General Physics: Mechanics

EPFL Swiss Plasma Center

PHYS-101(en)

Lecture 7a: Conservation of momentum, center of mass and continuous mass transfer

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Announcement

- Mini mock exam tomorrow
  - In-class (SG1) during normal lecture hours (10:1<u>5</u>-11:<u>00</u>)
  - The format is very similar to the exam. You will be given a “booklet” with the questions and space to write your solutions
  - “Cheat” sheet, notes, computer, calculator, etc. are all allowed, just no talking with neighbors
  - Take to your TA at this week’s Wednesday exercise session, to be graded and returned on Wednesday November 6th

Midterm course feedback

Swiss Plasma Center

- In general quite positive, but…

Midterm course feedback

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  - sound quality in classroom is not great

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  - my handwriting is difficult to understand
  - availability of blank lecture notes
  - example problems are too long or too short
  - problem sets are difficult

Today's agenda (Serway 6, 9 and MIT 8 -12)

1. **Brief review of last week**

2. Some examples
   - Conservation of momentum
   - Motion of Center of mass

3. Systems with variable mass
   - Rocket science and interstellar space travel!

DEMO (113)

Recoil of a cart

Review of last week — Momentum

- **Momentum**: A vector quantity defined as $\vec{p} = m\vec{v}$
- **Conservation of momentum**: Total momentum of a system stays constant, if the net external force on it is zero and matter is not exchanged
- **Impulse**: Integral of net force over a time interval, or change in momentum over a time interval

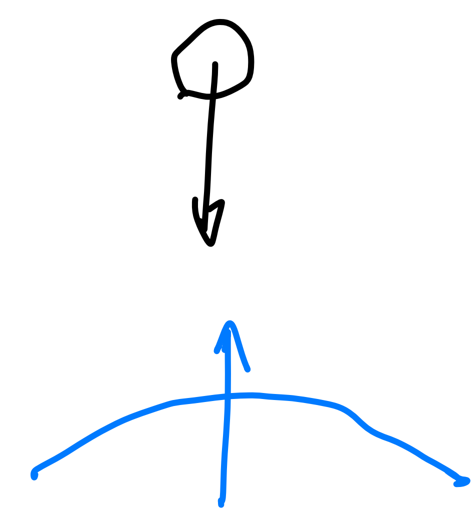
  $\vec{I} = \Delta\vec{p} = \int_{t_0}^{t_f} \vec{F}\,dt$
- **Center of mass**: For extended bodies, the one point that would move in the same path as a point mass subjected to the same net force

$$\overrightarrow{F}_{net}^{ext} = \frac{d\vec{p}_{sys}}{dt} = M\overrightarrow{A}_{CM}$$

Conceptual question

You drop a stone from the top of a high cliff. Consider the earth and stone as a system (neglecting the effects of the sun, moon, and other astronomical objects). As the stone falls, the momentum of this system…

A. increases in the downwards direction.
B. decreases in the downwards direction.
C. stays the same.



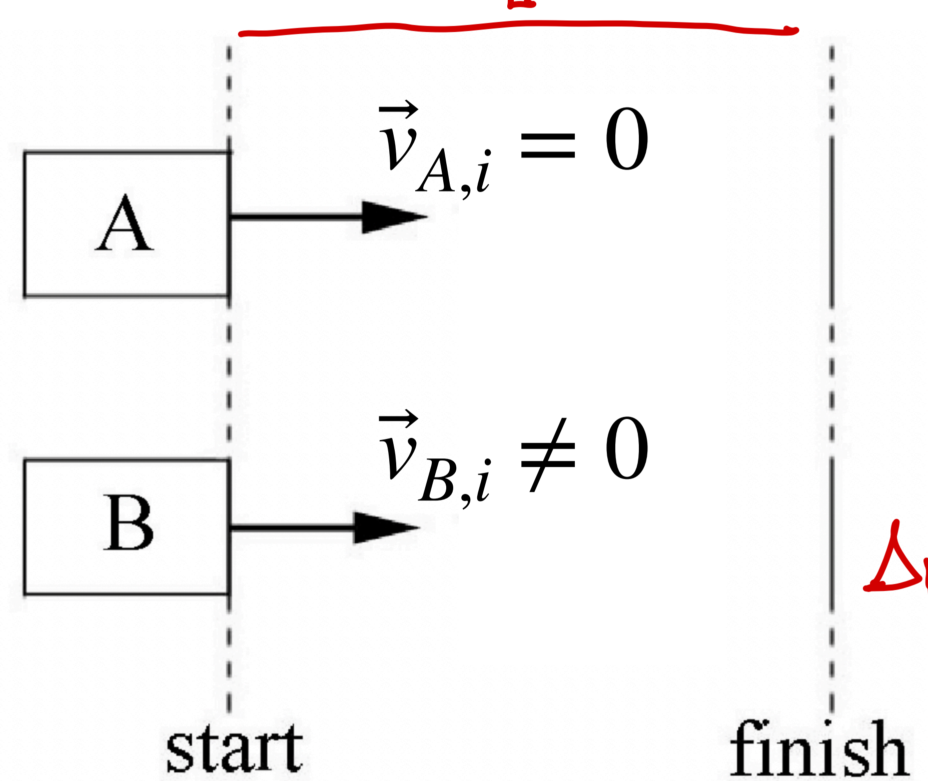
$$\vec{p}_i = m_s \vec{v}_s + m_E \vec{v}_E = 0$$

$$\vec{p}_f = 0 = m_s \vec{v}_s + m_E \vec{v}_E \Rightarrow \vec{v}_E = -\frac{m_s}{m_E} \vec{v}_s$$

Conceptual question

responseware.eu
Session ID: epflphys101en

Identical constant forces push two identical objects A and B continuously from a start line to a finish line. If A is initially at rest and B is initially moving to the right…



$$\Delta \vec{p} = \int_{t_0}^{t_f} \vec{F}(t)\,dt = \vec{I}$$
$$= \vec{F} \cdot \Delta t \qquad \Delta t = t_f - t_0$$
$$\Delta t_A = t_A^f - t_A^0$$
$$\Delta t_B = t_B^f - t_B^0 < \Delta t_A$$
$$\Delta p_B = F \cdot \Delta t_B < \Delta t_A = \Delta p_A$$

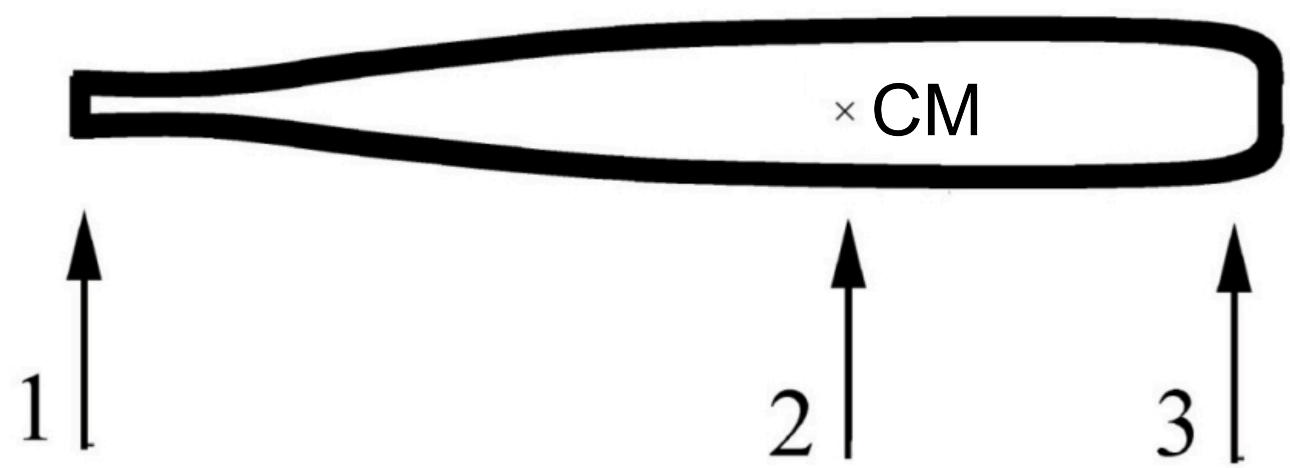
A. object A has the larger change in momentum.
B. object B has the larger change in momentum.
C. Both objects have the same change in momentum.

Conceptual question

responseware.eu
Session ID: epflphys101en

The greatest acceleration of the center of mass (CM) of the baseball bat will be produced by pushing with a force F at

A. Position 1
B. Position 2
C. Position 3
D. All are the same.



$$\sum \vec{F}_{ext} = \vec{F} = M\vec{A}_{CM} \Rightarrow \vec{A}_{CM} = \frac{1}{M}\vec{F}$$

Today's agenda (Serway 6, 9 and MIT 8 -12)

Swiss Plasma Center

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   - **Conservation of momentum**
   - Motion of Center of mass

3. Systems with variable mass
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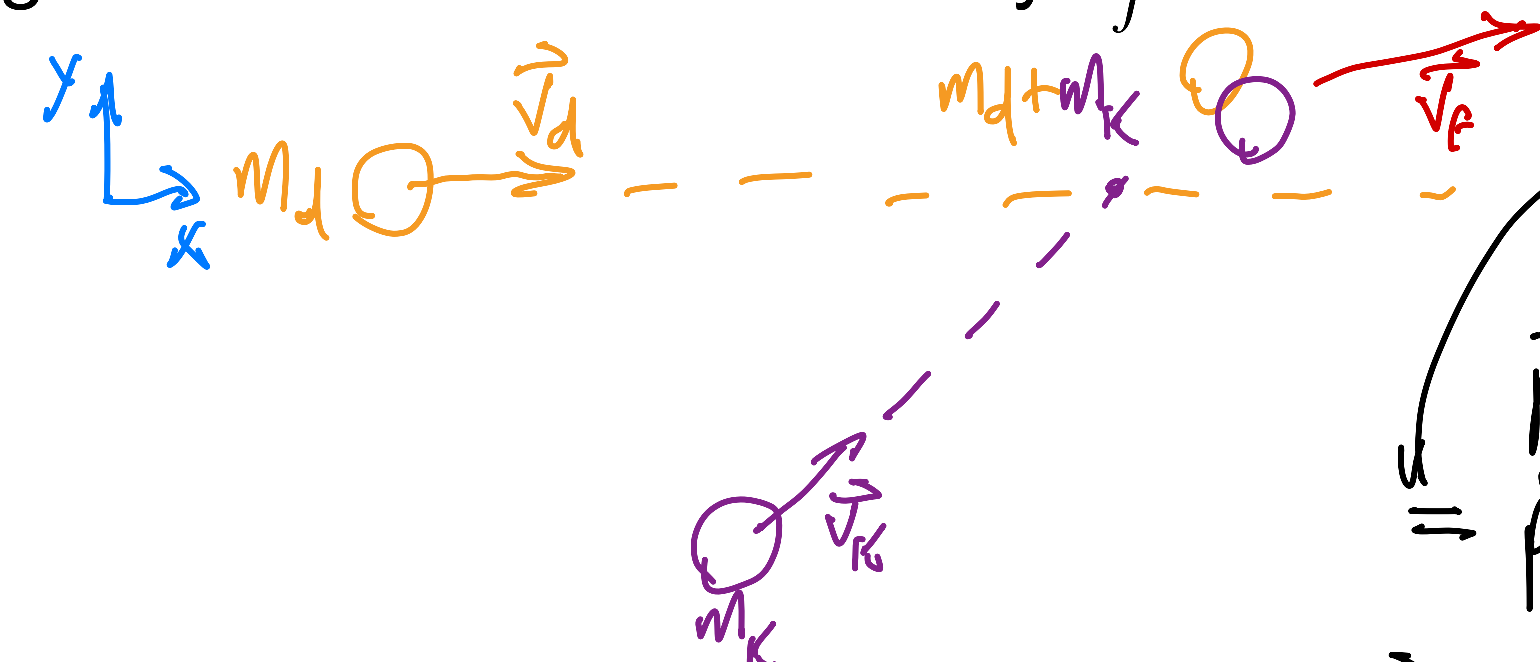
Example: Conservation of momentum

A kid of mass m_k sees that his dog (mass m_d) is playing on a flat icy surface. The dog suddenly falls and starts sliding with velocity $\vec{v}_d$. The kid runs after the dog, but upon touching the icy surface he also falls and slides with velocity $\vec{v}_k$. Fortunately, he has aimed right and reaches the dog.

Upon touching the dog the kid holds on to it and both keep sliding together. What is their velocity $\vec{v}_f$?



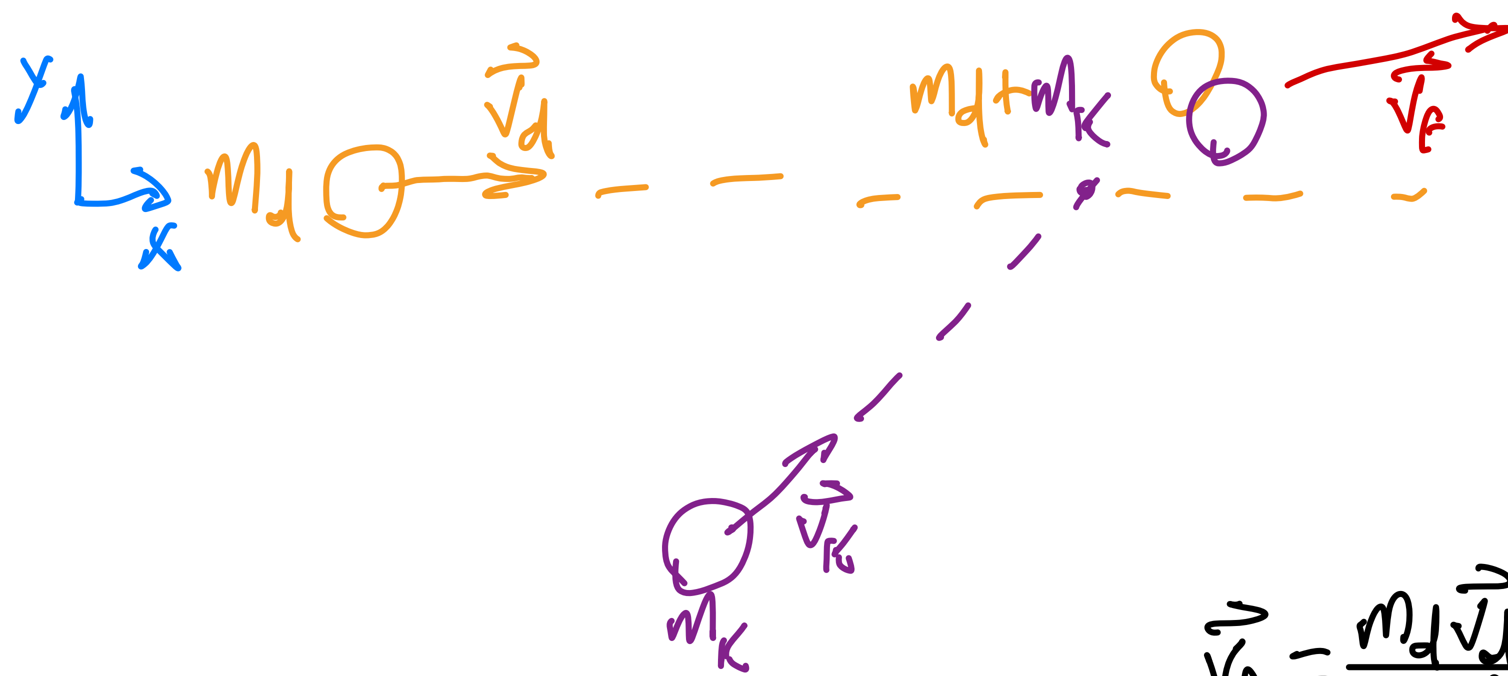
By cons. of momentum

$$\vec{p}_0 = m_d\vec{v}_d + m_k\vec{v}_k$$

$$= \vec{p}_f = (m_k+m_d)\vec{v}_f$$

$$\Rightarrow \vec{v}_f = \frac{m_d\vec{v}_d + m_k\vec{v}_k}{m_d+m_k}$$

Example: Conservation of momentum



$$\vec{v}_d = v_d \hat{x}$$

If, for example,

$$\vec{v}_K = v_K \hat{x}$$

$$\vec{v}_f = \frac{m_d \vec{v}_d + m_K \vec{v}_K}{m_d + m_K} = \frac{m_d v_d \hat{x} + m_K v_K \hat{x}}{m_d + m_K}$$

$$= \hat{x}\, \frac{m_d v_d + m_K v_K}{m_d + m_K}$$

If, for example, $v_d = 0$ and $m_d = m_K$, then

$$\vec{v}_f = \hat{x}\, \frac{m_K \cdot 0 + m_K v_K}{m_K + m_K} = \hat{x}\, \frac{m_K}{2 m_K} v_K = \hat{x}\, \frac{1}{2} v_K$$

DEMO (766)

Inelastic collision

Today’s agenda (Serway 6, 9 and MIT 8 -12)

Swiss Plasma Center

1. Brief review of last week

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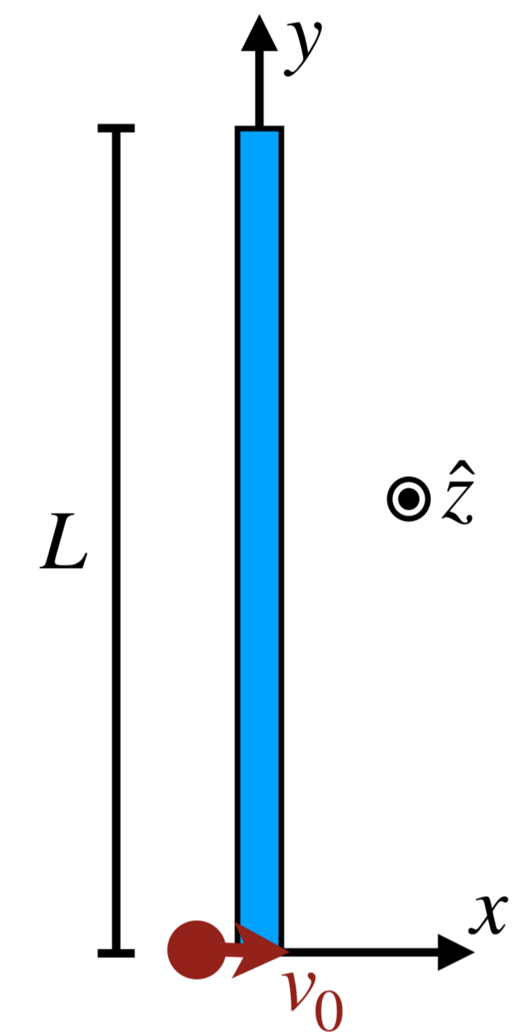
Example: Motion of center of mass (CM)

A slender rod of length L and mass M rests along the y-axis on a flat icy surface. The linear mass density of the rod $\lambda(y)$ varies quadratically with the distance from the origin.

A particle of mass $2M$ that moves along the x-axis with speed v_0 strikes the rod at the instant of time $t = t_0$.

Find the position $\overrightarrow{R}_{CM}$ of the center of mass as a function of t .

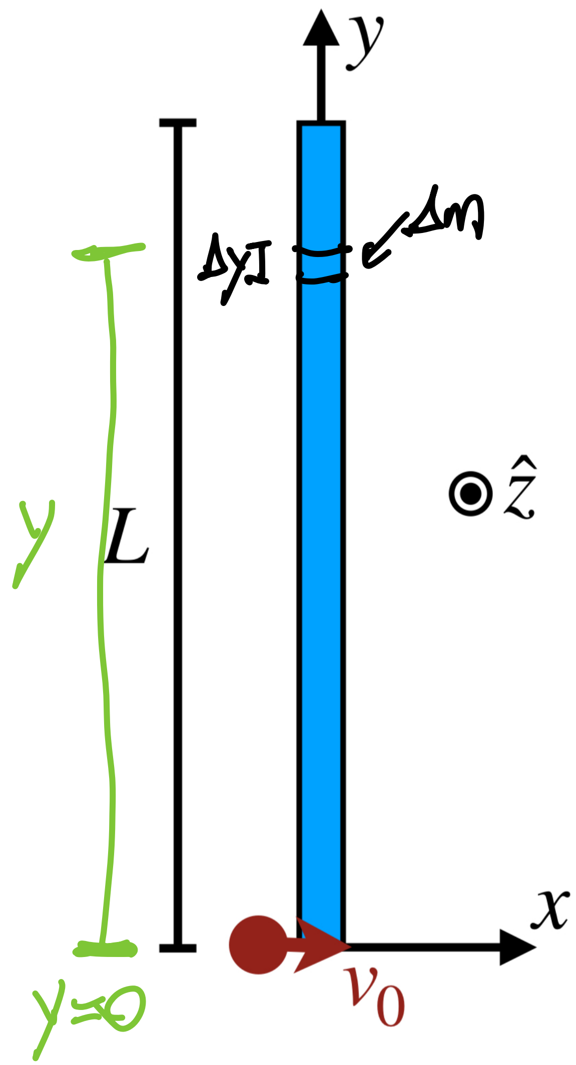


- System is composed of rod and particle

- $\sum \vec{F}_{ext} = 0 = M_{sys} \vec{A}_{CM} \Rightarrow \vec{A}_{CM} = 0 \Rightarrow \vec{V}_{CM} = \text{const. vector}$

$$\frac{d\vec{R}_{CM}}{dt} = \vec{V}_{CM} \qquad \vec{R}_{CM}(t) - \vec{R}_{CM}(t_0) = \int_{t_0}^{t} \vec{V}_{CM}\, dt = \vec{V}_{CM} \cdot \Delta t = \vec{V}_{CM} \cdot (t - t_0)$$

Example: Motion of center of mass (CM)



Let's find $\vec V_{CM}$. Choose some $t_1 < t_0$

$$\vec V_{CM}(t_1) = \frac{m_p \overset{V_0\hat x}{\vec V_p(t_1)} + m_r \overset{0}{\vec V_r(t_1)}}{m_p+m_r} = \frac{m_p}{m_p+m_r} V_0 \hat x = \vec V_{CM}$$

Let's now find $\vec R_{CM}(t_0)$

$$\vec R_{CM}(t_0) = \frac{m_p \overset{0}{\vec r_p(t_0)} + m_r \vec R^r_{CM}(t_0)}{m_p+m_r} = \frac{m_r}{m_p+m_r}\vec R^r_{CM}(t_0)$$

Now compute $\vec R^r_{CM}(t_0)$

$$\frac{\Delta m}{\Delta y} \xrightarrow{\Delta y \to 0} \frac{dm}{dy} = \lambda(y) = Ky^2 \Rightarrow \begin{aligned} dm &= \lambda(y)\,dy \\ &= Ky^2\,dy \end{aligned}$$

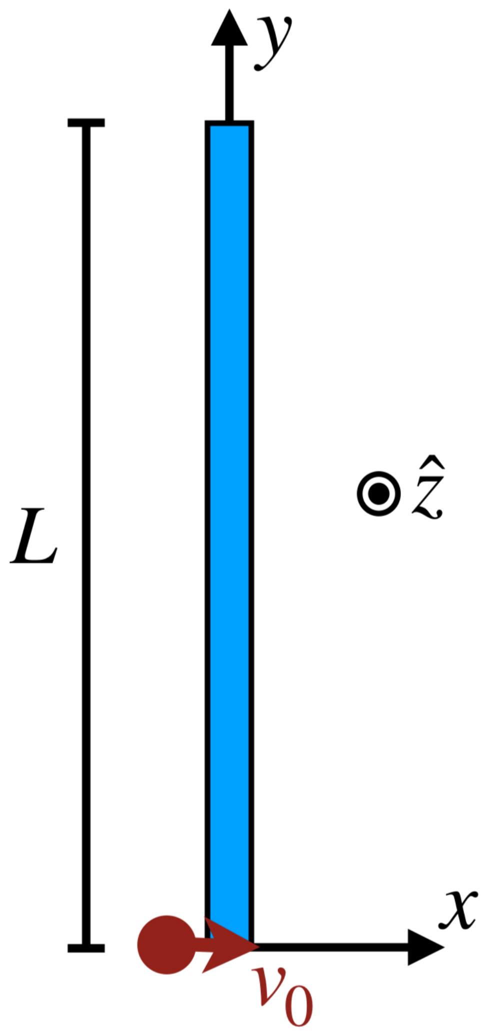
$$m_r = \int dm = \int_0^L Ky^2\,dy = K\int_0^L y^2\,dy = K\,\frac{y^3}{3}\Big|_0^L = K\frac{L^3}{3}$$

$$\Rightarrow K = \frac{3m_r}{L^3}$$

$$\vec R^r_{CM}(t_0) = \frac{1}{m_r}\int \vec r\,dm = \frac{1}{m_r}\int y\hat y \cdot Ky^2\,dy = \frac{K}{m_r}\hat y\int_0^L y^3\,dy = \hat y\,\frac{K}{m_r}\,\frac{y^4}{4}\Big|_0^L = \hat y\,\frac{K}{m_r}\,\frac{L^4}{4}$$

$$= \hat y\,\frac{1}{\cancel{m_r}}\,\frac{3\cancel{m_r}}{\cancel{L^3}}\,\frac{L^{\cancel{4}}}{4} = \hat y\,\frac{3L}{4}$$

Example: Motion of center of mass (CM)



$$\vec{R}_{CM}(t) = \vec{V}_{CM}(t-t_0) + \vec{R}_{CM}(t_0)$$

$$= \frac{m_p}{m_p+m_r} v_0 (t-t_0)\hat{x} + \underbrace{\frac{m_r}{m_p+m_r}\vec{R}^r_{CM}(t_0)}_{\frac{3L}{4}\hat{y}}$$

$$= \frac{2}{3} v_0 (t-t_0)\hat{x} + \frac{1}{3}\frac{3}{4}L\hat{y}$$

$$= \boxed{\frac{2}{3} v_0 (t-t_0)\hat{x} + \frac{1}{4}L\hat{y}}$$

Today's agenda (Serway 6, 9 and MIT 8 -12)

Swiss Plasma Center

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3. **Systems with variable mass**

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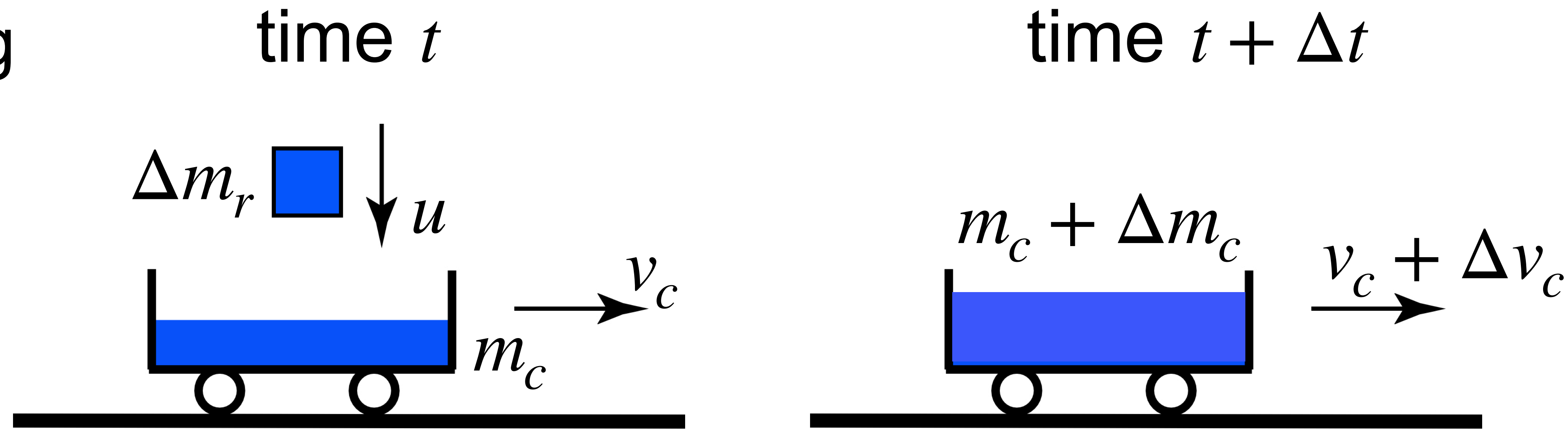
DEMO (178)

Falling chain

Types of systems with changing mass

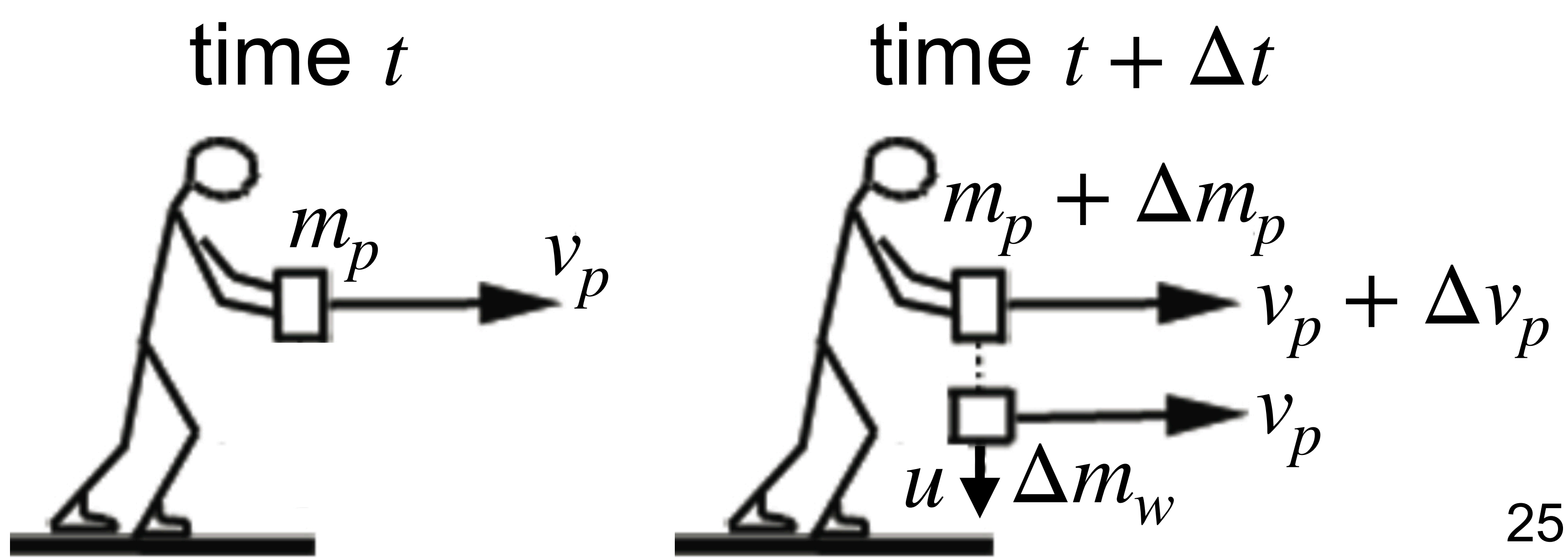
1. Transfer of mass <u>to</u> an object, but <u>no transfer of momentum</u> along the axis of motion

Rain falling into a cart:



2. Transfer of mass <u>from</u> an object, but <u>no transfer of momentum</u> along the axis of motion

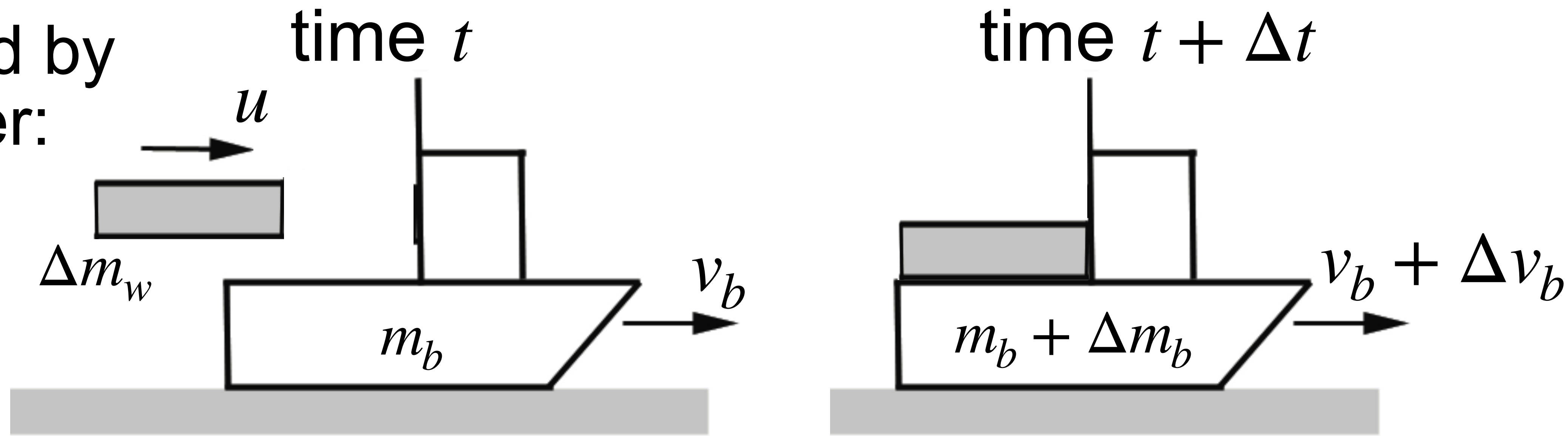
Ice skater with a leaky water bottle:



Types of systems with changing mass

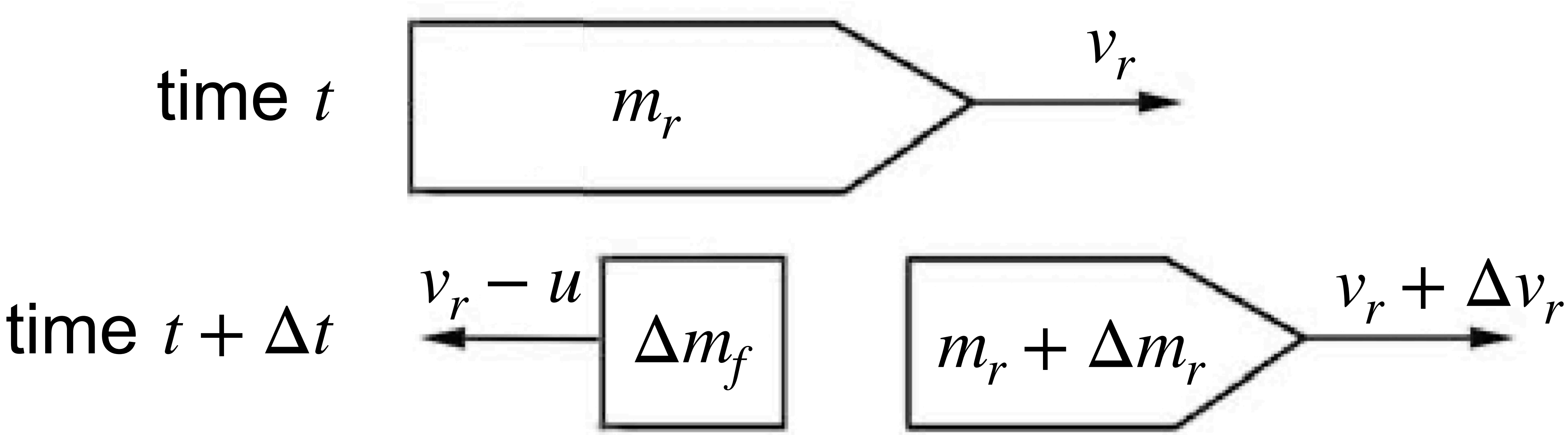
3. Mass hits an object, providing an impulse that transfers momentum along the direction of motion

Boat pushed by a jet of water:



4. Mass is ejected from an object, resulting in a recoil along the direction of motion

Rocket:



DEMO (113)

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Recoil of a cart
(ejection of gas)

Tackling systems with changing mass

- Determine the speed of an object with changing mass

Tackling systems with changing mass

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1. Choose reference frame and define system

2. Draw **momentum diagrams** at different times

3. Apply conservation of mass

Tackling systems with changing mass

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1. Choose reference frame and define system
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4. Use generalized version of Newton's 2nd law

$$\vec{F}^{ext}_{net} = \frac{d\vec{p}_{sys}}{dt}$$

Tackling systems with changing mass

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$$\vec{F}^{ext}_{net} = \frac{d\vec{p}_{sys}}{dt} = \boxed{\lim_{\Delta t \to 0} \frac{\vec{p}_{sys}(t+\Delta t) - \vec{p}_{sys}(t)}{\Delta t}}$$

Tackling systems with changing mass

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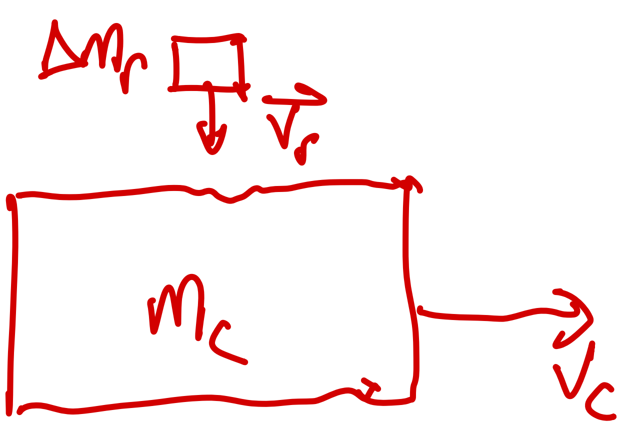
5. Find and solve the resulting differential equation

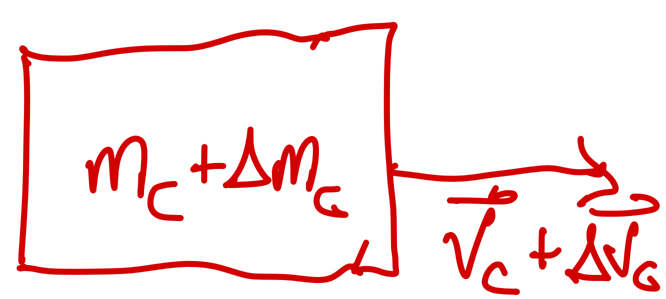
Momentum diagrams

- A simple graphical summary of ALL the masses and velocities in the system
- Similar to free body diagrams
- Draw them at important moments in time or at arbitrary moments

At time t



At $t+\Delta t$



Cons. of mass

$$m_c+\Delta m_r = m_c+\Delta m_c$$

$$\Rightarrow \Delta m_r = \Delta m_c$$

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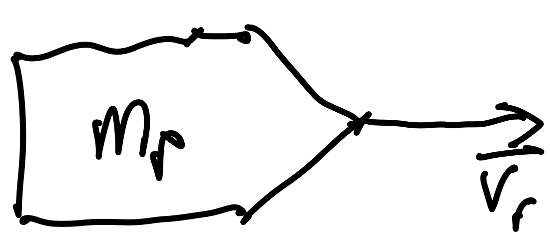
Derivation of rocket equation

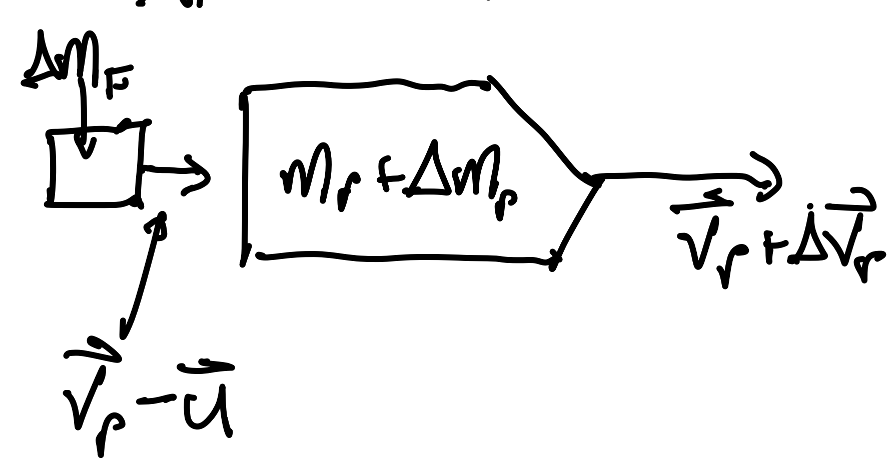
A rocket of dry mass M_0 starts from rest in our Solar System with a fuel mass M_f. It burns all the fuel, ejecting it backwards with velocity u relative to the rocket. This exhaust velocity u is independent of the velocity of the rocket. What is the final speed of the rocket? How long will it take to get to Proxima Centauri?

No external forces: $\sum \vec{F}_{ext} = 0$

At "t"



At "t+Δt"



Cons. of mass: $\cancel{m_r} = \cancel{m_r} + \Delta m_r + \Delta m_F \Rightarrow \Delta m_F = -\Delta m_r$

Derivation of rocket equation

Along $\hat{x}$:

$$p_{sys}(t) = m_r v_r$$

$$p_{sys}(t+\Delta t) = (m_r + \Delta m_r)(v_r + \Delta v_r) + \underbrace{\Delta m_E}_{= -\Delta m_r} \cdot (v_r - u)$$

$$= m_r v_r + m_r \Delta v_r + \cancel{\Delta m_r \cdot v_r} + \underbrace{\Delta m_r \cdot \Delta v_r}_{\approx 0} - \cancel{\Delta m_r v_r} + \Delta m_r \cdot u$$

$$\approx m_r v_r + m_r \Delta v_r + \Delta m_r \cdot u$$

$$\Rightarrow p_{sys}(t+\Delta t) - p_{sys}(t) = m_r \Delta v_r + \Delta m_r \cdot u$$

in this case

$$\sum F_{ext} = 0 = \frac{dp_{sys}}{dt} = \lim_{\Delta t \to 0}\left[\frac{p_{sys}(t+\Delta t) - p_{sys}(t)}{\Delta t}\right] = \lim_{\Delta t \to 0}\left[m_r \frac{\Delta v_r}{\Delta t} + u \frac{\Delta m_r}{\Delta t}\right]$$

$$= m_r \frac{dv_r}{dt} + u \frac{dm_r}{dt} = 0$$

$$\Rightarrow m_r \frac{dv_r}{dt} = -u \frac{dm_r}{dt} \Rightarrow \boxed{\frac{1}{u}\frac{dv_r}{dt} = -\frac{1}{m_r}\frac{dm_r}{dt}}$$

Derivation of rocket equation

$$\frac{1}{u}\frac{dv_r}{dt} = -\underbrace{\frac{1}{m_r}\frac{dm_r}{dt}}$$

$$\frac{d}{dt}\left[\ln\left(m_r(t)\right)\right] = \frac{1}{m_r(t)}\cdot\frac{dm_r}{dt}$$

$$\frac{d}{dt}\left(\frac{1}{u}v_r\right) = -\frac{d}{dt}\left[\ln(m_r)\right] \Rightarrow \frac{1}{u}v_r = -\ln(m_r) + C$$

$$\Rightarrow v_r = -u\ln(m_r) + C_1$$

$$v_r(t=0) = 0 = -u\ln\big(\underbrace{m_r(t=0)}_{=M_0+M_F}\big) + C_1 \Rightarrow C_1 = u\ln\left(M_0+M_F\right)$$

$$\Rightarrow v_r(t) = -u\ln\left(m_r\right) + u\ln\left(M_0+M_F\right) = u\cdot\ln\left[\frac{M_0+M_F}{m_r(t)}\right]$$

$$\Rightarrow \boxed{v_{rf} = u\cdot\ln\left[\frac{M_0+M_F}{M_0}\right] = u\cdot\ln\left(1+\frac{M_F}{M_0}\right)}$$

DEMO (172)

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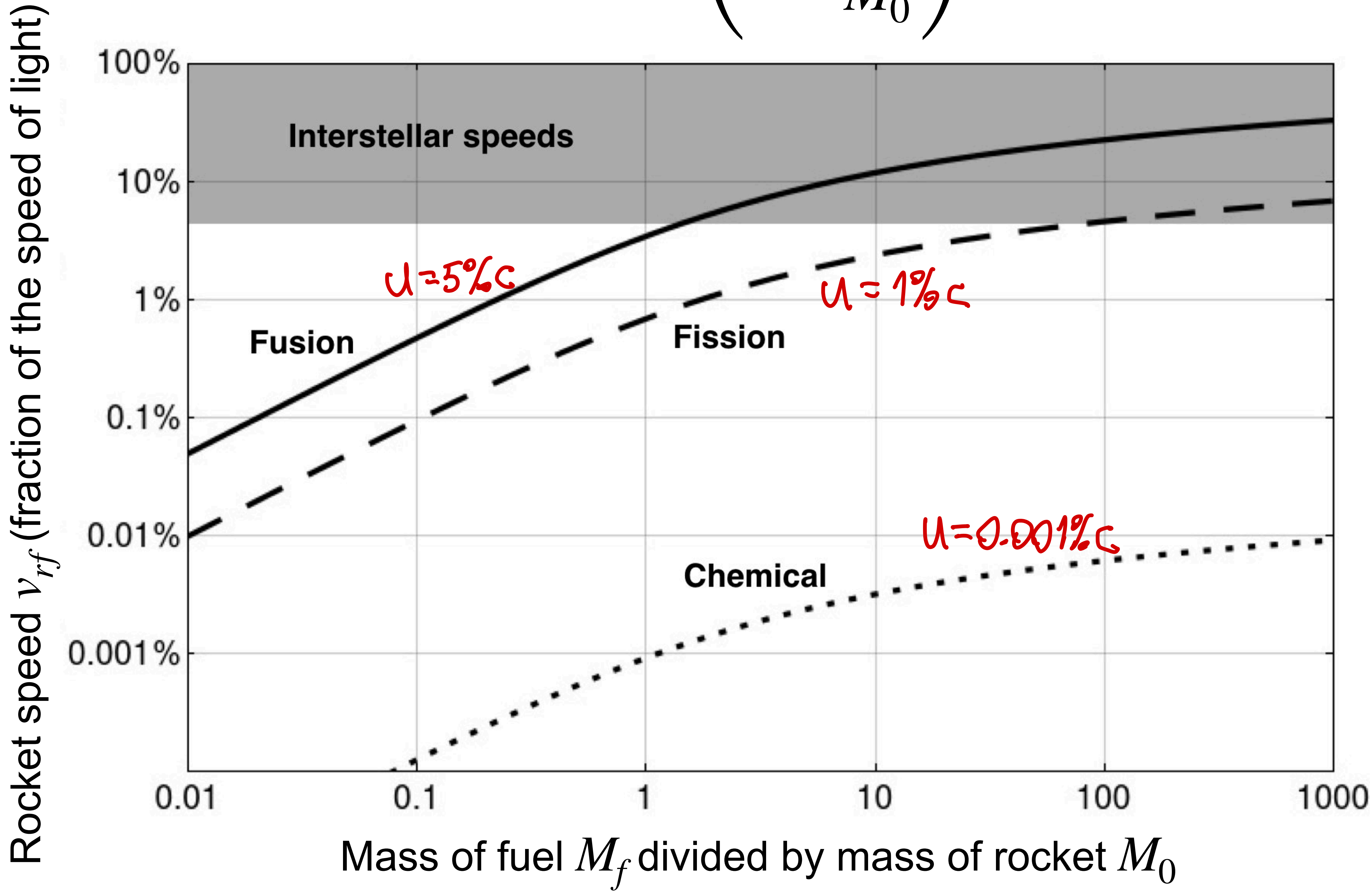
Rockets!

Significance of the rocket equation

$$v_{rf} = u \ln\left(1 + \frac{M_f}{M_0}\right)$$



Conceptual question

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A rocket is moving in outer space without gravity. It is burning fuel so that its “thrust” is constant (i.e. $u\dfrac{dm}{dt} = \text{const}$) at all times. Its acceleration is…

A. constant.
B. decreasing.
C. increasing.

Summary

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- Tackling systems with changing mass

1. Choose reference frame and define system

2. Apply conservation of mass

3. Use generalized Newton's 2nd law: $\vec{F}_{net}^{ext} = \lim_{\Delta t \to 0} \dfrac{\vec{p}_{sys}(t + \Delta t) - \vec{p}_{sys}(t)}{\Delta t}$

4. Find and solve the resulting differential equation

- Rocket equation: $v_{rf} = u \ln\left(1 + \dfrac{M_f}{M_0}\right)$

Mock exam tomorrow 😲