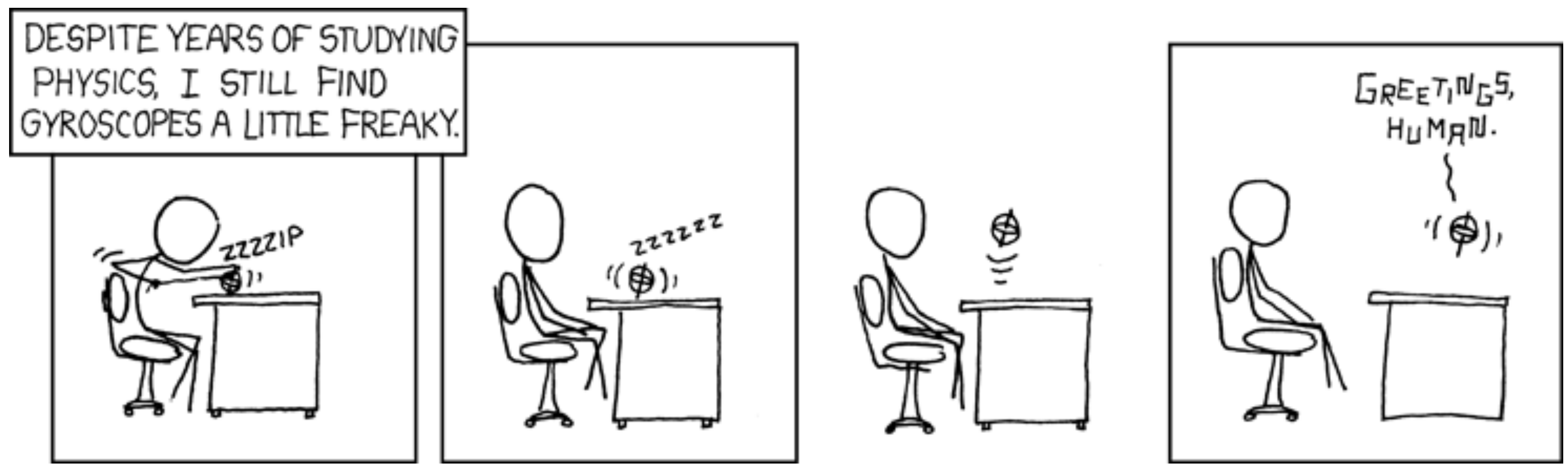
General Physics: Mechanics

EPFL
Swiss Plasma Center

PHYS-101(en)

Lecture 13a: Kepler's laws, gyroscopes and harmonic motion



xkcd.com/332

Dr. Marcelo Baquero
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December 9th, 2024

Announcements

- Next Monday (i.e. December 16th) we will start the lecture with written course feedback

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Swiss Plasma Center

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- Also next Monday I will give back the graded Mock exams for those of you who turned them in
- Final exams of some previous years will be made available in the Moodle for you to see and practice

Today's agenda (Serway 11,13; MIT 22,23)

EPFL Swiss Plasma Center

1. **Kepler's laws of planetary motion**
2. Gyroscopes
3. Harmonic motion
   - Simple harmonic motion

Kepler’s laws of planetary motion

- From 1610-1619 Johannes Kepler wrote:

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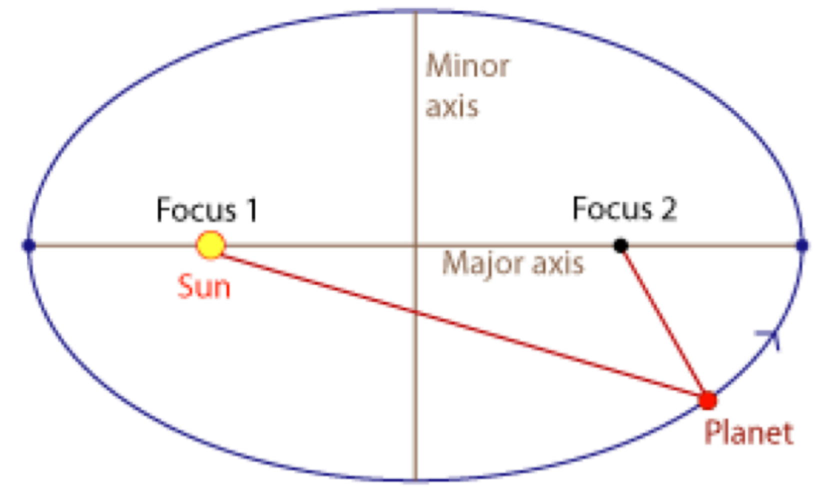
    1. The orbit of each planet is an ellipse, with the Sun at one focus.



Kepler's laws of planetary motion

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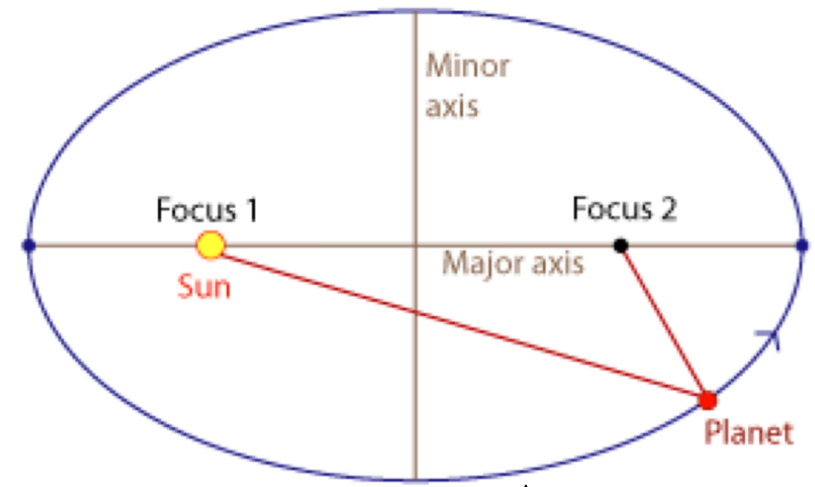
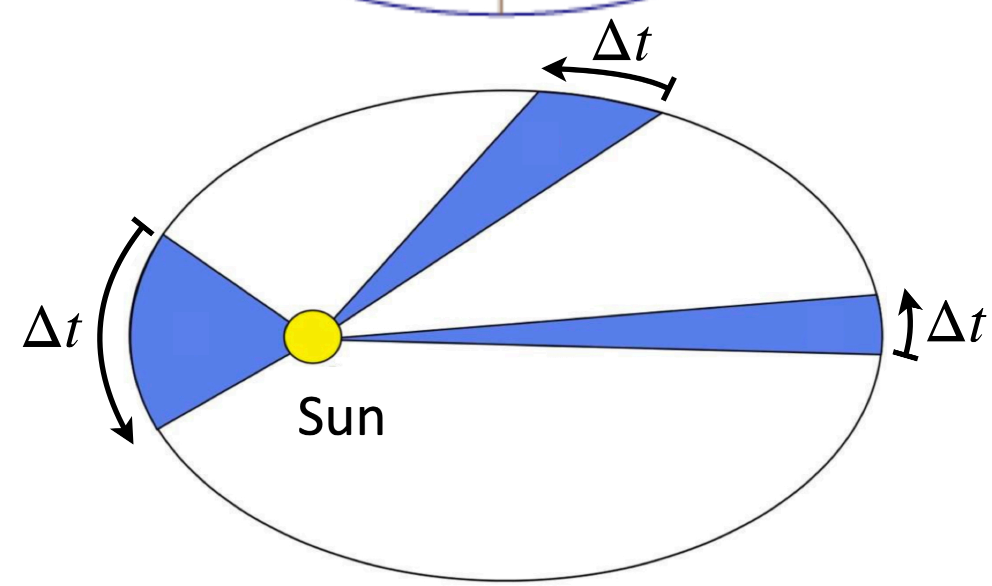
    1. The orbit of each planet is an ellipse, with the Sun at one focus.

    2. An imaginary line drawn from each planet to the Sun sweeps out equal areas in equal times.





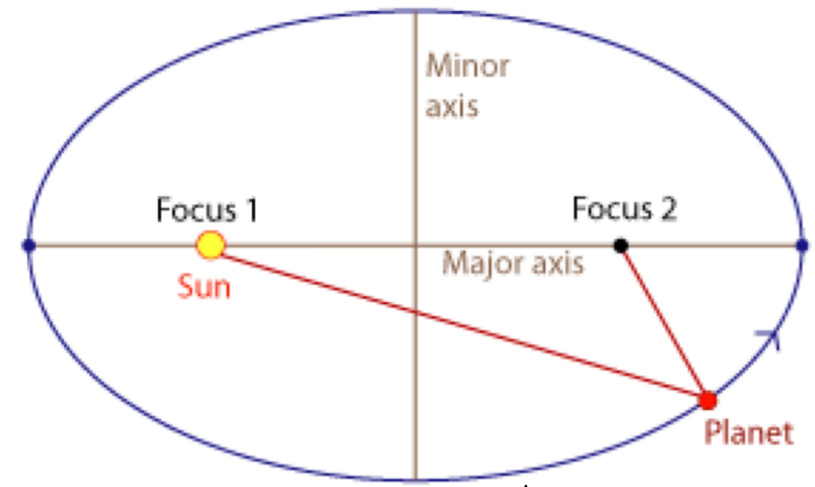
Kepler's laws of planetary motion

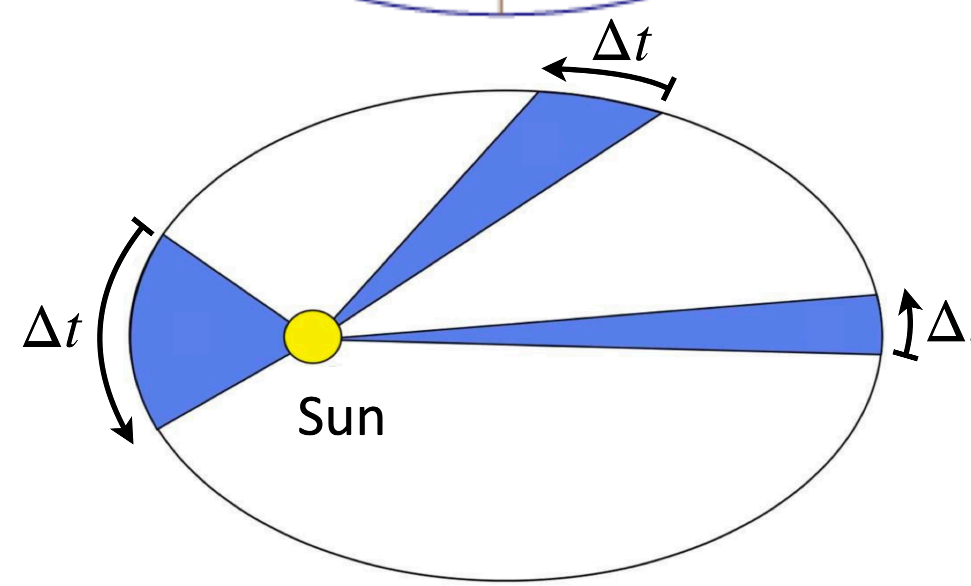
- From 1610-1619 Johannes Kepler wrote:

  1. The orbit of each planet is an ellipse, with the Sun at one focus.

  2. An imaginary line drawn from each planet to the Sun sweeps out equal areas in equal times.

  3. The square of a planet's orbital period is proportional to the cube of its mean distance from the Sun.





Planet	Period, T (Earth year)	Avg distance to Sun, r (10^6 km)	T^2/r^3 (10^{-25} yr^2/km^3)
Mercury	0.241	57.9	2.99
Venus	0.615	108.2	2.99
Earth	1	149.6	2.99
Mars	1.88	227.9	2.99
Jupiter	11.86	778.3	2.98
Saturn	29.5	1427	2.99
Uranus	84.0	2870	2.98
Neptune	165	4497	2.99

Kepler's laws of planetary motion

3. The square of a planet's orbital period is proportional to the cube of its mean distance from the Sun.

- Approximate orbits as circular and use

$$\vec{F}_G = -G\frac{m_p m_s}{r^2}\hat{r}$$

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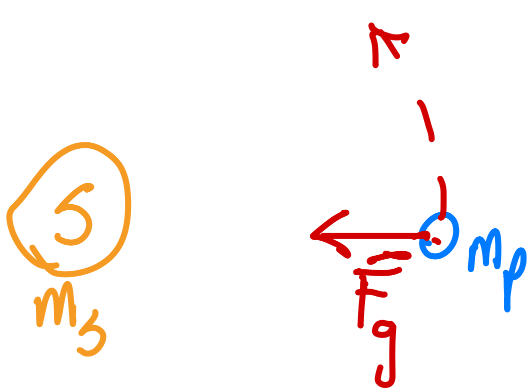
[redacted]			



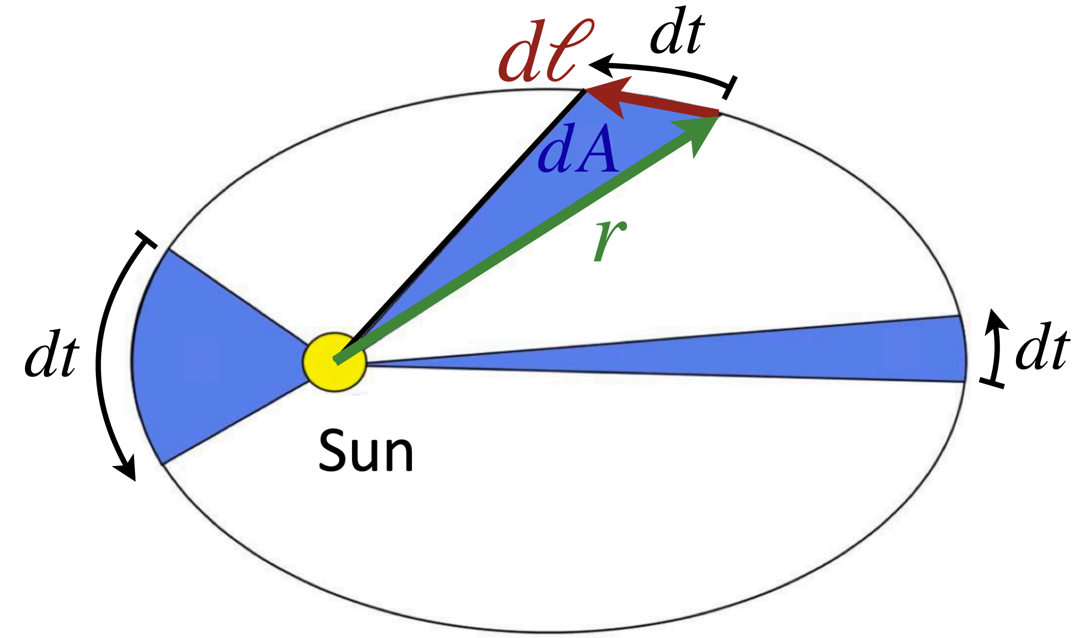
$$\vec{F}_g = \vec{F}_{cent} \Rightarrow -G\frac{m_p m_s}{r^2} = m_p a_c = m_p(-r\omega^2)$$

$$G\frac{\cancel{m_p} m_s}{r^2} = \cancel{m_p} r\omega^2 = r\left(\frac{2\pi}{T}\right)^2 = r\frac{4\pi^2}{T^2} \qquad \omega = \frac{2\pi}{T}$$

$$\boxed{T^2 = \frac{4\pi^2}{G m_s} r^3}$$

Kepler's laws of planetary motion

2. An imaginary line drawn from each planet to the Sun sweeps out equal areas in equal times.



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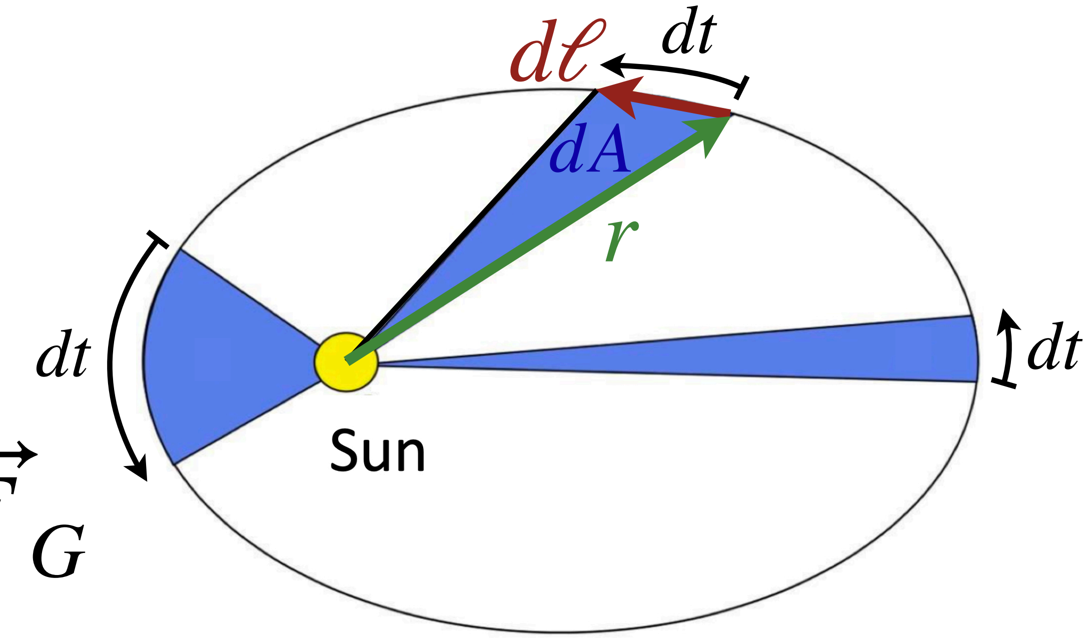
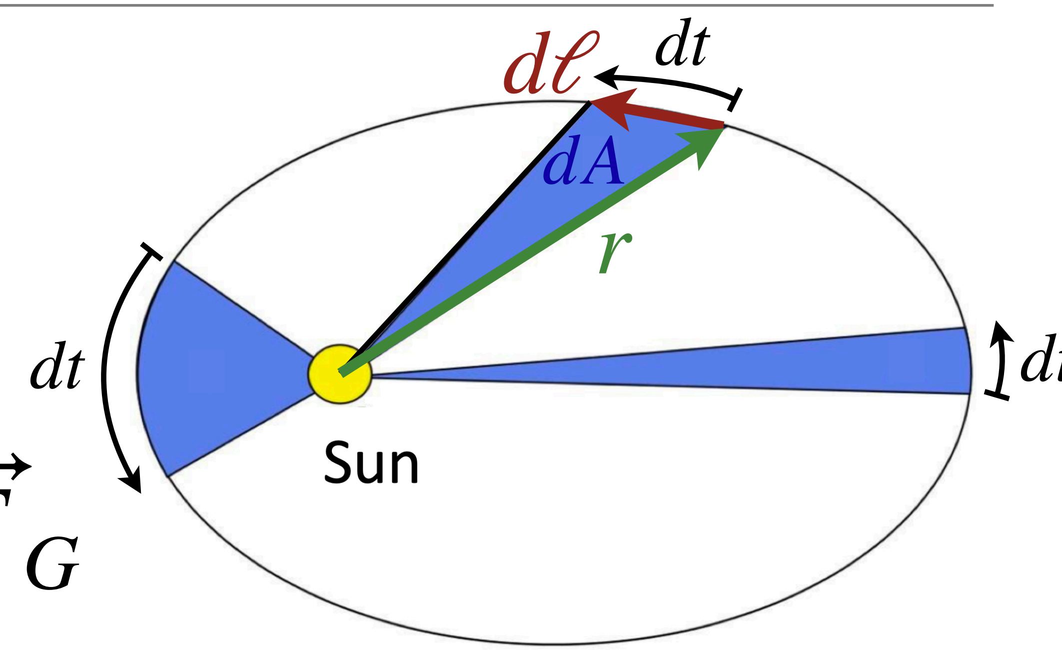


- Gravity is a central force, so $\vec{r} \,||\, \overrightarrow{F}_G$

Kepler's laws of planetary motion



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- Gravity is a central force, so $\vec{r} \,||\, \overrightarrow{F}_G$
- Thus, $\vec{\tau}_G = \vec{r} \times \overrightarrow{F}_G = 0$, so $\overrightarrow{L}_S = \vec{r} \times m_p \vec{v}$ is conserved

Kepler's laws of planetary motion

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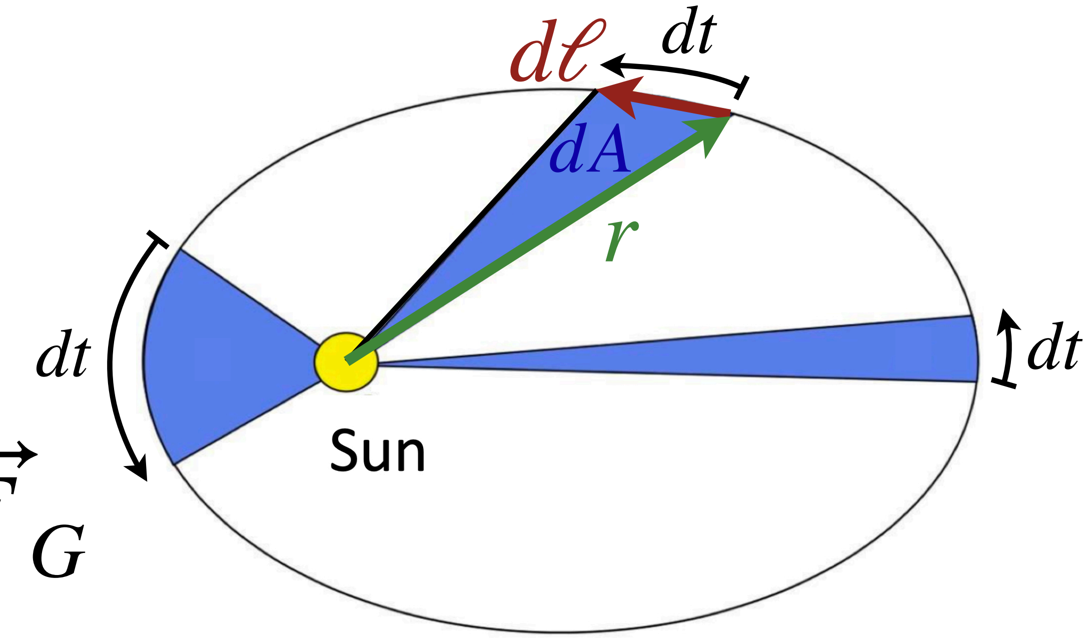

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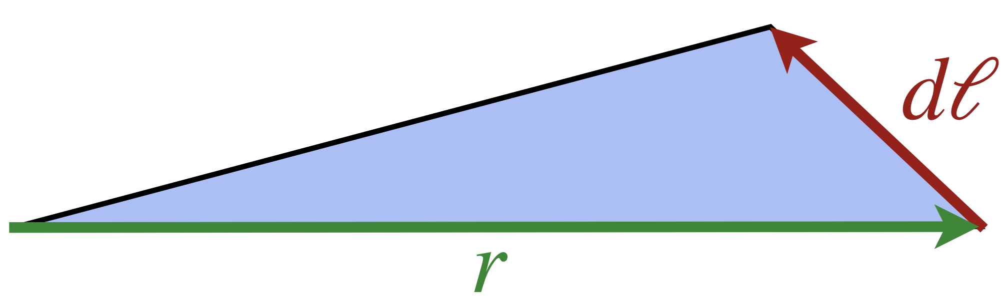
- How is this related to area?



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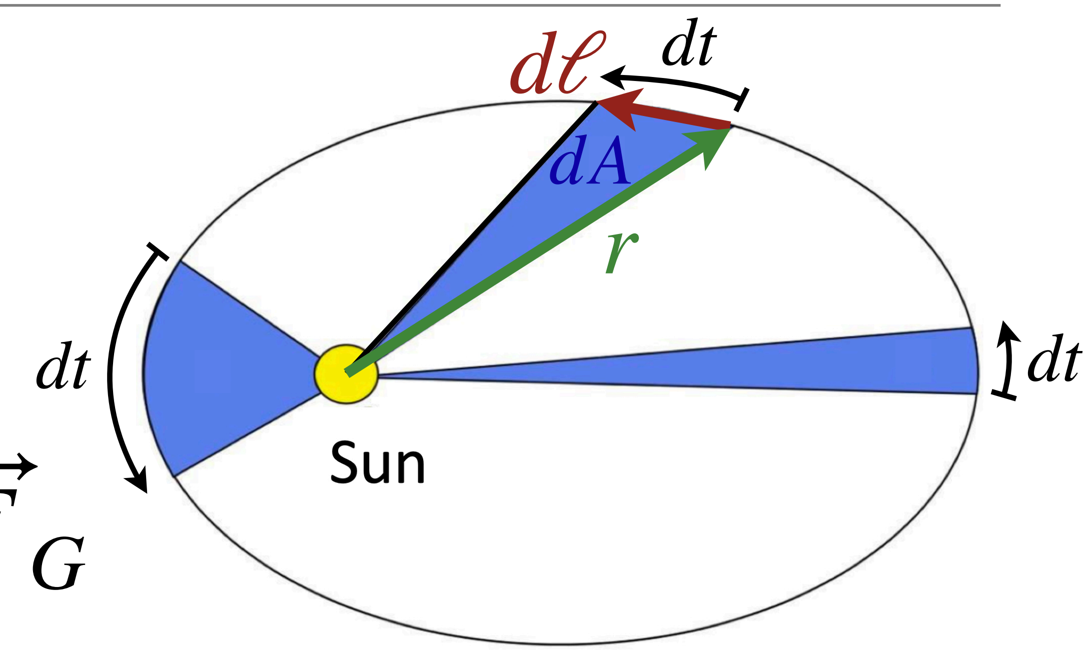

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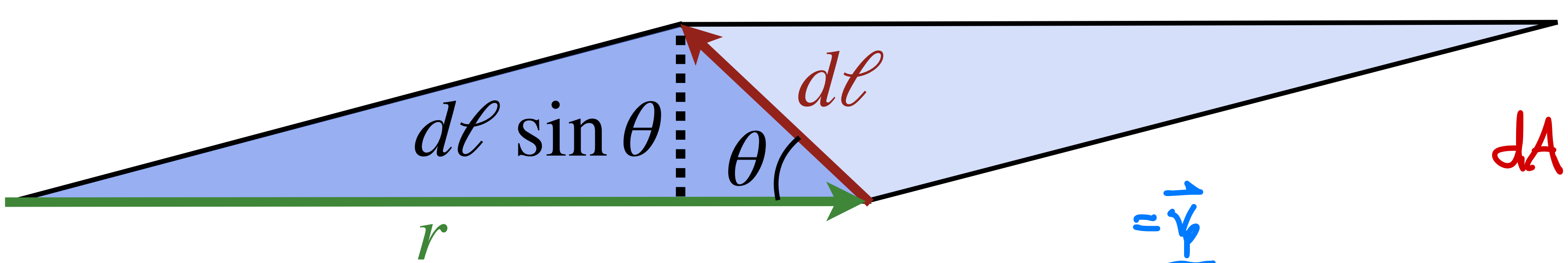

$A_p = r\, d\ell \sin(\theta)$

$dA = A_T = \frac{1}{2} A_p = \frac{1}{2} r\, d\ell \sin(\theta)$

$$dA = \frac{1}{2} \underline{r\, d\ell \sin(\theta)} = \frac{1}{2}\left|\vec{r} \times d\vec{\ell}\right| = \frac{1}{2}\left|\vec{r} \times \overbrace{\frac{d\vec{\ell}}{dt}}^{=\vec{v}_p} dt\right| = \frac{1}{2}\left|\vec{r} \times \vec{v}_p\right| dt = \frac{1}{2m_p}\left|\vec{r} \times m_p \vec{v}\right| dt$$

$$\boxed{\frac{dA}{dt} = \frac{1}{2m_p}\left|\vec{L}_S\right| = \text{constant}}$$

$$\frac{dA}{dt} = \text{const} = \frac{\Delta A}{\Delta t}$$

Kepler's laws of planetary motion

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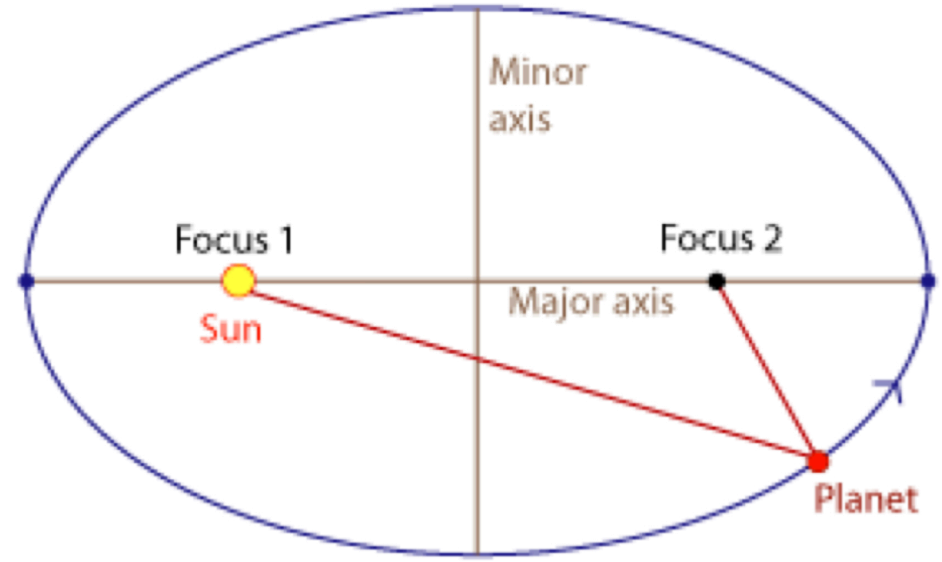


- Need to know the universal gravitational potential energy

$$U_G = -G\frac{m_p m_s}{r}$$

- Apply mechanical energy conservation:

$$E_m = K + U_G = \frac{1}{2}m_p v_p^2 - G\frac{m_p m_s}{r} = \text{constant}$$

- Apply conservation of angular momentum:

$$\vec{L} = \vec{r}\times\vec{p} = \vec{r}\times(m_p\vec{v}_p) = \text{const}$$

- Considerable mathematical magic

Today’s agenda (Serway 11,13; MIT 22,23)

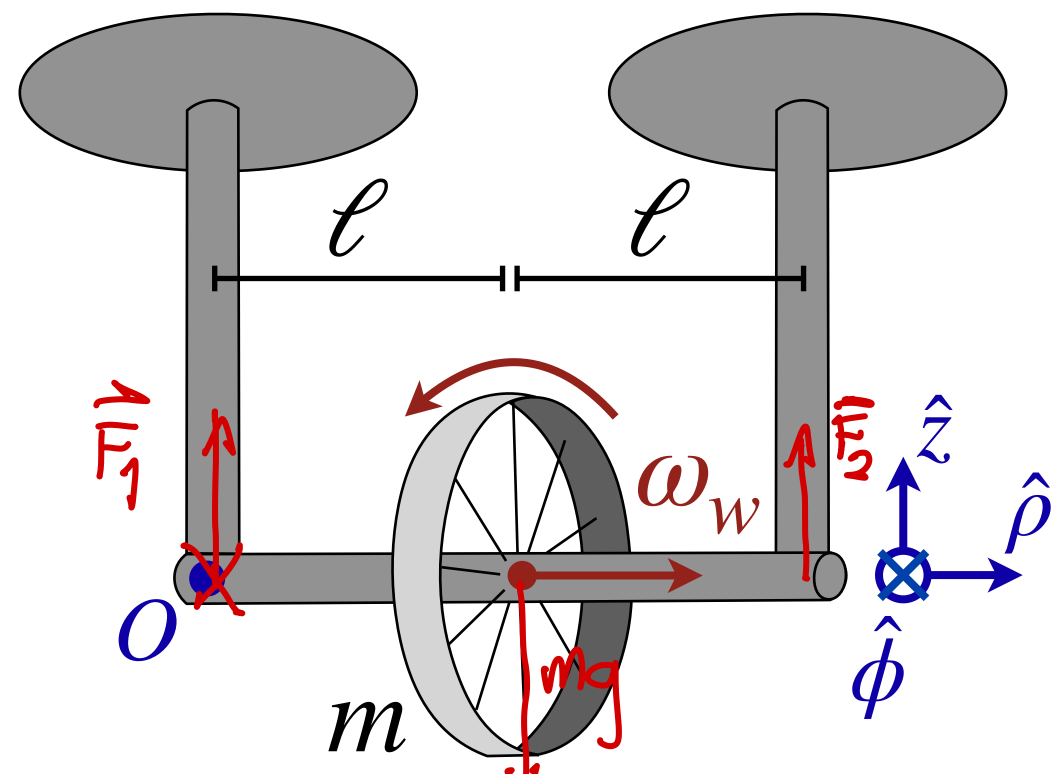
1. Kepler's laws of planetary motion
2. **Gyroscopes**
3. Harmonic motion
   - Simple harmonic motion

DEMO (50, 48)

Bicycle wheel

Analyzing fixed axis rotation

static eq. of beam: $\sum \vec{F} = 0 \ \wedge \ \sum \vec{\tau} = 0$



$\sum \vec{F} = 0$:

$$\vec{F}_1 + \vec{F}_2 - mg\hat{z} = F_1\hat{z} + F_2\hat{z} - mg\hat{z} = 0$$

Along $\hat{z}$: $F_1 + F_2 = mg \Rightarrow F_1 = mg - F_2$ *

$\sum \vec{\tau} = 0$: I take point O as the pivot

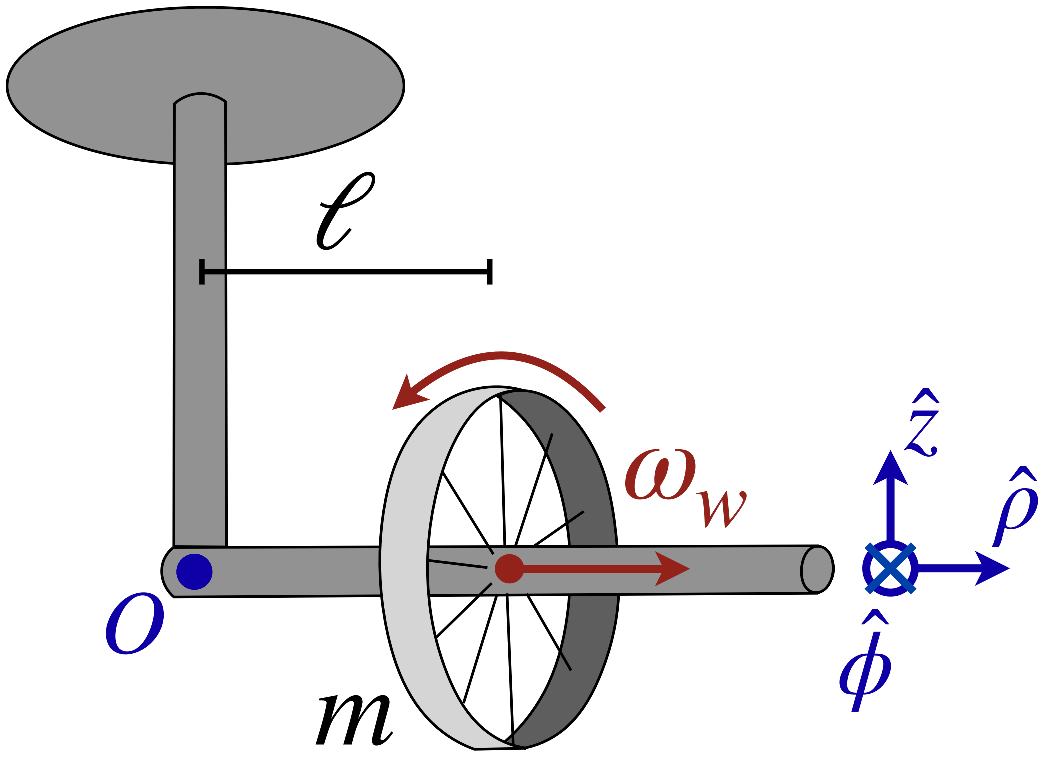
$$\sum \vec{\tau} = \cancel{\overset{0}{\vec{r}_1} \times \vec{F}_1} + \vec{r}_2 \times \vec{F}_2 + \vec{r}_g \times (-mg\hat{z}) = (2\ell\hat{\rho}) \times (F_2\hat{z}) + (\ell\hat{\rho}) \times (-mg\hat{z})$$

$$= 2\ell F_2 \underbrace{(\hat{\rho} \times \hat{z})}_{= -\hat{\phi}} - \ell mg(\hat{\rho} \times \hat{z}) = (2\ell F_2 - \ell mg)(-\hat{\phi}) = 0$$

$$\Rightarrow 2\cancel{\ell}F_2 - \cancel{\ell}mg = 0 \Rightarrow$$

$$\boxed{\begin{aligned} F_2 &= \tfrac{1}{2}mg \\ F_1 &= mg - F_2 = \tfrac{1}{2}mg \end{aligned}}$$

DEMO (50): A gyroscope

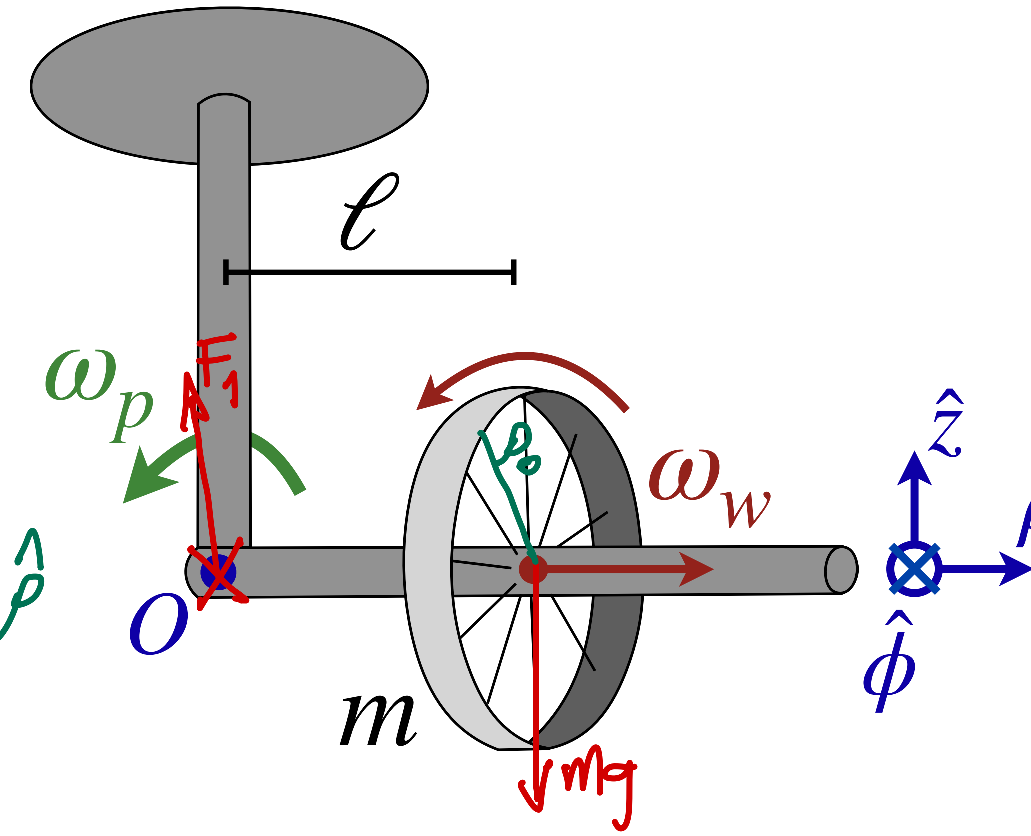


Analyzing a gyroscope

$$\sum \vec{\tau} = \frac{d\vec{L}_{tot}}{dt}$$

$$\sum \vec{\tau} = \cancelto{0}{\vec{r}_1 \times \vec{F}_1} + \vec{r}_g \times (-mg\hat{z}) = (\ell\hat{\rho}) \times (-mg\hat{z})$$

$$= -m\ell g(\hat{\rho} \times \hat{z}) = -m\ell g(-\hat{\phi}) = m\ell g\hat{\phi}$$



$$\vec{L}_{tot} = \vec{L}_w + \vec{L}_p$$

$$\vec{L}_w = I_w\vec{\omega}_w = m\rho_0^2\omega_w\hat{\rho}$$

$$\vec{L}_p = I_p\omega_p\hat{z}$$

$$\frac{d\vec{L}_{tot}}{dt} = \frac{d}{dt}[\vec{L}_w + \vec{L}_p] = \frac{d}{dt}(m\rho_0^2\omega_w\hat{\rho}) + \frac{d}{dt}(I_p\omega_p\hat{z})$$

$$\dot{\hat{\rho}} = \dot{\phi}\hat{\phi} = \omega_p\hat{\phi}$$

$$= m\rho_0^2\frac{d}{dt}(\omega_w\hat{\rho}) + I_p\frac{d}{dt}(\omega_p\hat{z}) = m\rho_0^2\left[\frac{d\omega_w}{dt}\hat{\rho} + \omega_w\overbrace{\frac{d\hat{\rho}}{dt}}\right] + I_p\left[\frac{d\omega_p}{dt}\hat{z} + \omega_p\cancelto{0}{\frac{d\hat{z}}{dt}}\right]$$

$$= \boxed{m\rho_0^2\frac{d\omega_w}{dt}\hat{\rho} + m\rho_0^2\omega_w\omega_p\hat{\phi} + I_p\frac{d\omega_p}{dt}\hat{z} = \sum\vec{\tau} = m\ell g\hat{\phi}}$$

Analyzing a gyroscope

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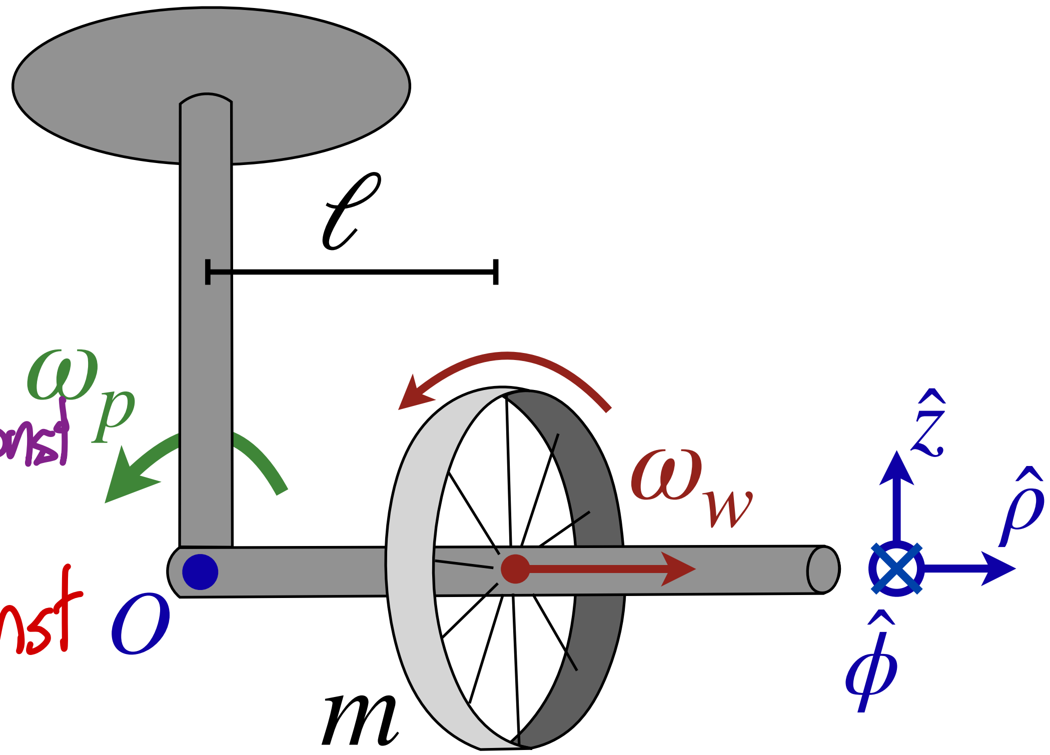
$$m\rho_0^2 \frac{d\omega_w}{dt}\hat{\rho} + m\rho_0^2 \omega_w \omega_p \hat{\phi} + I_p \frac{d\omega_p}{dt}\hat{z} = m\ell g\,\hat{\phi}$$

Along $\hat{\rho}$: $m\rho_0^2 \frac{d\omega_w}{dt} = 0 \Rightarrow \frac{d\omega_w}{dt} = 0 \Rightarrow \omega_w = \text{const}$

Along $\hat{z}$: $I_p \frac{d\omega_p}{dt} = 0 \Rightarrow \frac{d\omega_p}{dt} = 0 \Rightarrow \omega_p = \text{const}$

Along $\hat{\phi}$: $\cancel{m}\rho_0^2 \omega_w \omega_p = \cancel{m}\ell g$

$$\boxed{\omega_p = \frac{\ell g}{\rho_0^2 \omega_w}}$$



DEMO (501, 40)

Gyroscopes

Today’s agenda (Serway 11,13; MIT 22,23)

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1. Kepler's laws of planetary motion
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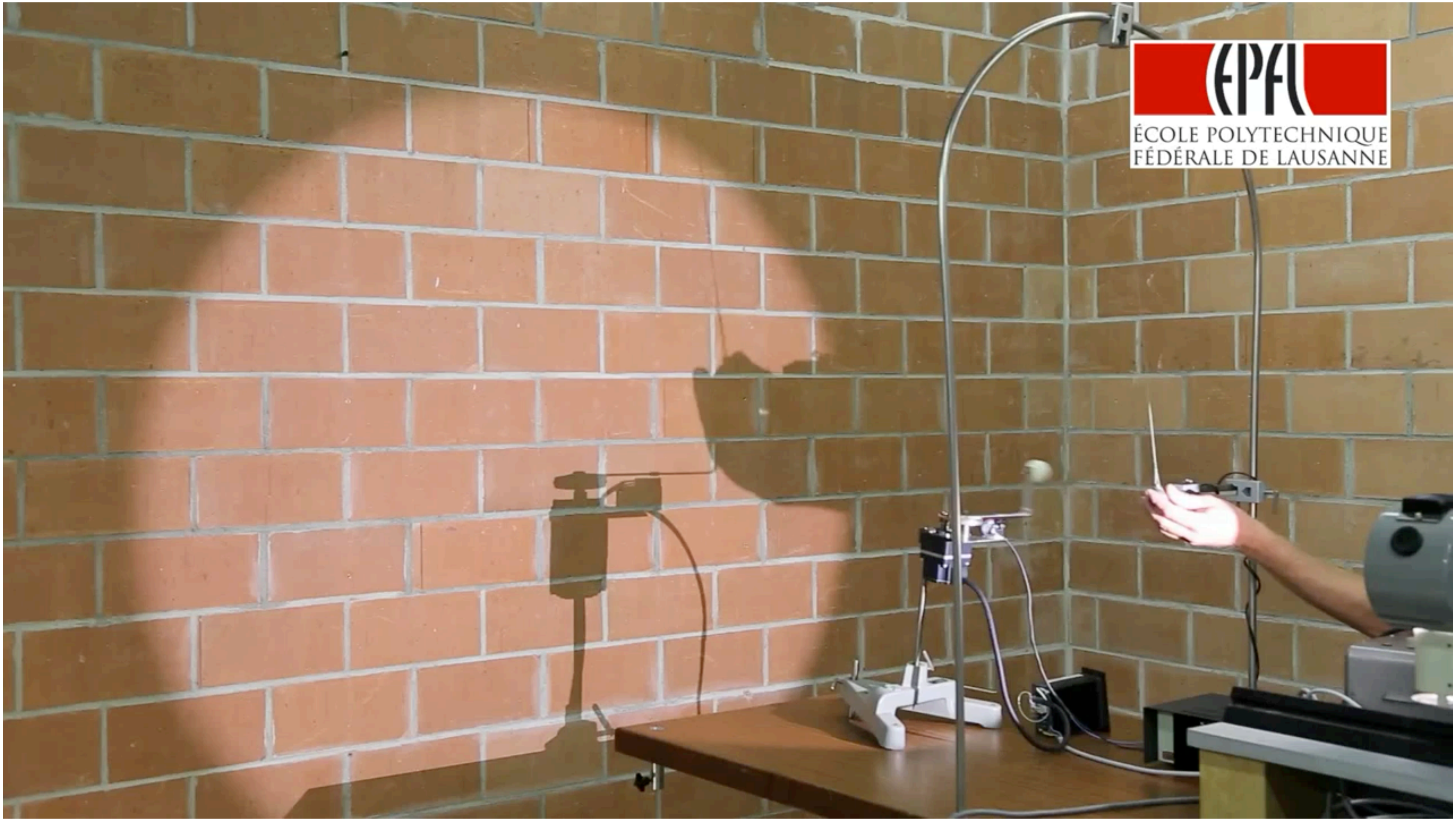
3. **Harmonic motion**
   - Simple harmonic motion

DEMO (190): Harmonic motion is like 1D circular motion



Harmonic motion

- Special type of periodic motion caused by forces of the form

$$\vec{F} = -k\,\Delta\vec{r} \qquad \Delta\vec{r} = \vec{r} - \vec{r}_0$$

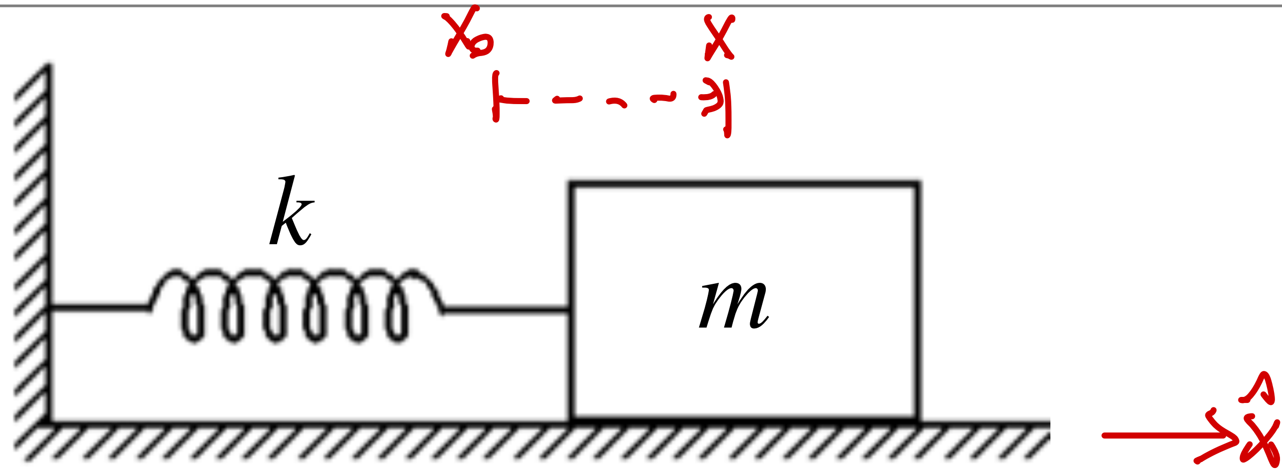
Harmonic motion

- Special type of periodic motion caused by forces of the form

$$\overrightarrow{F} = -\,k\,\Delta\vec{r}$$

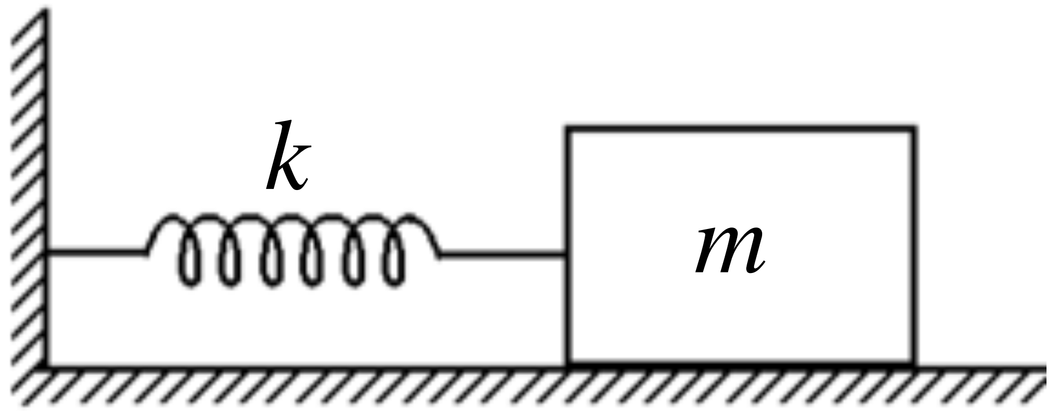
- This is has the same form as the *spring force*, which can represent many systems
  - e.g. atoms in crystals, pendulums, balls rolling in bowls

“Simple” harmonic oscillation in 1D



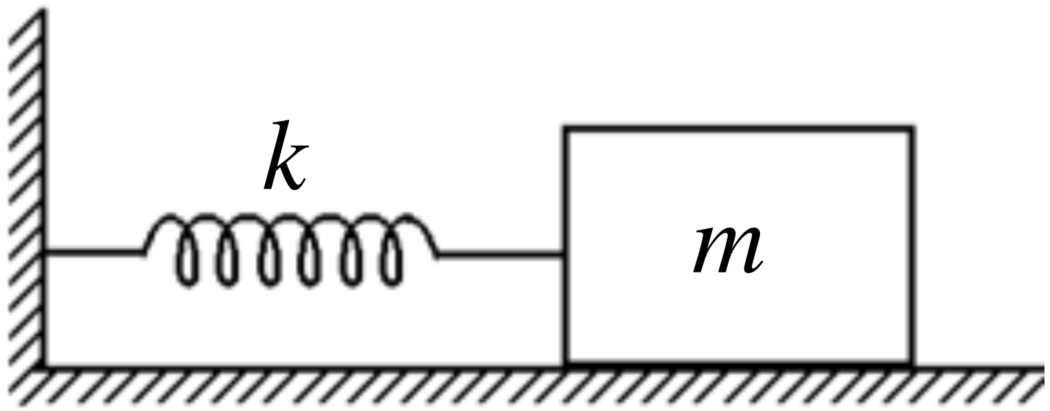
- Consider a frictionless mass-spring system $F = -K\Delta x = -K(x - x_0)$

“Simple” harmonic oscillation in 1D



- Consider a frictionless mass-spring system
- The spring has equilibrium position $x_0 = 0$ $\quad F = -K\Delta x = -K(x - \overset{0}{\cancel{x_0}}) = -Kx$

“Simple” harmonic oscillation in 1D



- Consider a frictionless mass-spring system
- The spring has equilibrium position $x_0 = 0$, so

$$F = -kx$$

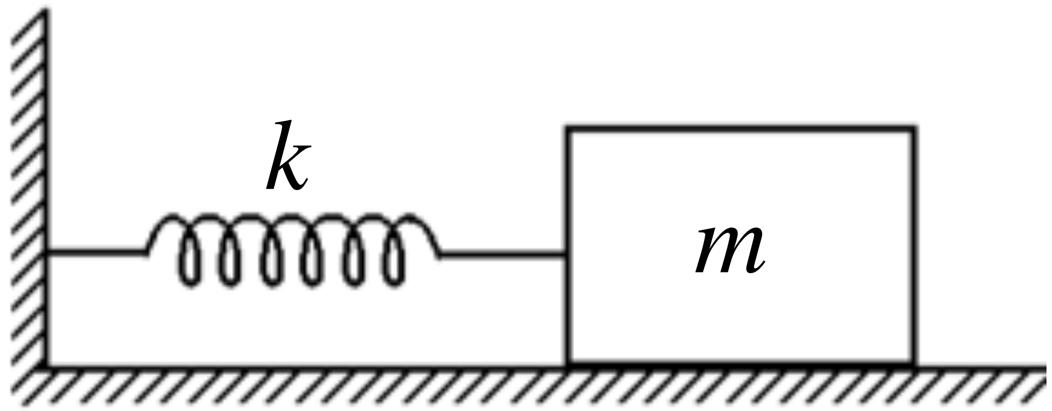
- Newtons 2nd law, $F = ma$, then yields

$$-kx = ma = m\frac{d^2x}{dt^2} \qquad a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\frac{K}{m}x$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{K}{m}x = 0$$

“Simple” harmonic oscillation in 1D



- Consider a frictionless mass-spring system
- The spring has equilibrium position $x_0 = 0$, so

$$F = -kx$$

- Newtons 2nd law, $F = ma$, then yields

$$-kx = ma$$

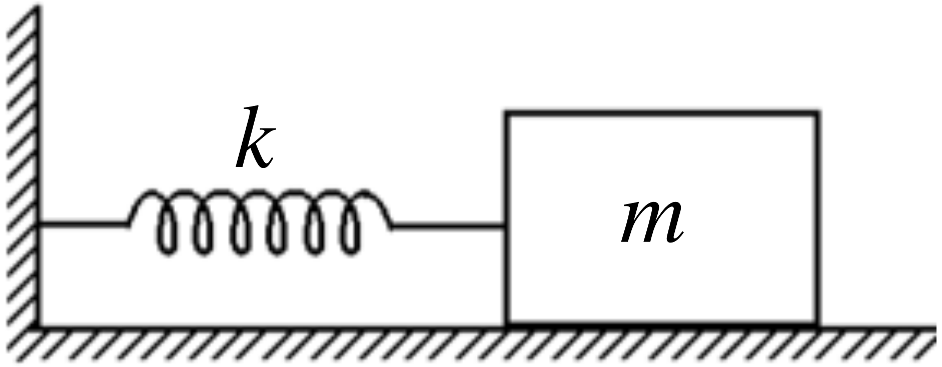
which leads to the differential equation

$$\boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0}$$

“Simple” harmonic oscillation in 1D

- The solution of

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$



is $x(t) = A_0 \cos(\omega_0 t) + B_0 \sin(\omega_0 t)$

"Simple" harmonic oscillation in 1D

- The solution of

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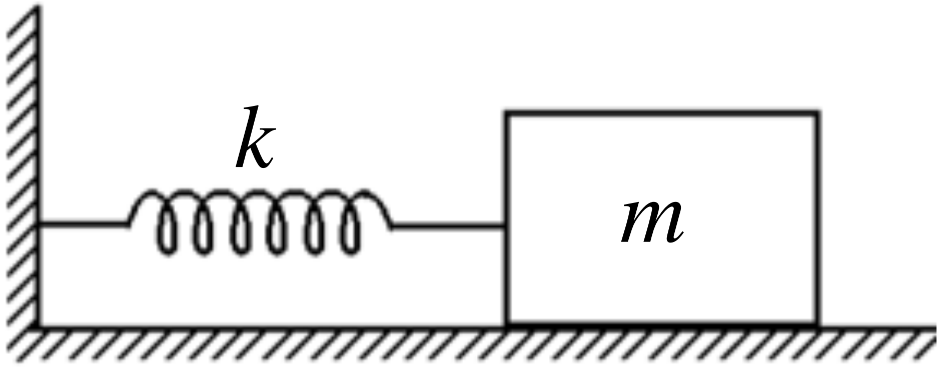


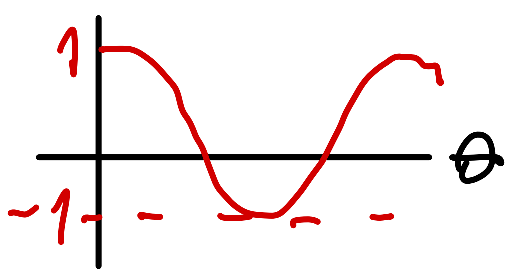
is $x(t) = A_0 \cos(\omega_0 t) + B_0 \sin(\omega_0 t)$

- Using trigonometric identities, this is equivalent to

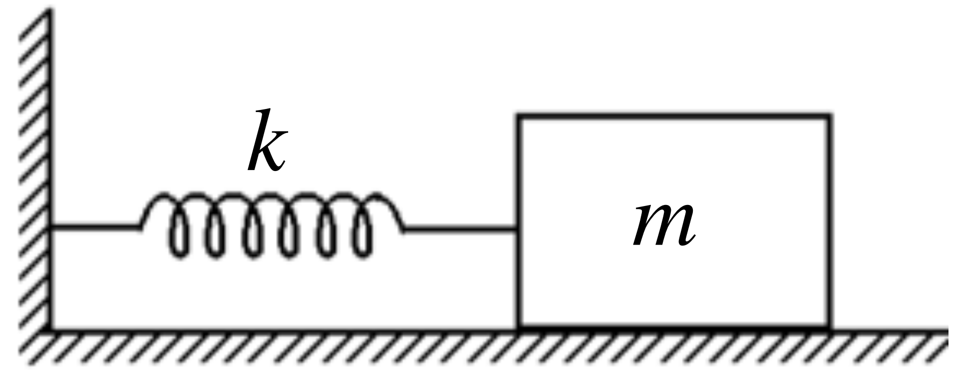
$$x(t) = A \cos(\omega_0 t + \varphi)$$

where

  - A is the amplitude of the oscillation

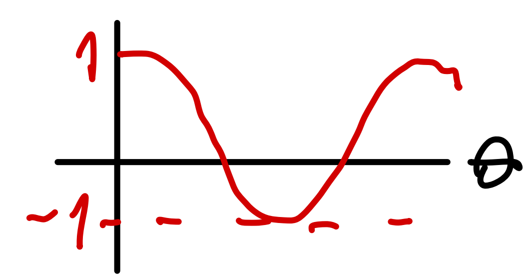


“Simple” harmonic oscillation in 1D

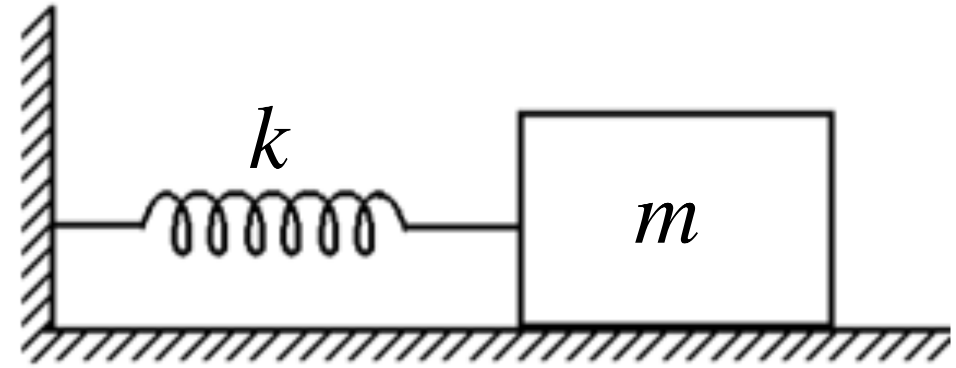


- The solution of
$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$
is $\quad x(t) = A_0 \cos(\omega_0 t) + B_0 \sin(\omega_0 t)$

- Using trigonometric identities, this is equivalent to
$$x(t) = A\cos(\omega_0 t + \varphi)$$
where
  - A is the amplitude of the oscillation
  - φ is the initial phase



“Simple” harmonic oscillation in 1D



- The solution of

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

  is $x(t) = A_0 \cos(\omega_0 t) + B_0 \sin(\omega_0 t)$

- Using trigonometric identities, this is equivalent to

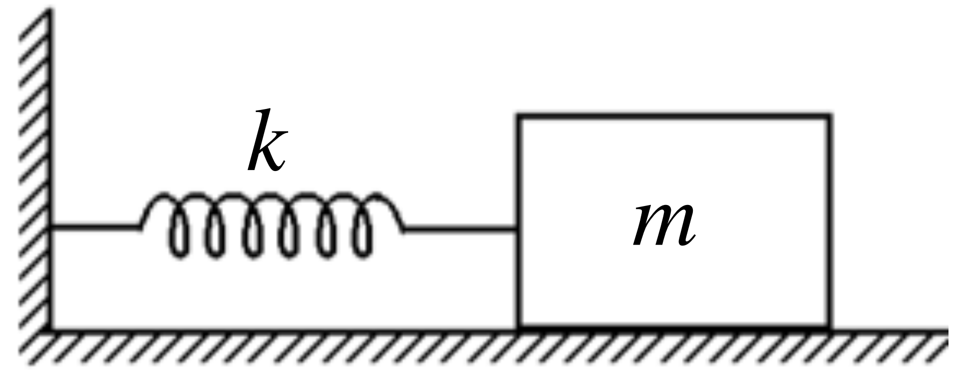
$$x(t) = A \cos(\omega_0 t + \varphi)$$

  where

  - A is the amplitude of the oscillation
  - φ is the initial phase
  - $\omega_0 = \sqrt{k/m} = 2\pi/T_0$ where T_0 is the period of the oscillation. ω_0 is called the *angular frequency.*

$\rightarrow f = \frac{1}{T_0}$

"Simple" harmonic oscillation in 1D



- The solution of

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

is $x(t) = A_0\cos(\omega_0 t) + B_0\sin(\omega_0 t)$

- Using trigonometric identities, this is equivalent to

$$x(t) = A\cos(\omega_0 t + \varphi)$$

$$v(t) = \dot{x}(t) = \frac{dx}{dt} = \frac{d}{dt}\left[A\cos(\omega_0 t+\varphi)\right] = A\frac{d}{dt}\left[\cos(\omega_0 t+\varphi)\right] = A\left[-\sin(\omega_0 t+\varphi)\right]\omega_0$$

$$= -A\omega_0\sin(\omega_0 t+\varphi)$$

$$\frac{d}{dt}(\omega_0 t+\varphi) = \omega_0$$

$$a(t) = \ddot{x}(t) = \frac{d^2x}{dt^2} = -A\omega_0\frac{d}{dt}\left[\sin(\omega_0 t+\varphi)\right] = -A\omega_0\cos(\omega_0 t+\varphi)\,\omega_0$$

$$= -A\omega_0^2\cos(\omega_0 t+\varphi) = -\omega_0^2 x$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = -\omega_0^2 x + \frac{k}{m}x = \underline{\left(-\omega_0^2 + \frac{k}{m}\right)}x = 0$$

$$\boxed{\omega_0^2 = \frac{k}{m}}$$

Kinetic energy of oscillation

- Kinetic energy is still $K = \dfrac{m}{2}v^2$

$$K = \frac{1}{2}m\left[-A\omega_0 \sin(\omega_0 t+\varphi)\right]^2 = \frac{1}{2}mA^2\omega_0^2\sin^2(\omega t+\varphi)$$

$$= \frac{1}{2}\cancel{m}A^2\frac{k}{\cancel{m}}\sin^2(\omega_0 t+\varphi) = \frac{1}{2}kA^2\sin^2(\omega t+\varphi)$$

$$\sin^2(\theta)+\cos^2(\theta)=1$$

$$= \frac{1}{2}kA^2\left[1-\cos^2(\omega_0 t+\varphi)\right]$$

$$= \frac{1}{2}kA^2 - \frac{1}{2}kA^2\cos^2(\omega_0 t+\varphi)$$

x^2

$$= \boxed{\frac{1}{2}kA^2 - \frac{1}{2}kx^2}$$

Total energy of oscillation

- Potential energy is still

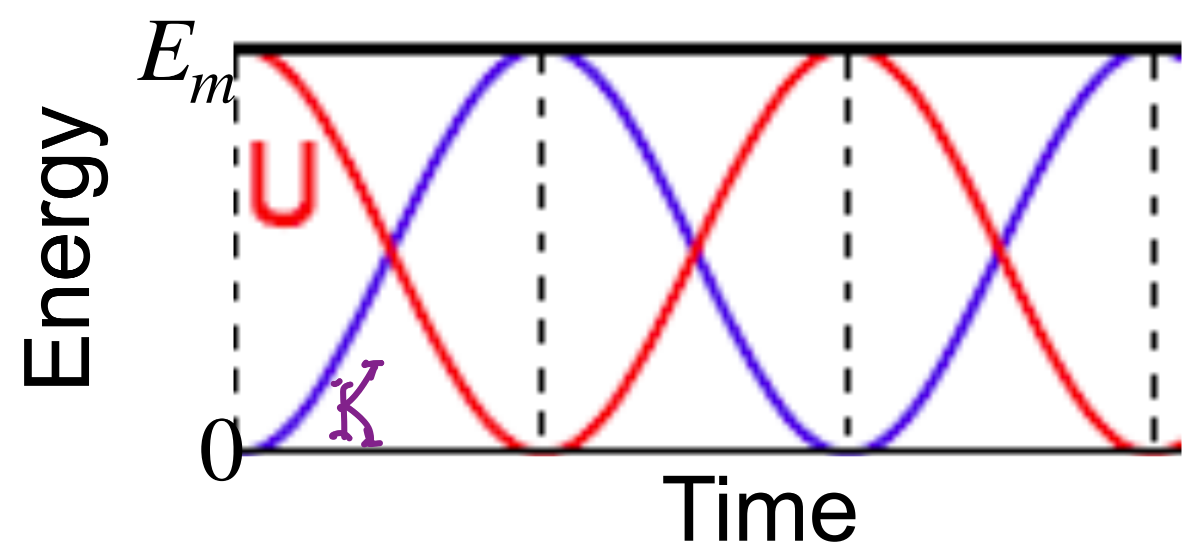
$$U = \frac{k}{2} x^2$$

Total energy of oscillation

- Potential energy is still

$$U = \frac{k}{2}x^2$$

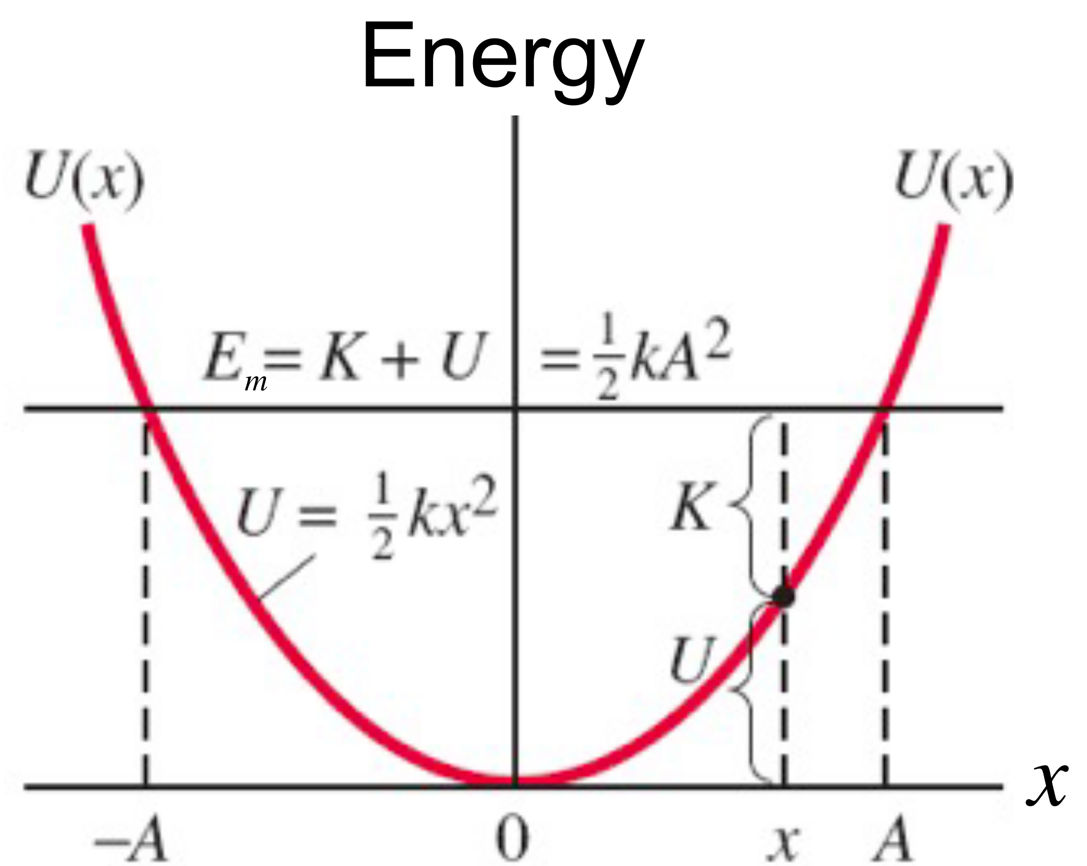
- Total mechanical energy is $E_m = K + U$



$E_m = K + U =$

$= \frac{1}{2}kA^2 - \cancel{\frac{1}{2}kx^2} + \cancel{\frac{1}{2}kx^2}$

$= \boxed{\frac{1}{2}kA^2}$

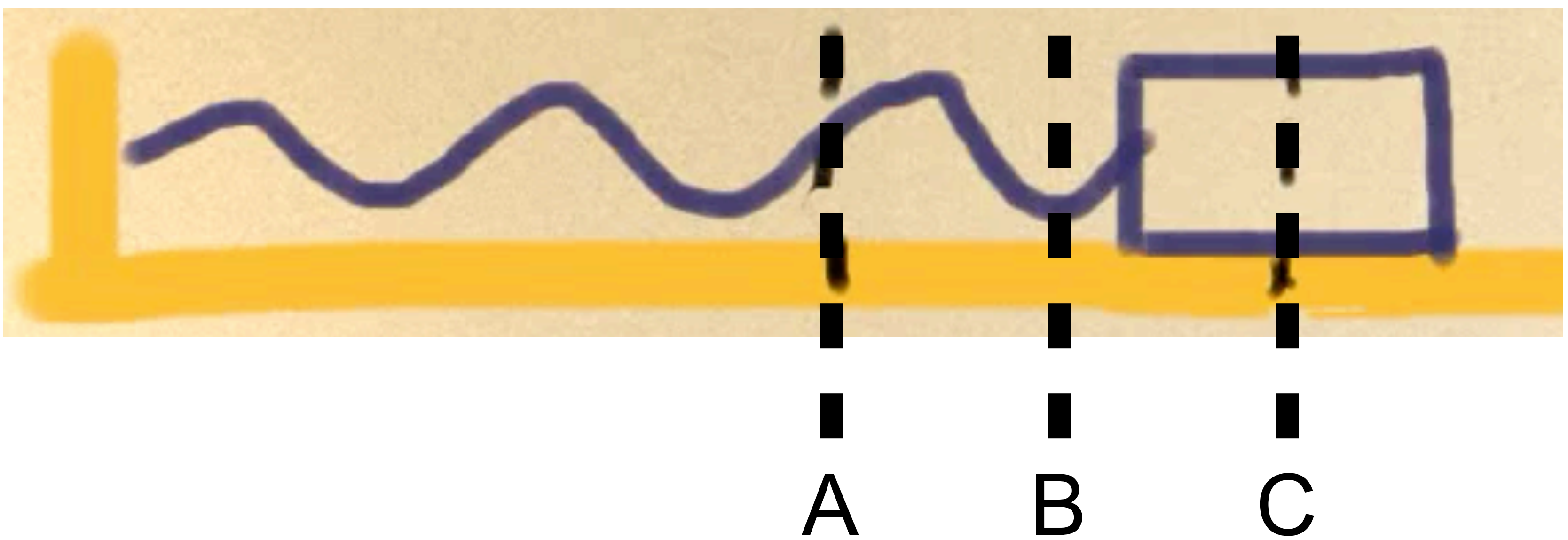


Conceptual question

A mass is oscillating back and forth on a spring about point A as shown. Point A is the equilibrium (unstretched) position of the mass. At which position is the magnitude of its acceleration the largest?

$a(t) = -\omega_0^2 x \Rightarrow |a(t)| = |-\omega_0^2 x(t)| = \omega_0^2 |x(t)|$

A. Point A
B. Point B
C. Point C



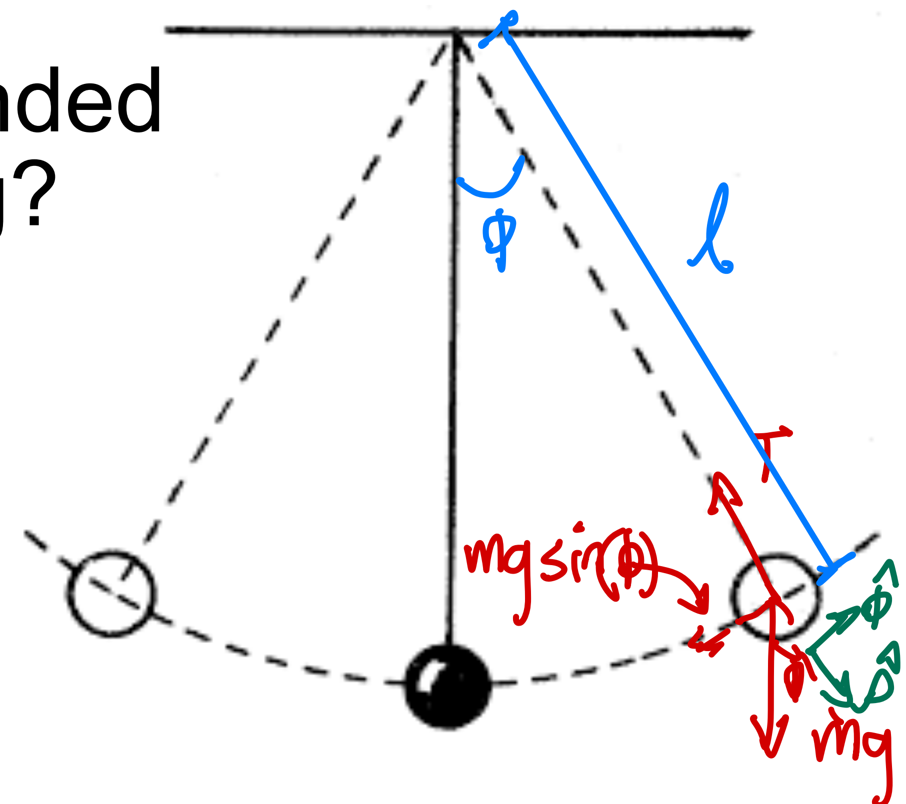
Conceptual question

An object can execute harmonic motion (i.e. oscillate) about…

A. any point.
B. any equilibrium point.
C. any stable equilibrium point.
D. any point, provided the forces exerted on the object obey Hooke's law.

Aside: Oscillation of a simple pendulum

- What is the motion of a mass suspended from a weightless, inextensible string?

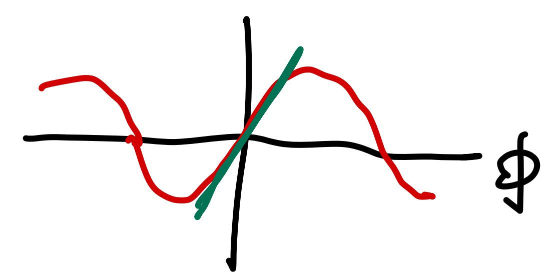


$\sum F_\phi : -\cancel{m}g\sin(\phi) = \cancel{m}a_T$

$$a_T = \ell\alpha = \ell\frac{d\omega}{dt} = \ell\frac{d}{dt}\left(\frac{d\phi}{dt}\right) = \ell\frac{d^2\phi}{dt^2}$$

$$\Rightarrow -g\sin(\phi) = \ell\frac{d^2\phi}{dt^2} \Rightarrow \frac{d^2\phi}{dt^2} + \frac{g}{\ell}\sin(\phi) = 0$$

If $\phi \ll 1$, then $\sin(\phi) \approx \phi$ $\left(\sin(\phi) = \phi - \frac{1}{6}\phi^3 + \dots\right)$



In that case, $\boxed{\frac{d^2\phi}{dt^2} + \underbrace{\frac{g}{\ell}}\phi = 0}$

$$\omega_0^2 = \frac{g}{\ell} \Rightarrow \frac{2\pi}{T_0} = \sqrt{\frac{g}{\ell}}$$

$$\Rightarrow T_0 = 2\pi\sqrt{\frac{\ell}{g}}$$

DEMO (286)

Mass and frequency of a pendulum

See you tomorrow!