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Plasma
Center

General Physics: Mechanics

PHYS-101(en)

Math review

$$e^{i\pi} + 1 = 0$$

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Math review

- You are expected to know how to perform standard derivatives and integrals, the most common of which are listed in slides 3 through 5
- The remaining slides contain a summary of all types of differential equations that are planned to appear in this course
- This is not a math course, so you can take these solutions as given and apply them without justification
- You can also copy them onto your final exam “cheat sheet”

List of common derivatives

- Polynomial function (where A and n are constants)

$$x(t) = At^n \;\Rightarrow\; \frac{dx}{dt} = nAt^{n-1}$$

- Exponential function (where A and b are constants)

$$x(t) = Ae^{bt} \;\Rightarrow\; \frac{dx}{dt} = bAe^{bt}$$

- Logarithmic function (where A, b, and c are constants)

$$x(t) = A\ln(bt + c) \;\Rightarrow\; \frac{dx}{dt} = \frac{Ab}{bt + c}$$

- Sine (where A, b, and c are constants)

$$x(t) = A\sin(bt + c) \;\Rightarrow\; \frac{dx}{dt} = bA\cos(bt + c)$$

- Cosine (where A, b, and c are constants)

$$x(t) = A\cos(bt + c) \;\Rightarrow\; \frac{dx}{dt} = -\,bA\sin(bt + c)$$

List of common indefinite integrals

- Polynomial function (where A and n are constants and D is an integration constant)

$$x(t) = At^n \;\Rightarrow\; \int x(t)dt = \frac{A}{n+1}t^{n+1} + D$$

- Exponential function (where A and b are constants and D is an integration constant)

$$x(t) = Ae^{bt} \;\Rightarrow\; \int x(t)dt = \frac{A}{b}e^{bt} + D$$

- Reciprocal function (where A, b, and c are constants and D is an integration constant)

$$x(t) = \frac{A}{bt+c} \;\Rightarrow\; \int x(t)dt = \frac{A}{b}\ln(bt+c) + D$$

- Sine (where A, b, and c are constants and D is an integration constant)

$$x(t) = A\sin(bt+c) \;\Rightarrow\; \int x(t)dt = -\frac{A}{b}\cos(bt+c) + D$$

- Cosine (where A, b, and c are constants and D is an integration constant)

$$x(t) = A\cos(bt+c) \;\Rightarrow\; \int x(t)dt = \frac{A}{b}\sin(bt+c) + D$$

List of common definite integrals

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- Polynomial function (where A and n are constants)

$$x(t) = At^n \;\Rightarrow\; \int_{t_i}^{t_f} x(t)dt = \frac{A}{n+1}(t_f^{n+1} - t_i^{n+1})$$

- Exponential function (where A and b are constants)

$$x(t) = Ae^{bt} \;\Rightarrow\; \int_{t_i}^{t_f} x(t)dt = \frac{A}{b}(e^{bt_f} - e^{bt_i})$$

- Reciprocal function (where A, b, and c are constants)

$$x(t) = \frac{A}{bt+c} \;\Rightarrow\; \int_{t_i}^{t_f} x(t)dt = \frac{A}{b}(\ln(bt_f + c) - \ln(bt_i + c)) = \frac{A}{b}\ln\left(\frac{bt_f + c}{bt_i + c}\right)$$

- Sine (where A, b, and c are constants)

$$x(t) = A\sin(bt+c) \;\Rightarrow\; \int_{t_i}^{t_f} x(t)dt = -\frac{A}{b}\cos(bt_f + c) + \frac{A}{b}\cos(bt_i + c)$$

- Cosine (where A, b, and c are constants)

$$x(t) = A\cos(bt+c) \;\Rightarrow\; \int_{t_i}^{t_f} x(t)dt = \frac{A}{b}\sin(bt_f + c) - \frac{A}{b}\sin(bt_i + c)$$

Direct integration

[Lectures 1b, 2a, 2b, 7a, 9a]

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1. $\dfrac{dx}{dt} = f(t)$

- For example: $\dfrac{dx}{dt} = -gt + v_0$ (ballistic motion)

  $\dfrac{dv}{dt} = \dfrac{F(t)}{m}$ (Newton's second law)

  $F(x) = -\dfrac{dU}{dx}$ (potential energy)

- Solved by direct integration: $x(t) = \int f(t)dt$

Separation of variables

[Lectures 6a, 9a]

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2. $$\frac{dx}{dt} = f(t)x^n$$

- For example: $$m\frac{dv}{dt} = -\beta v$$ (laminar air drag)

$$m\frac{dv}{dt} = -\beta v^2$$ (turbulent air drag)

- Solved by separation of variables: $$\frac{1}{x^n}\frac{dx}{dt} = f(t)$$

$$\int \frac{1}{x^n}\frac{dx}{dt}dt = \int f(t)dt \quad \Rightarrow \quad \int \frac{1}{x^n}dx = \int f(t)dt$$

Inhomogeneous with constant coeff. [Lecture 6b]

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3. $$\frac{dx}{dt} + A_1 x = A_2$$

- E.g. $m\dfrac{dv}{dt} = -\beta v + mg$ (laminar drag with gravity)
- Solved by substituting $x(t) = X(t) + A_2/A_1$ to get
$$\frac{dX}{dt} + A_1 X = 0$$
- Then $X(t)$ can be found according to the previous slide
- Lastly, take this solution for $X(t)$ and substitute it back into $x(t) = X(t) + A_2/A_1$ to arrive at the final answer

Rocket equation

[Lectures 7a, 12a]

4. $A\dfrac{dx}{dt} + x\dfrac{dy}{dt} = 0$

- E.g. $m_r\dfrac{dv_r}{dt} + \dfrac{dm_r}{dt}u = 0$ (rocket equation)

- Solved by separation of variables:

$$\frac{1}{x}\frac{dx}{dt} = -\frac{1}{A}\frac{dy}{dt} \quad \Rightarrow \quad \frac{d}{dt}\left(\ln(x)\right) = -\frac{d}{dt}\left(\frac{y}{A}\right)$$

$$\int \frac{d}{dt}\left(\ln(x)\right) dt = -\int \frac{d}{dt}\left(\frac{y}{A}\right) dt \quad \Rightarrow \quad \ln(x) = -\frac{y}{A} + C$$

Harmonic equation

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5. $$\frac{d^2x}{dt^2} + \omega_0^2 x = 0$$

- E.g. $$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$ (spring harmonic oscillator)

  $$\frac{d^2x}{dt^2} + \frac{g}{\ell}x = 0$$ (oscillating pendulum)

- Solved with the form: $x(t) = A\cos\left(\omega_0 t + \varphi\right)$

- Use A and φ to satisfy initial conditions

Damped harmonic eq.

6. $$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega_0^2 x = 0$$

- E.g. $-\beta\frac{dx}{dt} - kx = m\frac{d^2x}{dt^2}$ (damped harmonic motion)
- Solved with the form:

$$x(t) = e^{-\lambda t}\left(A_1 e^{t\sqrt{\lambda^2-\omega_0^2}} + A_2 e^{-t\sqrt{\lambda^2-\omega_0^2}}\right)$$

- Use A_1 and A_2 to satisfy initial conditions

Forced damped harmonic eq.

7. $$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega_0^2 x = \frac{F_d}{m}\cos(\omega_d t)$$

- E.g. Forced damped harmonic motion
- Solved with the form:

$$x(t) = e^{-\lambda t}\left(A_1 e^{t\sqrt{\lambda^2-\omega_0^2}} + A_2 e^{-t\sqrt{\lambda^2-\omega_0^2}}\right) + A_d(\omega_d, F_d)\cos(\omega_d t + \varphi(\omega_d))$$

where

$$A_d(\omega_d, F_d) = \frac{F_d/m}{\sqrt{(2\lambda\omega_d)^2 + (\omega_0^2 - \omega_d^2)^2}} \text{ and } \varphi(\omega_d) = \tan^{-1}\left(\frac{2\lambda\omega_d}{\omega_d^2 - \omega_0^2}\right)$$

- Use A_1 and A_2 to satisfy initial conditions

Forced damped harmonic eq.

[Problem set 13]

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8. $$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega_0^2 x = \frac{F_d}{m}\cos(\omega_d t) + C$$

- E.g. damped harmonic motion with a constant forcing
- Solved with the form:

$$x(t) = e^{-\lambda t}\left(A_1 e^{t\sqrt{\lambda^2 - \omega_0^2}} + A_2 e^{-t\sqrt{\lambda^2 - \omega_0^2}}\right)$$

$$+A_d(\omega_d, F_d)\cos(\omega_d t + \varphi(\omega_d)) + \frac{C}{\omega_0^2}$$

where $A_d(\omega_d, F_d)$ and $\varphi(\omega_d)$ are shown on previous slide

- Use A_1 and A_2 to satisfy initial conditions
- C term can be viewed as forcing term with $\omega_d = 0$