0$, we find that (0.5 points):

$$F_{s,B} = \beta_s N_B = \beta_s M_B g \quad (28)$$

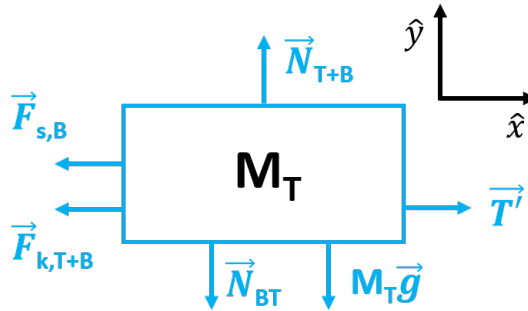
$$a_{x,B} = \frac{F_{s,B}}{M_B} = \beta_s g \quad (29)$$

We can now equate the acceleration of the table + book system to $a_{x,B}$ and find the condition to keep the book from sliding on the table (0.5 points):

$$a = \frac{g(M_P - \mu_k(M_T + M_B))}{(M_T + M_B) + M_P} = \beta_s g \quad (30)$$

$$\frac{(M_P - \mu_k(M_T + M_B))}{(M_T + M_B) + M_P} = \beta_s \quad (31)$$

The alternative approach involves individually analyzing the three bodies in the system: the table, book, and person. The free-body diagrams for the person and book will remain unchanged, but the free-body diagram for the table needs to be reconsidered:



There are several differences with the table + book free-body diagram from part a) (0.5 points):

- In the $-\hat{y}$ direction, the mass of the book has been replaced with the normal force of the book on the table, N_{BT} .
- The mass of the table and book ($M_T + M_B$) has also been replaced with just the mass of the table, M_T .
- In the $-\hat{x}$ direction, there is an additional force of static friction from the book on the table, $F_{s,B}$. This is the action-reaction partner of $F_{s,B}$ acting on the book from the table.
- The direction of three of the forces are unchanged: T' , N_{T+B} , and $F_{k,T+B}$.

We can now apply Newton's second law to the person, book, and table. For the person, this is given by equation 15 (0.5 points). For the book, the equations are the same as equations 26 and 27. For the table (0.5 points):

$$\Sigma F_{x,T} = M_T a_{x,T} = T' - F_{s,B} - F_{k,T+B} \quad (32)$$

$$\Sigma F_{y,T} = M_T a_{y,T} = N_{T+B} - M_T g - N_{BT} \quad (33)$$

Solving for the normal forces using the fact that $a_y = 0$ for the book and table allows us to rewrite equation 32:

$$T' = M_T a_{x,T} + F_{s,B} + \mu_k(M_T + M_B) g \quad (34)$$

As with parts a) and b), $T = T'$. Solving for T using the sum of forces on the person (equation 15) and recalling that $a = a_{x,T} = -a_{y,P}$, we find (0.5 points):

$$a(M_T + M_P) = M_P g - F_{s,B} - \mu_k(M_T + M_B) g \quad (35)$$

Revisiting equation 26 and the condition that the acceleration must be the same for the table, book, and person, we find that (0.5 points):

$$a = a_{x,B} = \frac{F_{s,B}}{M_B} \quad (36)$$

Rearranging, we find that:

$$F_{s,B} = \frac{M_B(M_P g - \mu_k(M_B + M_T))}{M_T + M_B + M_P} \quad (37)$$

Finally, substituting in the result of equation 28, we arrive at the same answer as the first approach for the value of β_s necessary to keep the book from sliding on the table (0.5 points):

$$\frac{(M_P - \mu_k(M_T + M_B))}{M_T + M_B + M_P} = \beta_s \quad (38)$$