General Physics: Mechanics

Swiss Plasma Center

PHYS-101(en)

Lecture 8a:
Kinetic energy
and work

Émilie du Châtelet
(circa 1740)

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November 4th, 2024

Today’s agenda (Serway 7-8, MIT 13-14)

1. Energy and work
    - Kinetic energy
    - Work done by a force
    - Work-kinetic energy theorem
    - To be continued next week…

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The skiers

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The skiers

1) What are the final speeds of the two skiers?

2) Which skier would win in a race?

Energy

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- **Energy** is one of the most fundamental and important concepts in physics
- It is possessed by objects, in various forms, and can be transferred between them
- Can be understood intuitively from its Greek origin ἐνέργεια, meaning “activity”
- It is given an exact and rigorous definition in the language of mathematics
- It is so important because it is conserved (as we will see next lecture)
- Similar to momentum, but more complex as it has many different forms (like force)

Kinetic energy

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- **Kinetic energy** is a type of energy associated with the motion of an object

$$K = \frac{m}{2} v^2$$

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- Émilie du Châtelet used observations of brass weights, dropped into clay from various heights, to motivate the importance of a quantity proportional to mv^2

- All forms of energy have units of Joules (J) = [kg·m^2/s^2] = [N·m]

Conceptual question

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Review

Consider two carts, of mass m ① and $2m$ ②, at rest on an air track. If you push the first cart for 3 s and then the other for the same length of time, exerting equal force on each, the **momentum** of the light cart is…

A. 4 times…
B. 2 times…
C. equal to…
D. half…
E. one-fourth…

$$\vec{p}_1(t_0) = \vec{p}_2(t_0) = 0$$

$$\vec{p}_1(t_f) = \Delta\vec{p}_1 = \vec{I}_1 = \int_{t_0}^{t_f} \vec{F}\,dt$$

$$\vec{p}_2(t_f) = \Delta\vec{p}_2 = \vec{I}_2 = \int_{t_0}^{t_f} \vec{F}\,dt$$

$=$

the momentum of the heavy cart.

Conceptual question

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Consider two carts, of mass m ① and $2m$ ②, at rest on an air track. If you push the first cart for 3 s and then the other for the same length of time, exerting equal force on each, the **kinetic energy** of the light cart is…

A. greater than…
B. equal to…
C. less than…

the kinetic energy of the heavy cart.

$\vec{p}_1(t_f) = \vec{p}_2(t_f) \Rightarrow |\vec{p}_1(t_f)| = |\vec{p}_2(t_f)| \equiv p$

$\overbrace{|\vec{p}_1(t_f)|}^{=p} = m v_1 \qquad \overbrace{|\vec{p}_2(t_f)|}^{p} = 2m v_2$

$\Rightarrow v_1 = \frac{p}{m} \qquad \Rightarrow v_2 = \frac{p}{2m}$

$K_1 = \frac{1}{2} m v_1^2 = \frac{1}{2} m \left(\frac{p}{m}\right)^2 = \frac{1}{2m} p^2$

$K_2 = \frac{1}{2}(2m) v_2^2 = m \left(\frac{p}{2m}\right)^2 = \frac{1}{4m} p^2$

$K_1 = 2K_2$

Work

- **Work** arises when a force is applied along the direction of the displacement of an object
- It is a scalar quantity, typically represented by W, that has units of force times distance [N·m] = [kg·m^2/s^2] = Joule (J)
- Work is said to be done by a force (or the object exerting the force) on an object

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- Work is said to be done by a force (or the object exerting the force) on an object
- This can be positive if the force is in the direction of motion or negative if the force opposes the direction of motion
- Work is to kinetic energy as impulse is to momentum

Work done by a constant force

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- The **work** done by a constant force is defined as

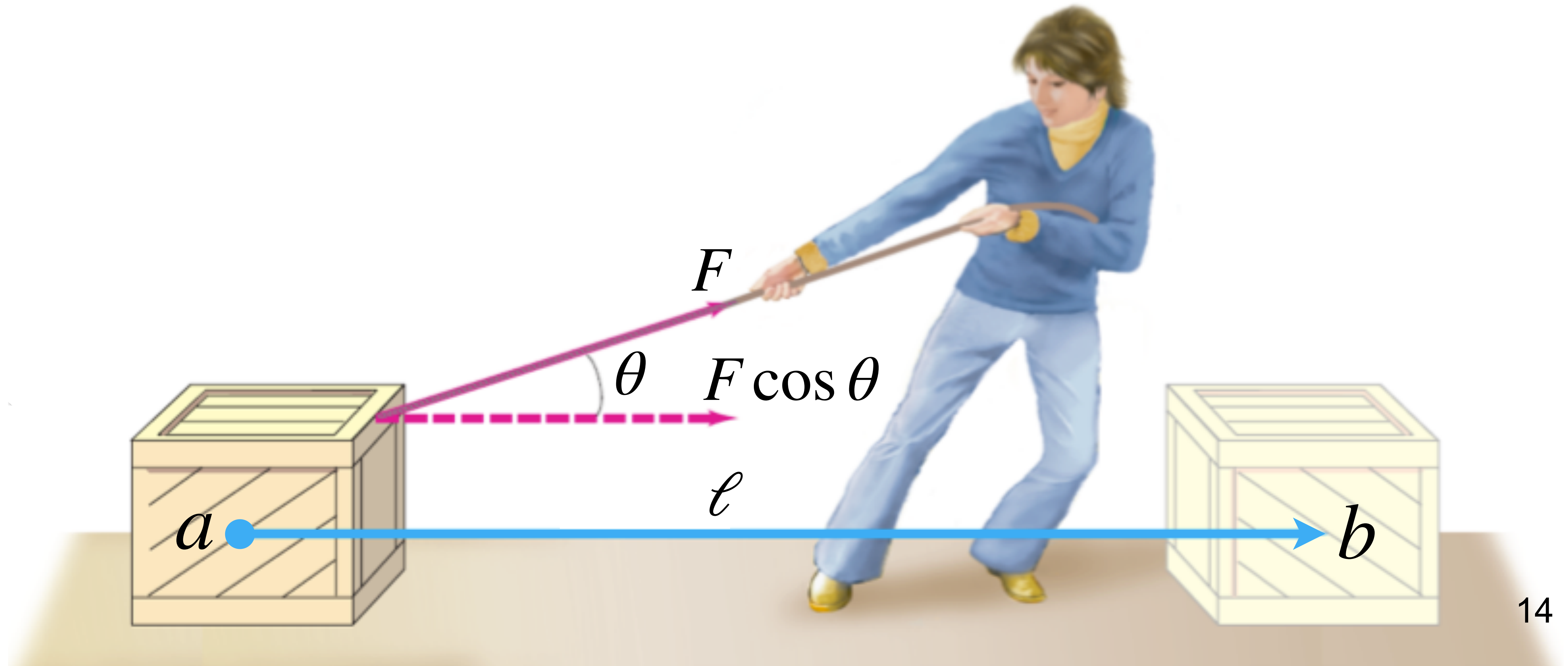
the distance traveled multiplied by the component of the force in the direction of the displacement

Work done by a constant force

- The **work** done by a <u>constant</u> force is defined as

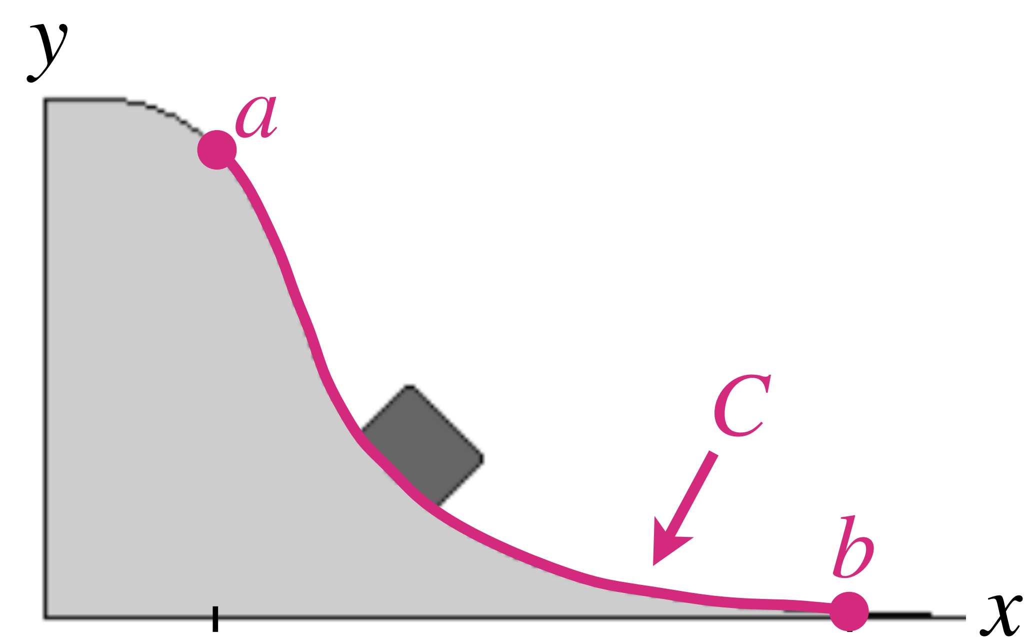
  the distance traveled multiplied by the component of the force in the direction of the displacement

- Mathematically, this is $W = \vec{F} \cdot \vec{\ell} = F\ell\cos\theta$



Work done by a variable force

- The **work** done by a variable force can be found by adding up constant forces over a lot of infinitesimally small distances

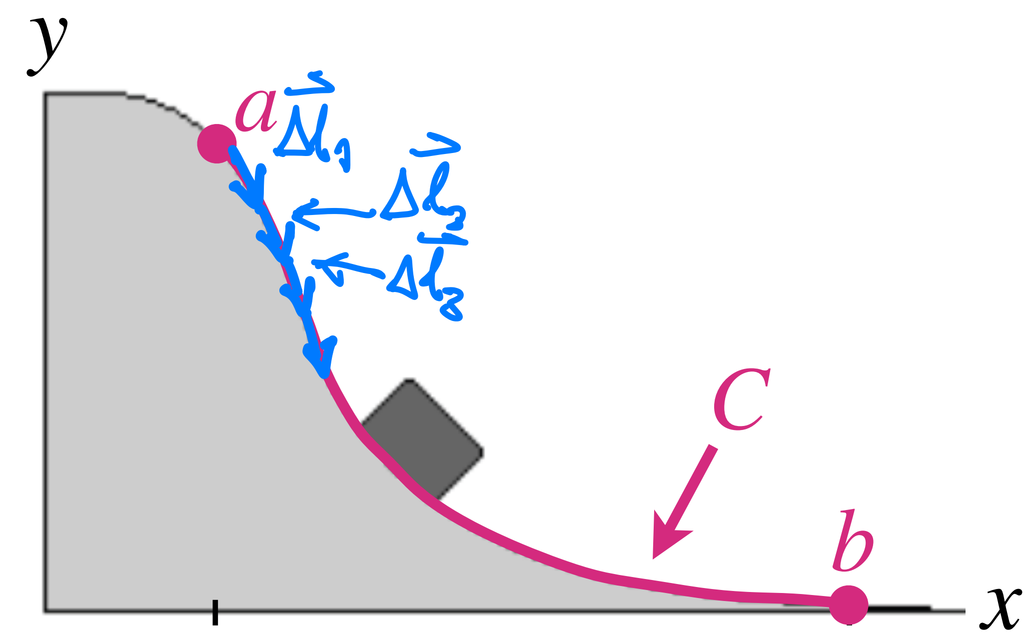


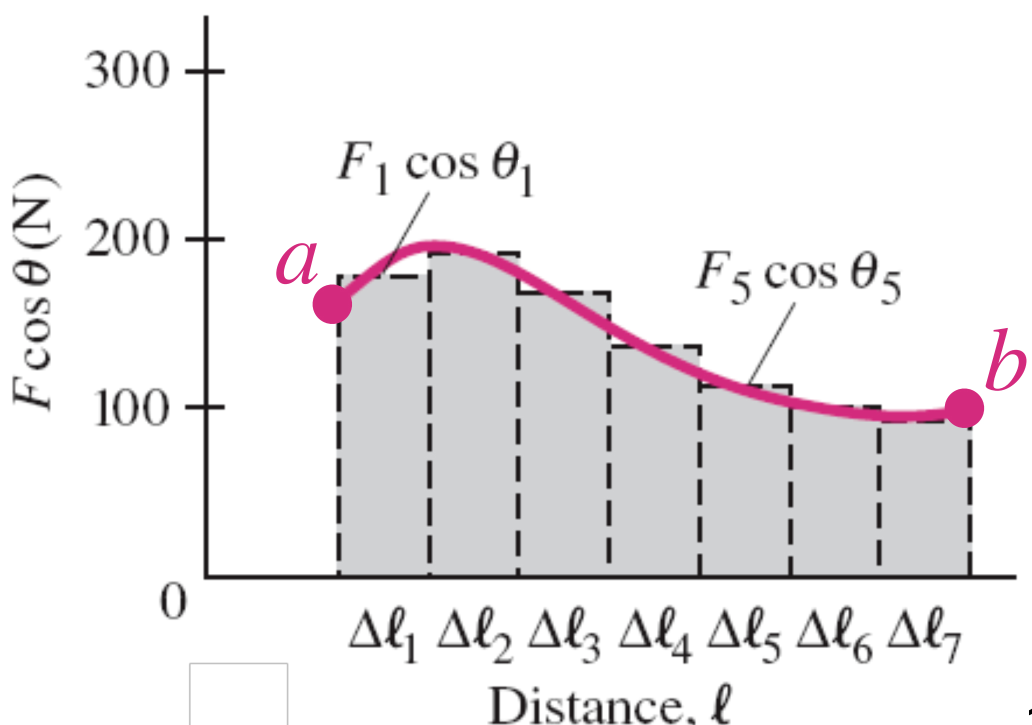
Work done by a variable force

- The **work** done by a <u>variable</u> force can be found by adding up constant forces over a lot of infinitesimally small distances

- Mathematically, each segment contributes

$$\Delta W_i = \vec{F}_i \cdot \Delta \vec{\ell}_i$$





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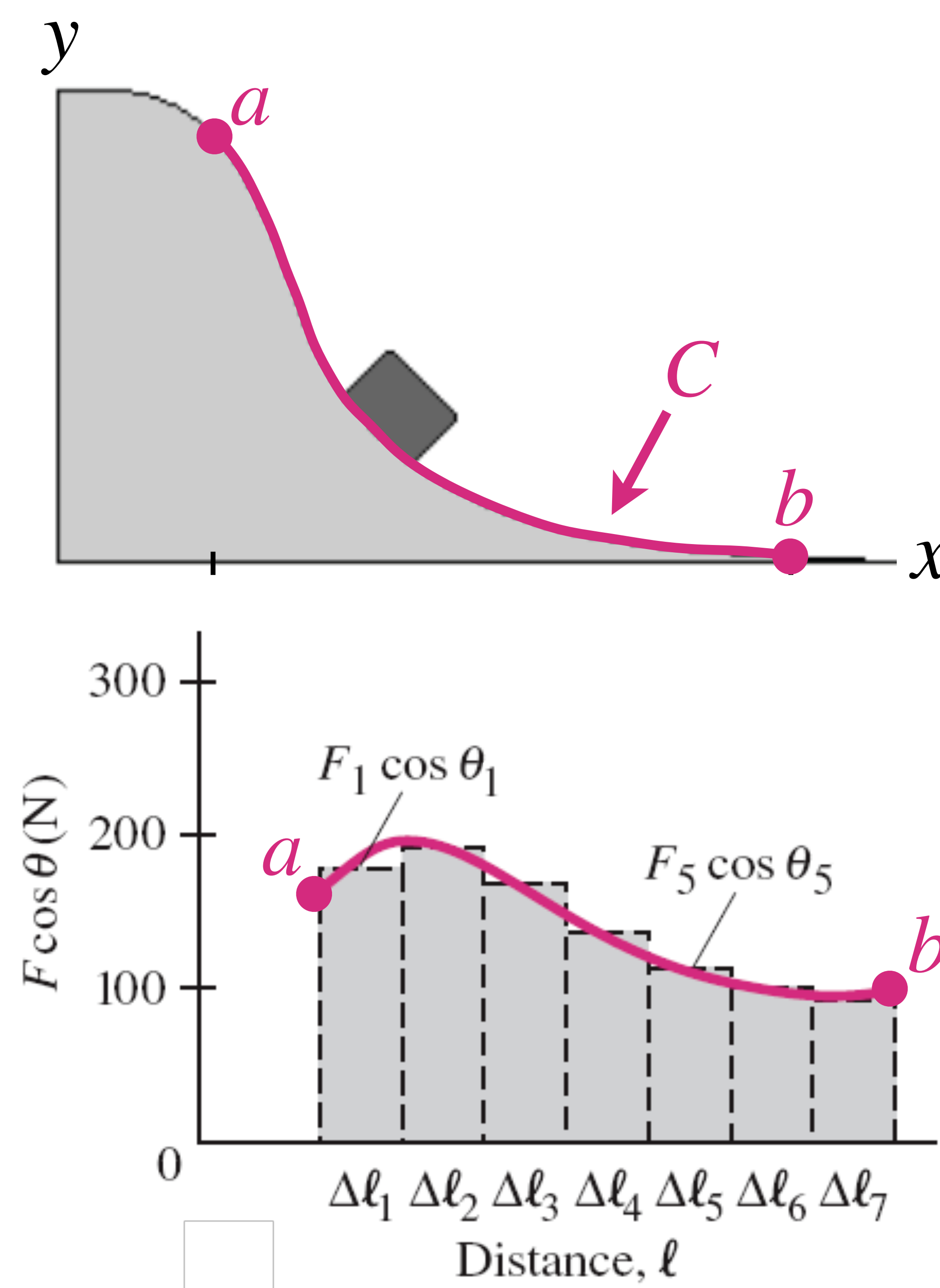
$$\Delta W_i = \vec{F}_i \cdot \Delta \vec{\ell}_i$$

- The sum is

$$W = \lim_{\Delta \ell_i \to 0} \sum_i \vec{F}_i \cdot \Delta \vec{\ell}_i$$

$$= \int_a^b \vec{F} \cdot d\vec{\ell}$$



Work done by a variable force

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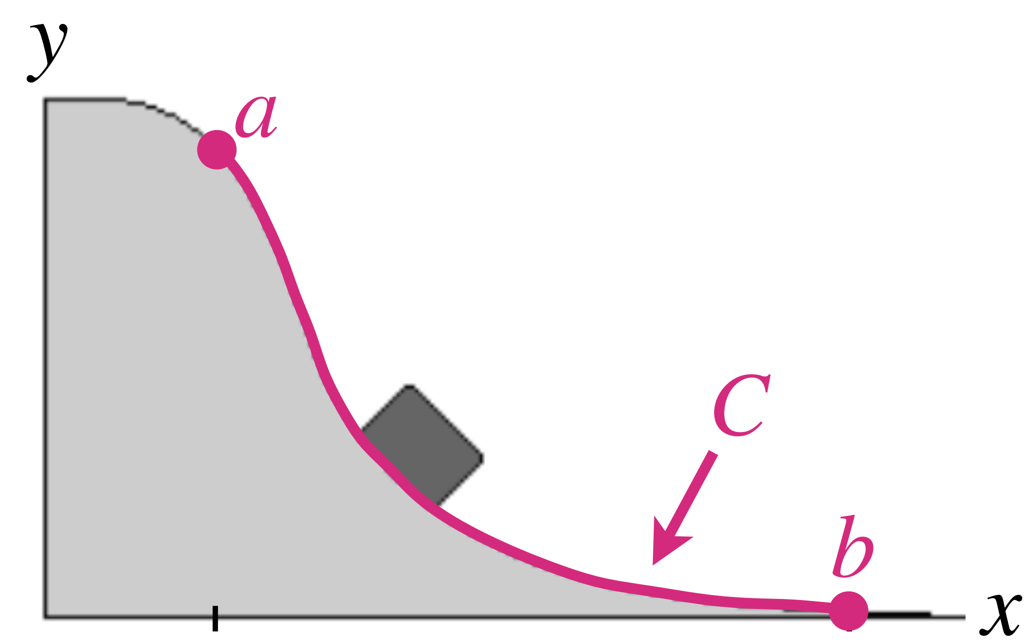
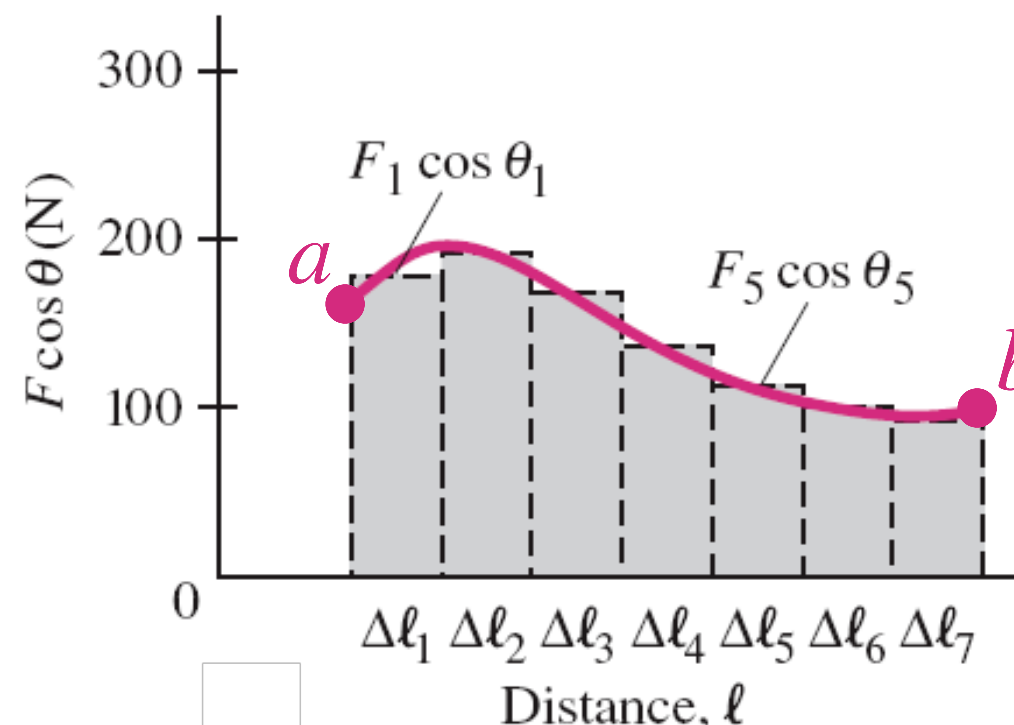
$$W = \lim_{\Delta \ell_i \to 0} \sum_i \vec{F}_i \cdot \Delta \vec{\ell}_i$$

$$= \int_a^b \vec{F} \cdot d\vec{\ell} \boxed{= \int_C \vec{F} \cdot d\vec{\ell}}$$



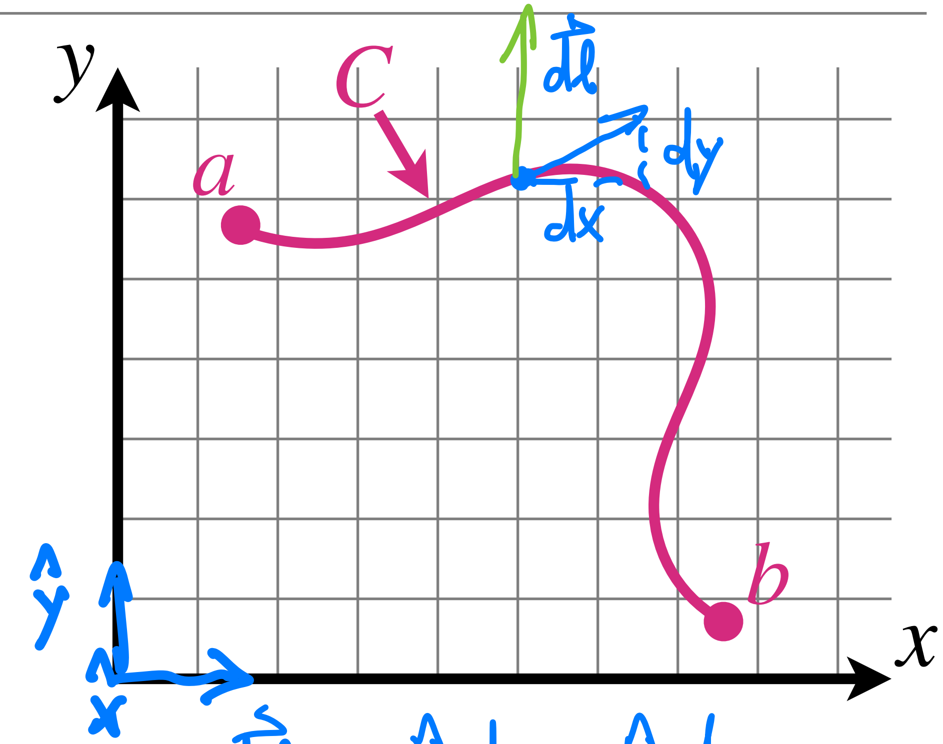


Work in various coordinate systems

- Cartesian:

$$\hat{x}\cdot\hat{x} = \hat{y}\cdot\hat{y} = 1$$
$$\hat{x}\cdot\hat{y} = \hat{y}\cdot\hat{x} = 0$$

$$W = \int_C \vec{F}\cdot d\vec{l}$$
$$= \int_C (F_x\hat{x} + F_y\hat{y})\cdot(\hat{x}\,dx + \hat{y}\,dy)$$
$$= \int_C F_x\,dx + F_y\,dy$$
$$= \int_C F_x\,dx + \int_C F_y\,dy + \int_C F_z\,dz$$



$$d\vec{l} = \hat{x}\,dx + \hat{y}\,dy$$

Work in various coordinate systems

- Polar (and cylindrical):

$\hat{\rho}\cdot\hat{\rho} = \hat{\phi}\cdot\hat{\phi} = 1$

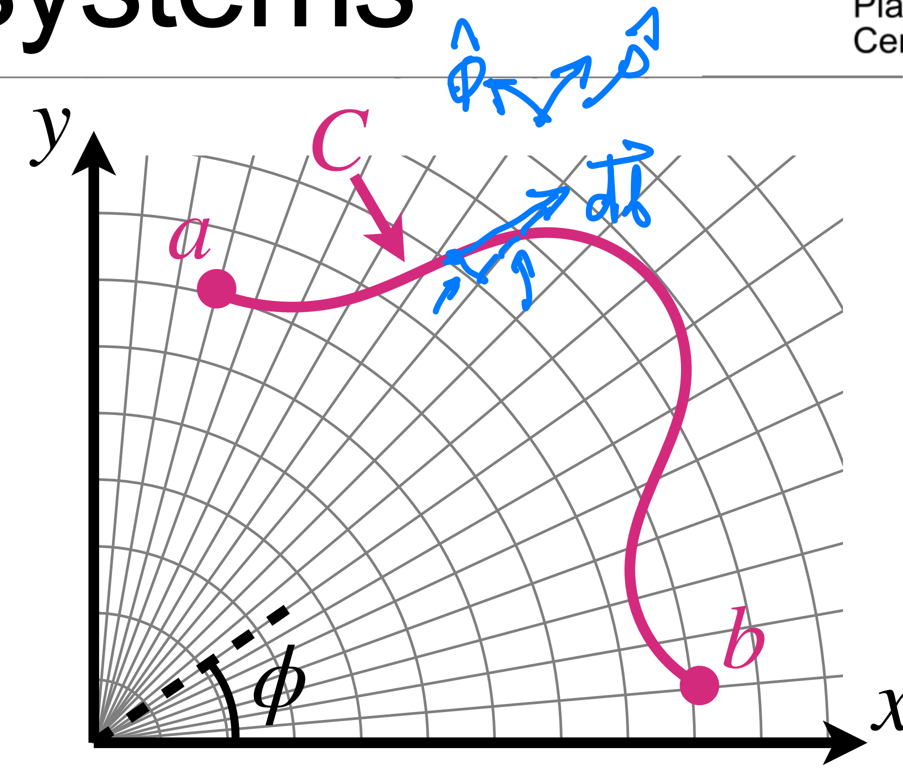
$\hat{\rho}\cdot\hat{\phi} = \hat{\phi}\cdot\hat{\rho} = 0$

$$W = \int_C \vec{F}\cdot\vec{dl}$$

$$= \int_C (F_\rho\hat{\rho} + F_\phi\hat{\phi})\cdot(\hat{\rho}\,d\rho + \hat{\phi}\,\rho\,d\phi)$$

$$= \int_C F_\rho\,d\rho + F_\phi\,\rho\,d\phi$$

$$= \int_C F_\rho\,d\rho + \int_C F_\phi\,\rho\,d\phi + \int_C F_z\,dz$$



$\vec{dl} = \hat{\rho}\,d\rho + \hat{\phi}\,d\phi\cdot\rho$

Conceptual question

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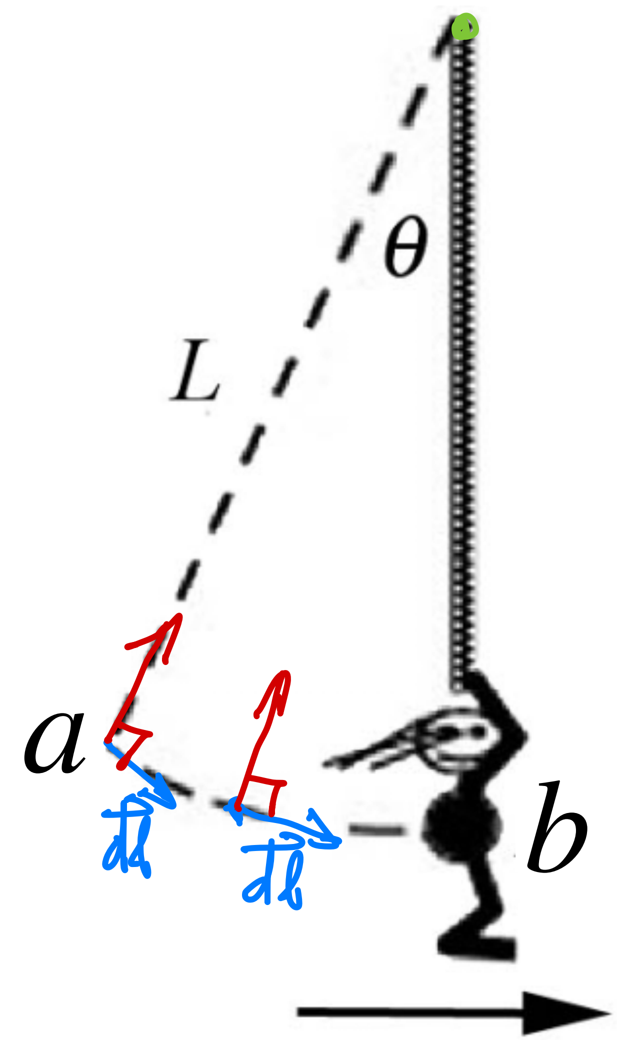
A person swings on an inextensible rope that is attached to a fixed point. The rope exerts a tension T on the person. The work done by tension on the person as she moves from a to b is:

A. T
B. TL
C. $TL\theta$
D. $mgL(1-\cos\theta)$
E. 0



$\vec{F}\cdot d\vec{l} = 0$
in this case
$\Rightarrow W=0$

Conceptual question

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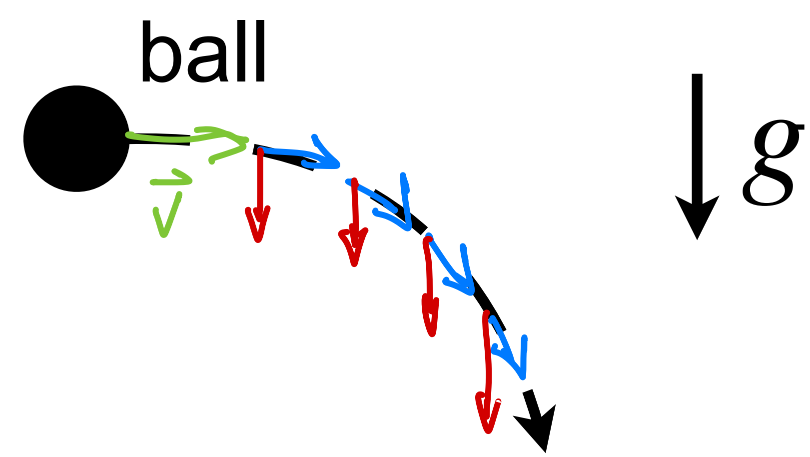
A ball is given an initial horizontal velocity and allowed to fall under the influence of gravity, as shown below.

The work done by the force of gravity on the ball is…

A. positive.
B. zero.
C. negative.



$\vec{F}\cdot\vec{dl} = F\,dl\cos(\theta) > 0$

SPEED! conceptual question

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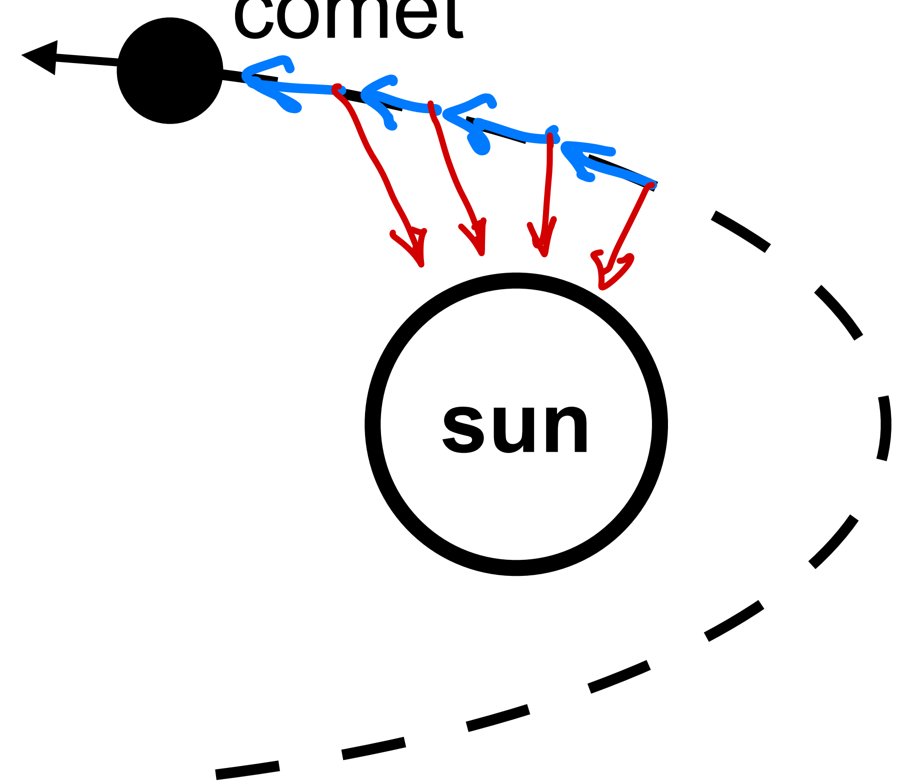
A comet is on a hyperbolic orbit around the Sun. While the comet is moving <u>away</u> from the Sun, the work done by the Sun on the comet is…



A. positive.
B. zero.
C. negative.

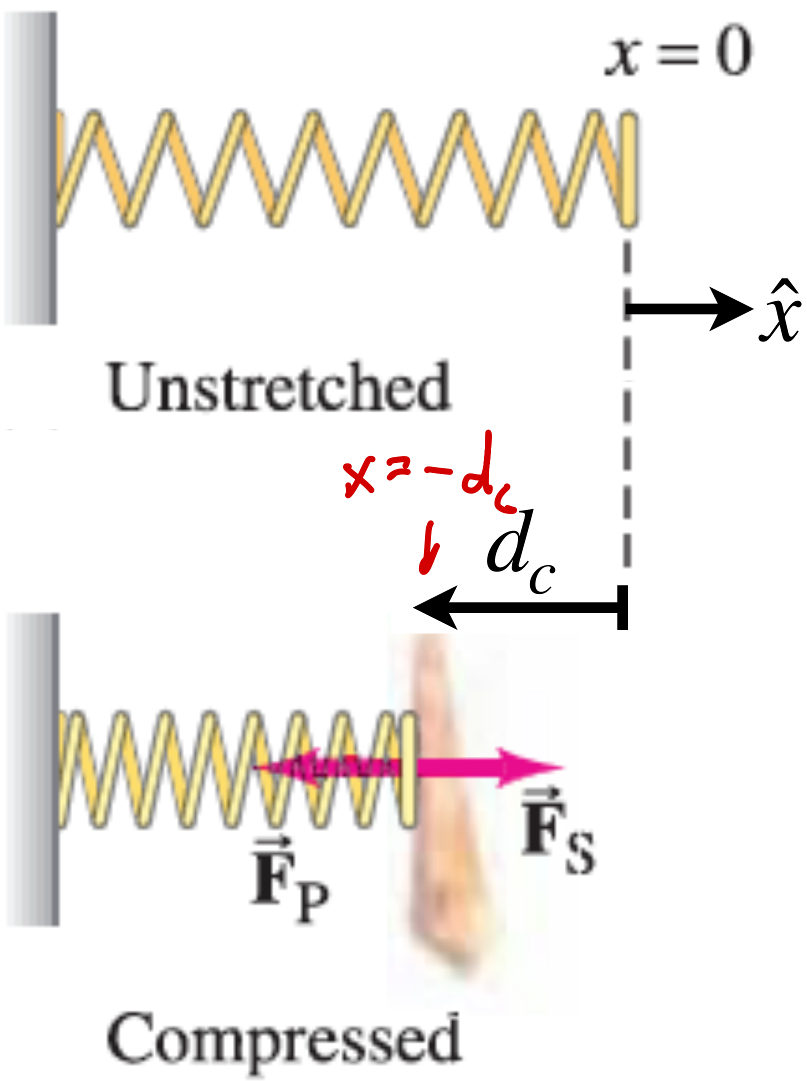
If $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$

$\cos(\theta) < 0$

$\vec{F} \cdot \vec{dl} < 0$

Work done by a spring

- A person compresses a spring by a distance d_c from its equilibrium position. What is the work done by the spring on the person as it stretches back towards the equilibrium position?



$$\vec{F}_s = -k(x - \cancel{x_0})\hat{x} = -kx\hat{x} \qquad \vec{dl} = \hat{x}\,dx$$

$$W_s = \int_{-d_c}^{0} \vec{F}_s \cdot \vec{dl} = \int_{-d_c}^{0} -kx\,\hat{x}\cdot\hat{x}\,dx$$

$$= \int_{-d_c}^{0} -kx\,dx = -k\int_{-d_c}^{0} x\,dx$$

$$= -k\left[\frac{x^2}{2}\right]_{-d_c}^{0} = -k\left(0 - \frac{(-d_c)^2}{2}\right)$$

$$= \boxed{\frac{1}{2}kd_c^2}$$

Work-kinetic energy theorem

- Doing work on a system changes its **kinetic energy**
- Seems sensible, given colloquial definitions of the words

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- Work-kinetic energy theorem:

For a constant mass system,
the <u>total</u> work done by <u>all</u> the forces equals
the change in the kinetic energy of the system.

Work-kinetic energy theorem

- Doing work on a system changes its **kinetic energy**
- Seems sensible, given colloquial definitions of the words
- Work-kinetic energy theorem:

For a constant mass system,
the total work done by all the forces equals
the change in the kinetic energy of the system.

- In mathematics, this is

$$\boxed{W_{net} = \Delta K}$$

Proving the work-kinetic energy theorem

$$K = \frac{1}{2} m v^2 = \frac{1}{2} m \vec{v}\cdot\vec{v} \qquad \vec{v}\cdot\vec{v} = |\vec{v}|\cdot|\vec{v}|\cos(0) = v^2 \quad (v = |\vec{v}|)$$

$$\frac{d}{dt}K = \frac{d}{dt}\left(\frac{1}{2} m \vec{v}\cdot\vec{v}\right) = \frac{1}{2} m \frac{d}{dt}(\vec{v}\cdot\vec{v}) = \frac{1}{2} m \left[\frac{d\vec{v}}{dt}\cdot\vec{v} + \vec{v}\cdot\frac{d\vec{v}}{dt}\right]$$

$$= \frac{1}{2} m \left(2\frac{d\vec{v}}{dt}\cdot\vec{v}\right) = m \underbrace{\frac{d\vec{v}}{dt}}_{\vec{a}}\cdot\vec{v} = m\vec{a}\cdot\vec{v}$$

$$\vec{v} = \frac{d\vec{l}}{dt}$$

Newton's 2nd law, $m\vec{a} = \vec{F}_{net}$

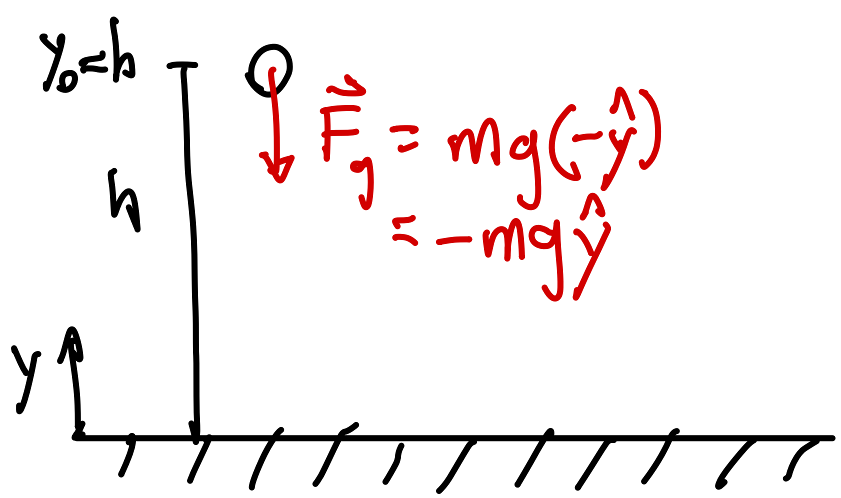
$$\frac{d}{dt}K = \vec{F}_{net}\cdot\frac{d\vec{l}}{dt}$$

$$\int_{t_a}^{t_b} \frac{d}{dt}K\,dt = \int_{t_a}^{t_b} \vec{F}_{net}\cdot\frac{d\vec{l}}{\cancel{dt}}\,\cancel{dt} = \int_a^b \vec{F}_{net}\cdot d\vec{l} = W_{net}$$

$$\boxed{K(t_b) - K(t_a) = W_{net}}$$

Work done by gravity

- An object at rest falls directly downwards from a height $y = h$ to the ground at $y = 0$ under the sole influence of gravity. What is the work done by gravity? What is the change in kinetic energy?



$$\vec{F}_g = -mg\hat{y} \qquad \vec{dl} = \hat{y}\,dy$$

$$W = \int_h^0 \vec{F}_g \cdot \vec{dl} = \int_h^0 -mg\hat{y}\cdot\hat{y}\,dy$$

$$= \int_h^0 -mg\,dy = -mg\int_h^0 dy = -mg(0-h)$$

$$= \boxed{mgh}$$

Work done by gravity

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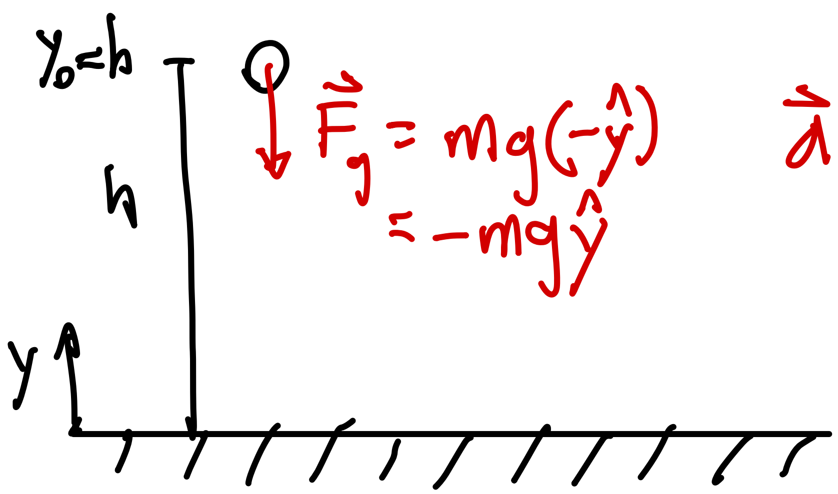

$$\vec{F}_g = mg(-\hat{y}) = -mg\hat{y}$$

$$\vec{a} = g(-\hat{y})$$

$$v(t) = \cancel{v_0}^{\,0} - gt = -gt$$

$$-v_f = -gt_f$$

$$\Rightarrow t_f = \frac{v_f}{g}$$

$$y(t) = \cancel{y_0}^{\,h} + \cancel{v_0}^{\,0}t - \frac{g}{2}t^2 = h - \frac{1}{2}gt^2$$

$$0 = h - \frac{1}{2}g(t_f)^2 = h - \frac{1}{2}g\left(\frac{v_f}{g}\right)^2 = h - \frac{1}{2g}v_f^2$$

$$m \cdot \frac{1}{2}v_f^2 = gh \cdot m$$

$$K_f = \boxed{\Delta K = mgh}$$

Because $K_0 = 0$

Conceptual question

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The same horizontal force, of magnitude F, is applied to two different blocks, of mass $\overset{①}{m}$ and $3\overset{②}{m}$ respectively. The blocks move on a frictionless surface and both blocks begin from rest. If each block moves the same distance as the force is applied, the heavier block acquires…

A. 9 times as much…
B. 3 times as much…
C. the same…
D. 1/3 times as much…
E. 1/9 times as much…

$$\Delta K_1 = K_1^f - \cancel{K_1^i}^{\,0} = W_{net} = \int_{x_a}^{x_b} F\,dx$$

$$\Delta K_2 = K_2^f = W_{net}$$

$$\Rightarrow K_1^f = K_2^f$$

kinetic energy as the lighter block.

Conceptual question

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Compared to the amount of energy required to accelerate a car from rest to 10 km/h, the work required to accelerate the same car from 10 km/h to 20 km/h is…

A. the same.
B. twice as much.
C. three times as much.
D. four times as much.

$$\Delta K = \frac{1}{2} m v_f^2 - \cancel{\frac{1}{2} m v_0^2} = $$
$$= \frac{1}{2} m \left(10 \frac{km}{h}\right)^2$$
$$= \frac{1}{2} m \cdot 100 \left(\frac{km}{h}\right)^2$$

$$\Delta K = \frac{1}{2} m \left(20 \frac{km}{h}\right)^2 - \frac{1}{2} m \left(10 \frac{km}{h}\right)^2$$
$$= \frac{1}{2} m \left(400 \frac{km^2}{h^2} - 100 \frac{km^2}{h^2}\right)$$
$$= \frac{1}{2} m \cdot 300 \frac{km^2}{h^2}$$

Conceptual question

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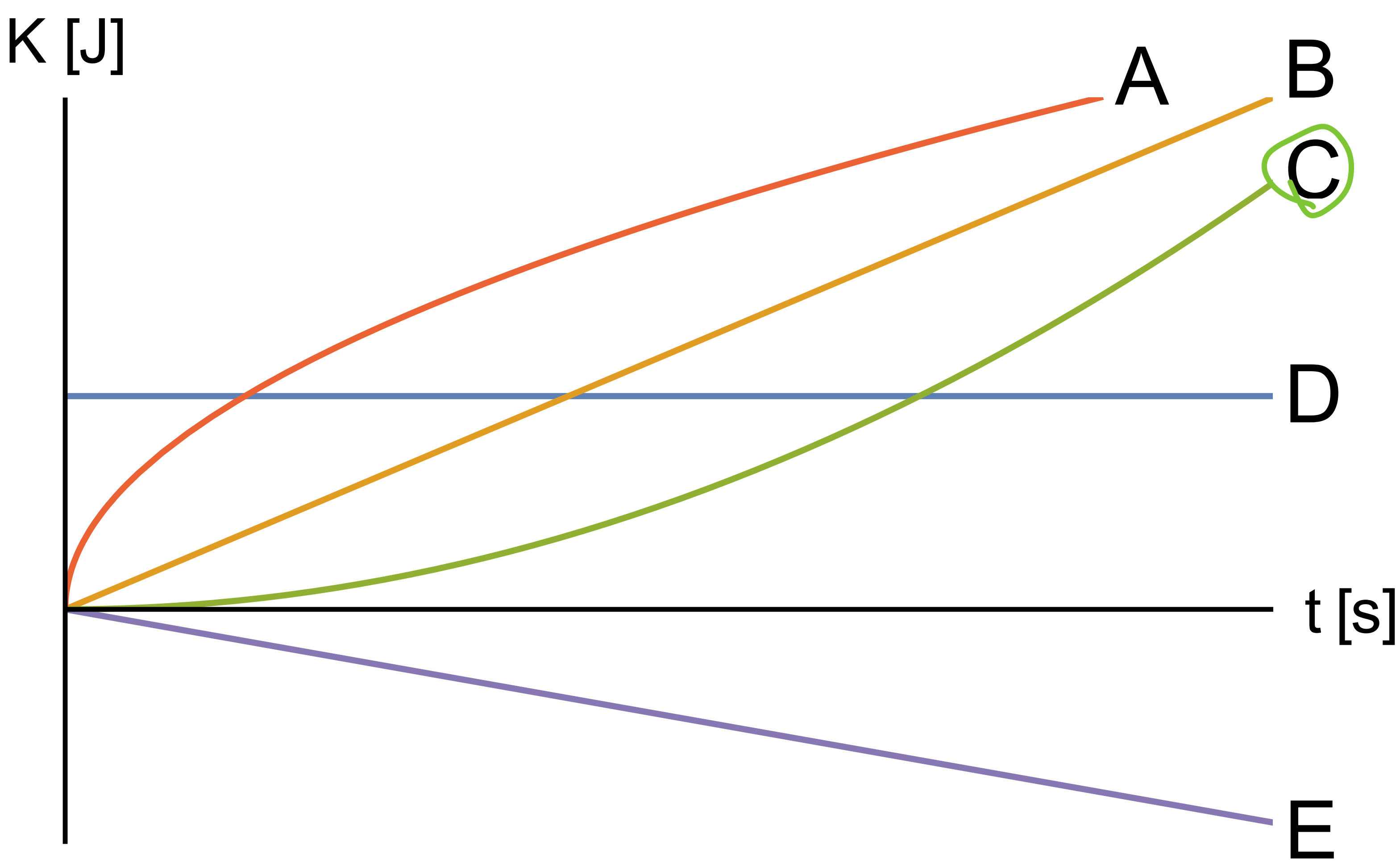
An acorn falls to the earth from a tree. Which of the following graphs best represents the time dependence of the acorn's kinetic energy? Neglect air resistance.



$v(t) = \cancel{v_0}^{0} - gt = -gt$

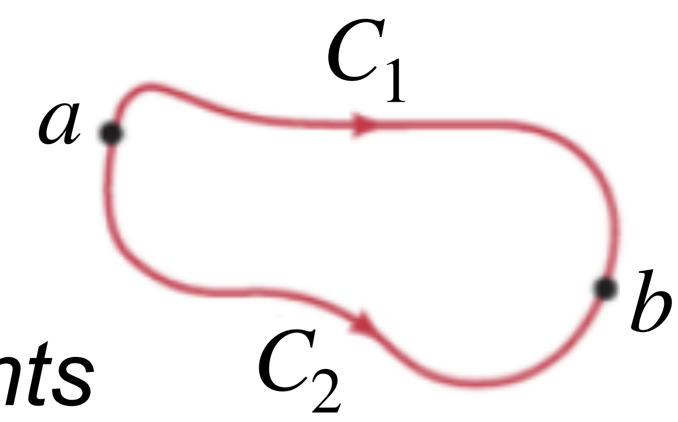
$K = \frac{1}{2}mv^2$

$= \frac{1}{2}m(-gt)^2$

$= \frac{1}{2}mg^2t^2$

Conservative and nonconservative forces

- A force is called *conservative* if



The work done by the force on
a particle moving between any two points
is independent of the path taken by the particle.

Conservative and nonconservative forces

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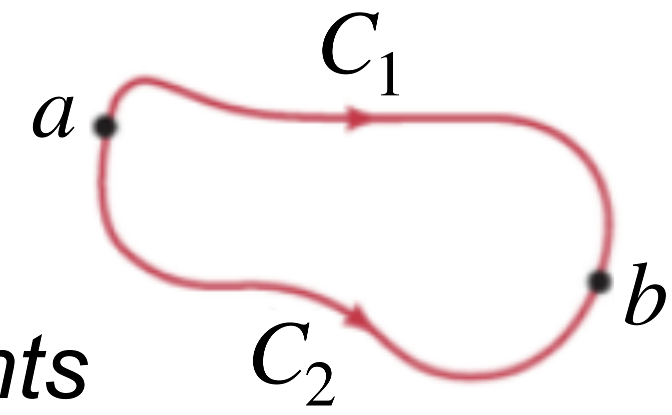
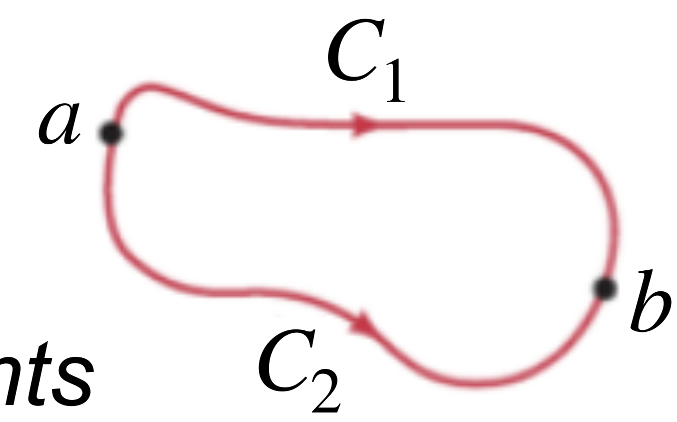


The work done by the force on
a particle moving between any two points
is independent of the path taken by the particle.

or equivalently

The net work done by the force on
a particle moving around any closed path is zero.

Conservative and nonconservative forces



- A force is called *conservative* if

  The work done by the force on a particle moving between any two points is independent of the path taken by the particle.

  or equivalently

  The net work done by the force on a particle moving around any closed path is zero.

- Otherwise the force is *non-conservative*

- Gravity and springs are conservative, while friction and air drag are non-conservative

Path independence of gravity

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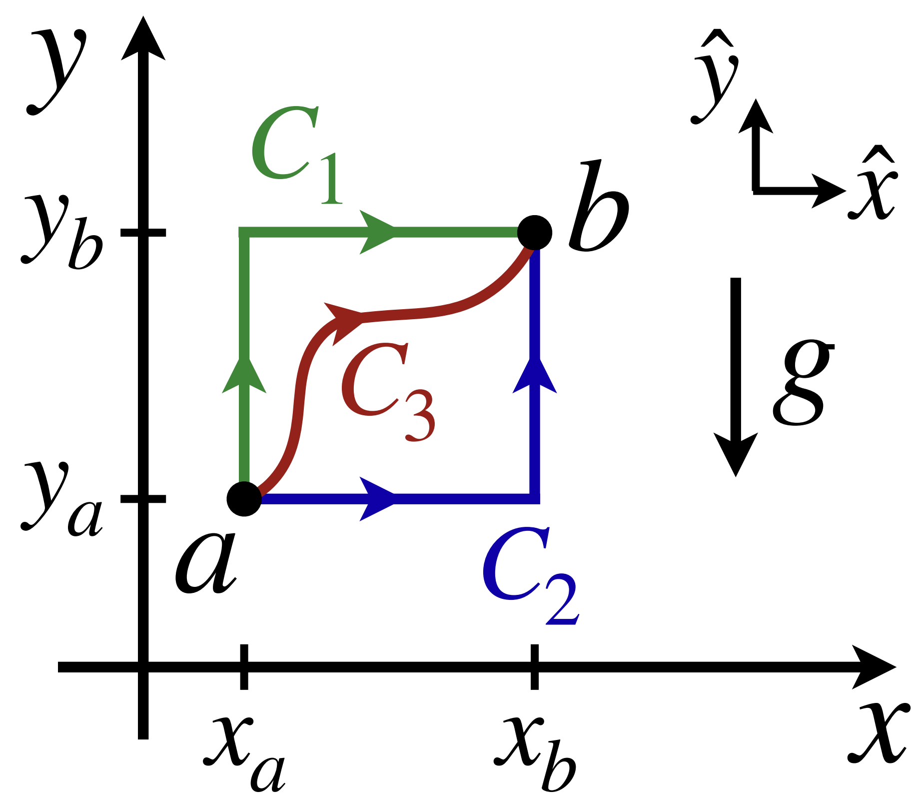
- Show that the gravitational force $\vec{F}_g = -mg\hat{y}$ is conservative



$$W_g = \int_{C_1} \vec{F}_g \cdot \vec{dl} = \int_{y_a}^{y_b} \vec{F}_g \cdot \vec{dl} + \int_{x_a}^{x_b} \vec{F}_g \cdot \vec{dl}$$
$$= \int_{y_a}^{y_b} -mg\hat{y} \cdot (\hat{y}\, dy) + \int_{x_a}^{x_b} -mg\hat{y} \cdot (\hat{x}\, dx)$$
$$= \int_{y_a}^{y_b} -mg\, dy = -mg(y_b - y_a)$$

$$W_g = \int_{C_2} \vec{F}_g \cdot \vec{dl} = \int_{x_a}^{x_b} \cancel{-mg\hat{y} \cdot \hat{x}}\, dx + \int_{y_a}^{y_b} -mg\hat{y}\, dy$$
$$= -mg(y_b - y_a)$$

$$W_g = \int_{C_3} \vec{F}_g \cdot \vec{dl} = \int_{C_3} \vec{F}_g \cdot (\cancel{\hat{x}}\, dx + \hat{y}\, dy) = \int_{C_3} -mg\hat{y} \cdot (\cancel{\hat{x}}\, dx + \hat{y}\, dy) = \int_{y_a}^{y_b} -mg\, dy$$
$$= -mg(y_b - y_a)$$

$$W_{net} = \Delta K$$

DEMO (635)

1) What are the final speeds of the two skiers?

DEMO (556)

Brachistochrone problem

2) Which skier would win in a race?

Rules of thumb for mechanics problems

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- Do you need to know information at particular times or at particular locations?
  - Care about timings: Use Newton’s laws (as it requires the equation of motion)
  - Care about locations: Use work and energy
- Are the forces constant?
  - Yes: Use Newton’s laws
  - No: Use work and energy
- Work and energy are scalar quantities, so they cannot determine the direction of velocity or acceleration

Summary

- Kinetic energy is defined by $K = \dfrac{m}{2}v^2$ and has units of Joules

- The work done by a force $\vec{F}$ is

$$W = \int_a^b \vec{F} \cdot d\vec{\ell}$$

- Work-kinetic energy theorem: The total (net) work done on an object equals the change in its kinetic energy

$$W_{net} = \Delta K = \frac{m}{2}v_b^2 - \frac{m}{2}v_a^2$$

- The work done by conservative forces *does not* depend on the path, while it *does* for nonconservative forces