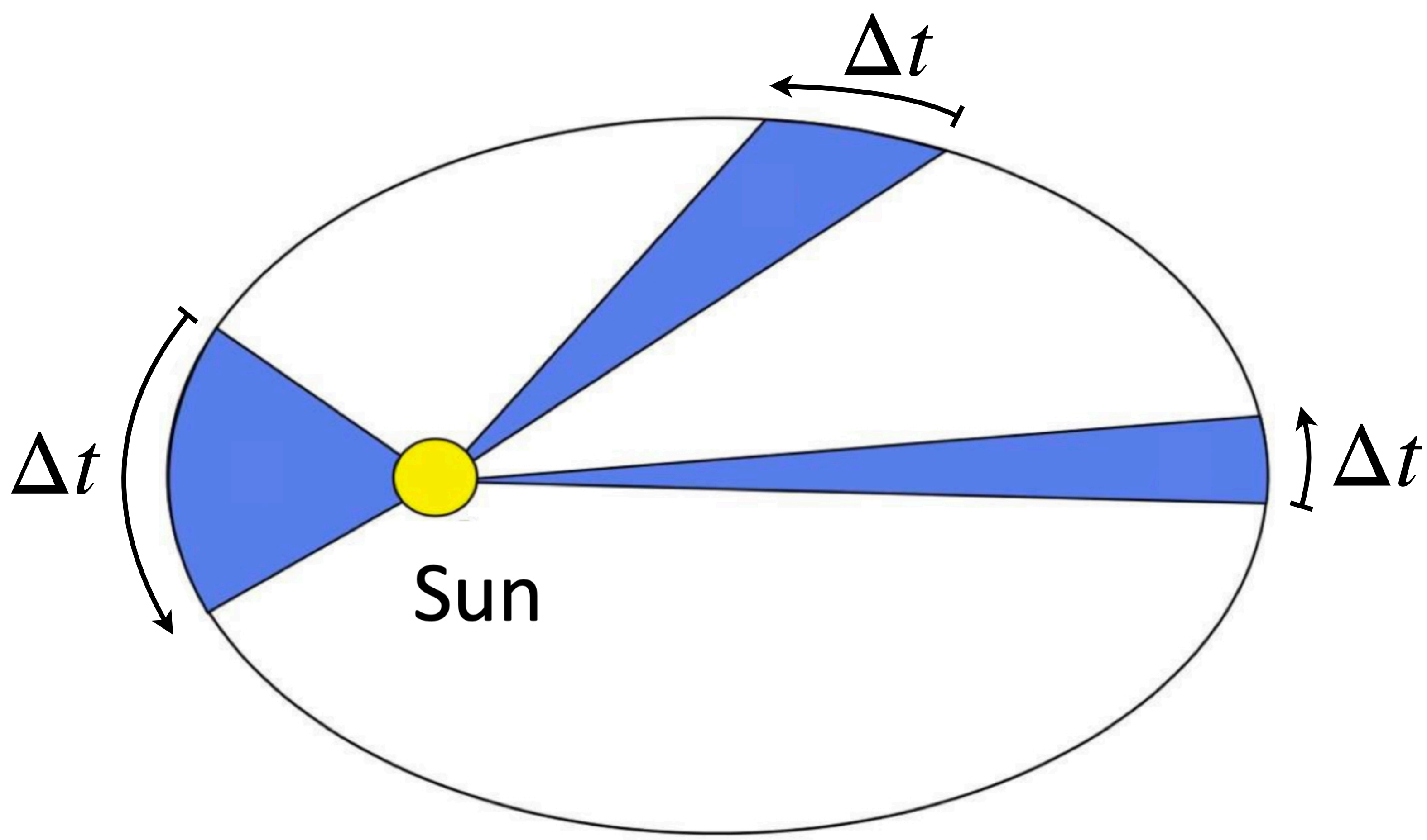
General Physics: Mechanics

PHYS-101(en)

Lecture 12a:
Angular momentum



Dr. Marcelo Baquero
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December 2nd, 2024

Announcements

- We'll hold another ***mock exam*** tomorrow
  - You can bring a "cheat" sheet containing formulas or all of your notes, as you wish
  - Turn in at the end if you want exam to be graded (optional). Graded exams will be returned on Monday December 16th
  - Solutions will be published on the Moodle
  - Does **<u>not</u>** matter at all for your final grade

Today's agenda (Serway 11,13, MIT 19)

1. Conclusion of rotation of rigid objects about a fixed axis
   - Work and power for rotation
   - Work-kinetic energy theorem for rotation
   - Angular momentum and its conservation
2. Kepler's laws of planetary motion

Summary of rotation and translation

Rotational motion (about a fixed axis)		Translational motion (in one dimension)	
Angular position	ϕ	Position	x
Angular speed	$\omega = d\phi/dt$	Speed	$v = dx/dt$
Angular acceleration	$\alpha = d\omega/dt$	Acceleration	$a = dv/dt$
Moment of inertia	$I = \int \rho^2 dm$	Mass	m
Net torque	$\Sigma\tau_{ext} = I\alpha$	Net force	$\Sigma F_{ext} = ma$
Rotational kinetic energy	$K^{rot} = I\omega^2/2$	Translational kinetic energy	$K^{trans} = mv^2/2$
Work	?	Work	$W = \int_{x_a}^{x_b} F dx$
Power	?	Power	$P = Fv$
Angular momentum	?	Momentum	$p = mv$
Net torque	?	Net force	$\Sigma F_{ext} = dp/dt$

Conceptual question

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You are using a wrench and trying to loosen a rusty nut. Which of the arrangements shown is most effective in loosening the nut?

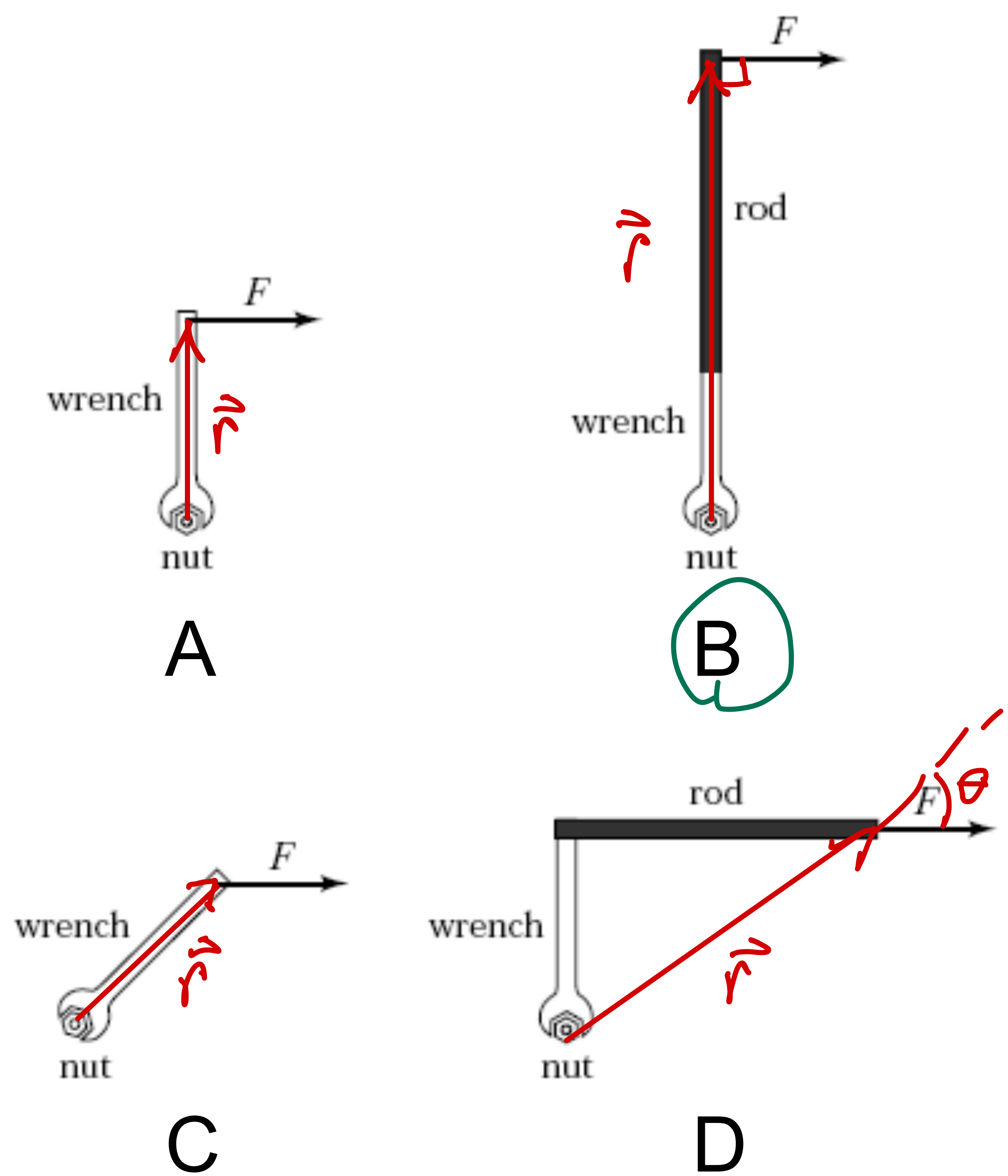
$\vec{\tau} = \vec{r} \times \vec{F}$

$|\vec{\tau}| = |\vec{r} \times \vec{F}| = |\vec{r}||\vec{F}|\sin(\theta)$

Large $|\vec{r}|$

~~Large $|\vec{F}|$~~

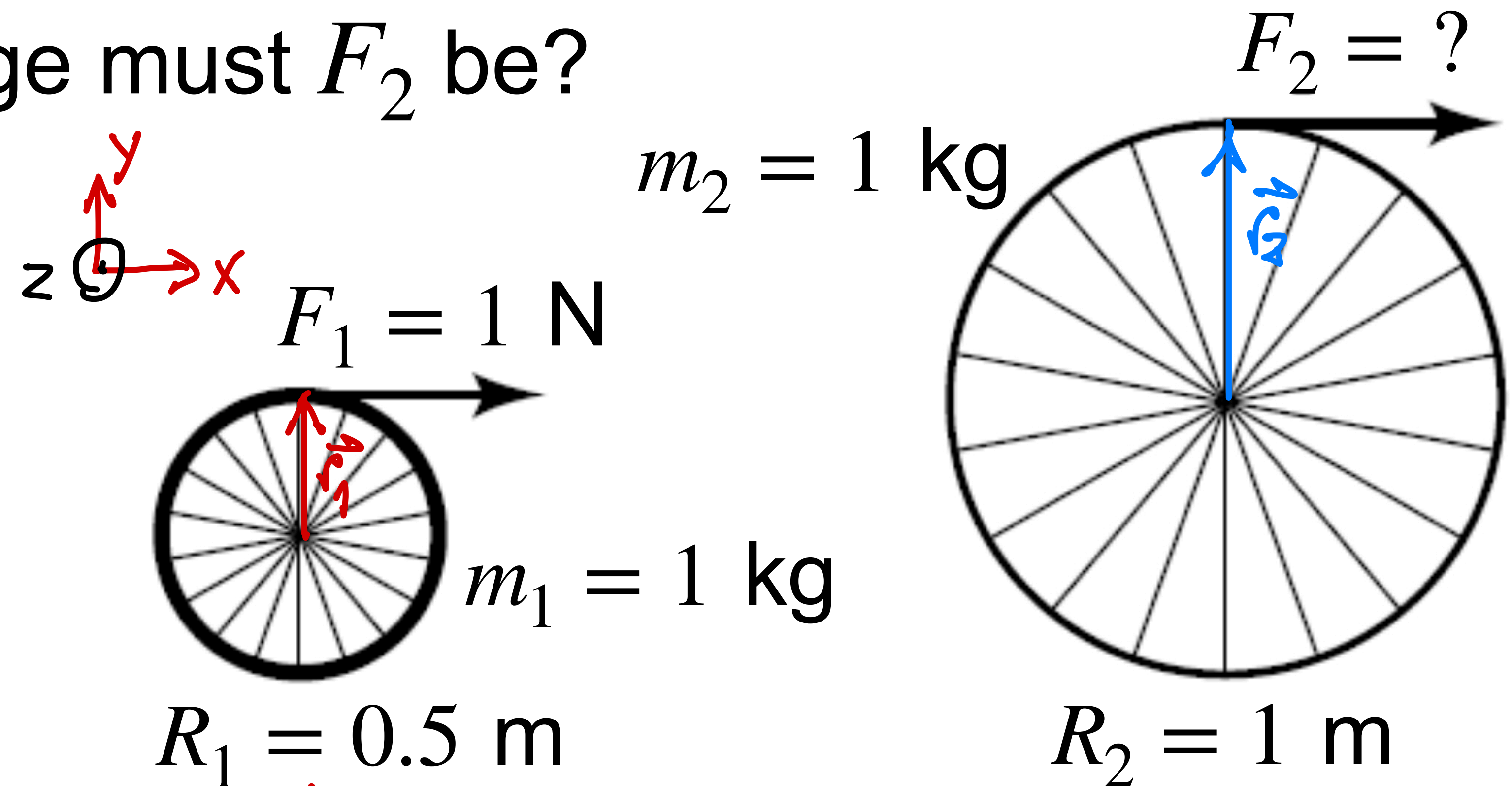
Large $\sin(\theta)$ $(\theta \approx \frac{\pi}{2})$



Conceptual question

Two wheels with fixed hubs, each having a mass of 1 kg, start from rest, and forces are applied as shown. Assume the hubs and spokes are massless, so that the moment of inertia is $I = mR^2$. In order to impart identical angular accelerations, how large must F_2 be?

A. 0.25 N
B. 0.5 N
C. 1.0 N
D. 2.0 N
E. 4.0 N



$\vec{\tau}_1 = \vec{r}_1 \times \vec{F}_1 = (R_1\hat{y}) \times (F_1\hat{x}) = R_1F_1\,\hat{y}\times\hat{x} = R_1F_1(-\hat{z}) = -R_1F_1\hat{z} = I_1\vec{\alpha}_1 = I_1\alpha_1(-\hat{z})$

$\Rightarrow R_1F_1 = I_1\alpha_1 \Rightarrow \alpha_1 = \frac{R_1F_1}{I_1} = \frac{R_1F_1}{m_1R_1^2} = \frac{F_1}{m_1R_1}$

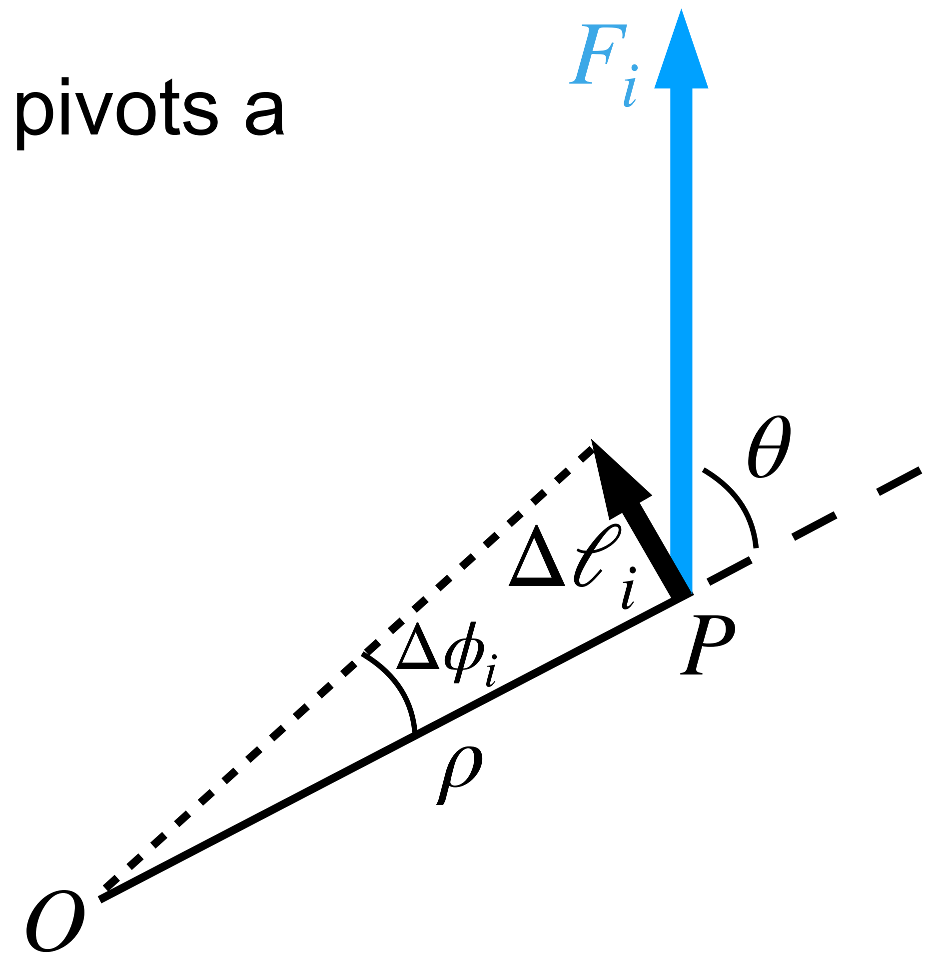
$\vec{\tau}_2 = \vec{r}_2 \times \vec{F}_2 = I_2\vec{\alpha}_2 \Rightarrow \alpha_2 = \frac{F_2}{m_2R_2}$

$\alpha_1 = \alpha_2 \Rightarrow \frac{F_1}{m_1R_1} = \frac{F_2}{m_2R_2}$

$F_2 = \frac{\cancel{m_2}R_2}{\cancel{m_1}R_1}F_1$

Work and power for rotation

- A force $\vec{F}_i$ is applied at a point P, which pivots a small distance $\Delta \ell_i$ about point O



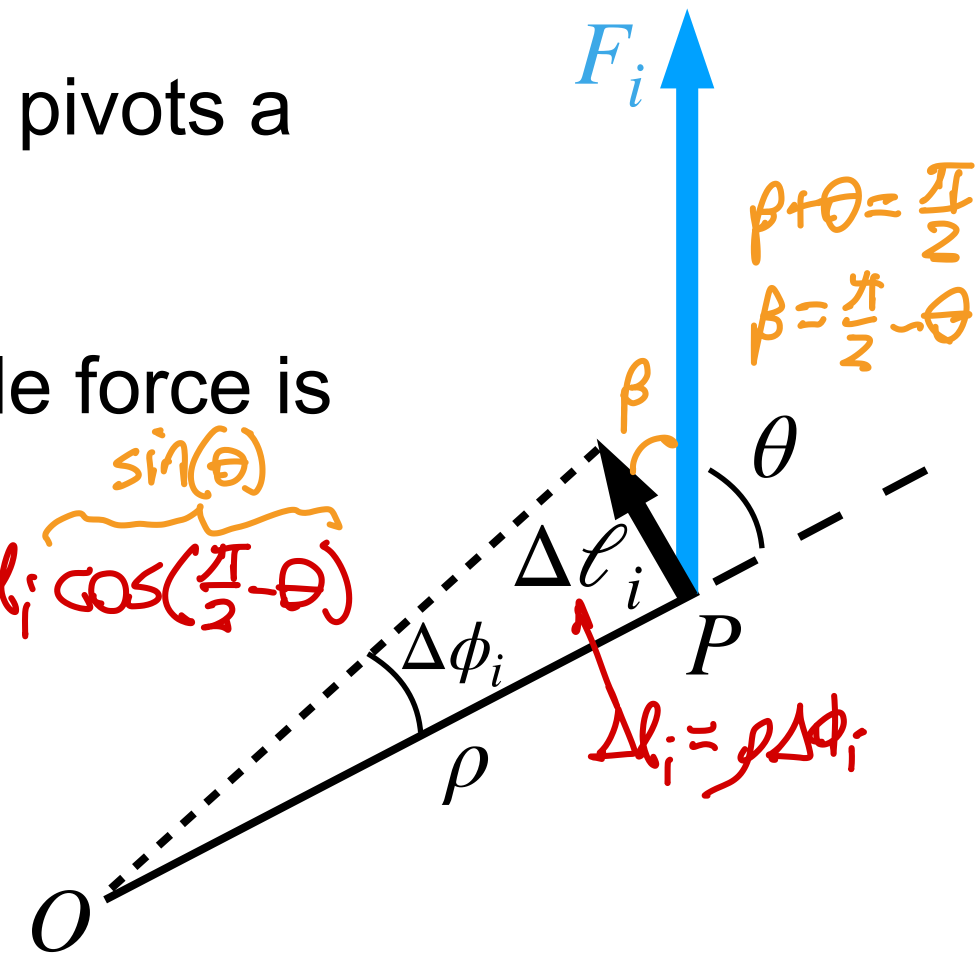
Work and power for rotation

- A force $\vec{F}_i$ is applied at a point P, which pivots a small distance $\Delta \ell_i$ about point O

- As in week 8, the work done by a variable force is

$$\Delta W_i = \vec{F}_i \cdot \Delta \vec{\ell}_i = F_i \Delta \ell_i \cos(\beta) = F_i \Delta \ell_i \overbrace{\cos\left(\frac{\pi}{2} - \theta\right)}^{\sin(\theta)}$$

$$= F_i \Delta \ell_i \sin(\theta)$$



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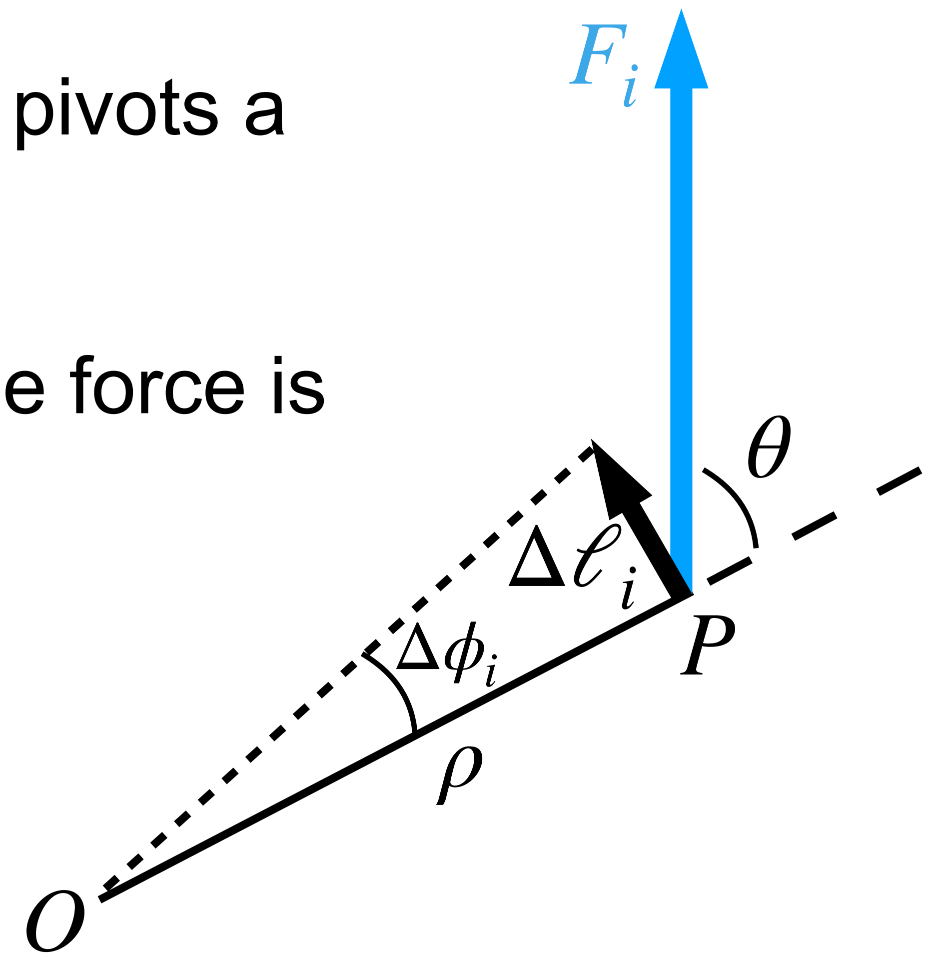
$$= \underbrace{F_i \sin\theta\,(\rho}_{r_i}\Delta\phi_i) = \tau_i\,\Delta\phi_i$$



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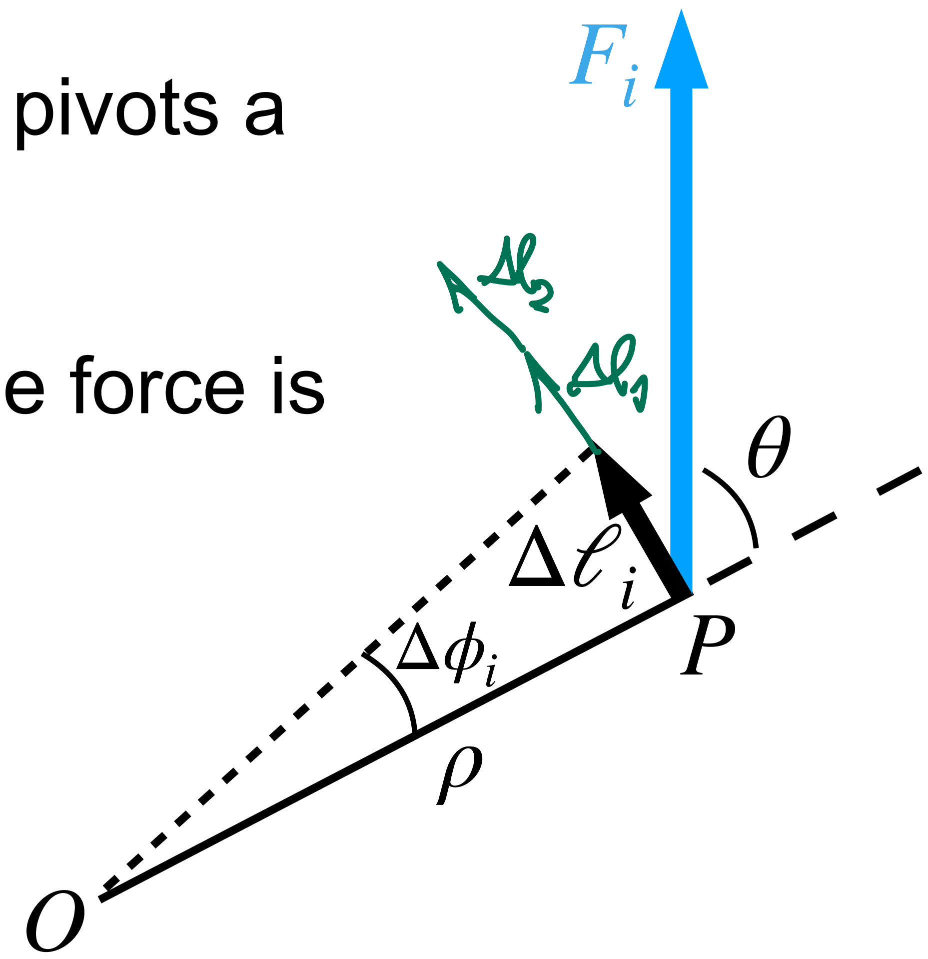

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$$= F_i \sin\theta \left(\rho\Delta\phi_i\right) = \tau_i \,\Delta\phi_i$$

- Total work is the sum over differential changes in angle

$$W = \lim_{\Delta\phi_i \to 0} \sum_i \tau_i \,\Delta\phi_i = \int_{\phi_a}^{\phi_b} \tau \, d\phi = \int_{t_a}^{t_b} \tau \overbrace{\frac{d\phi}{dt}}^{\omega} dt = \int_{t_a}^{t_b} \tau\omega \, dt$$

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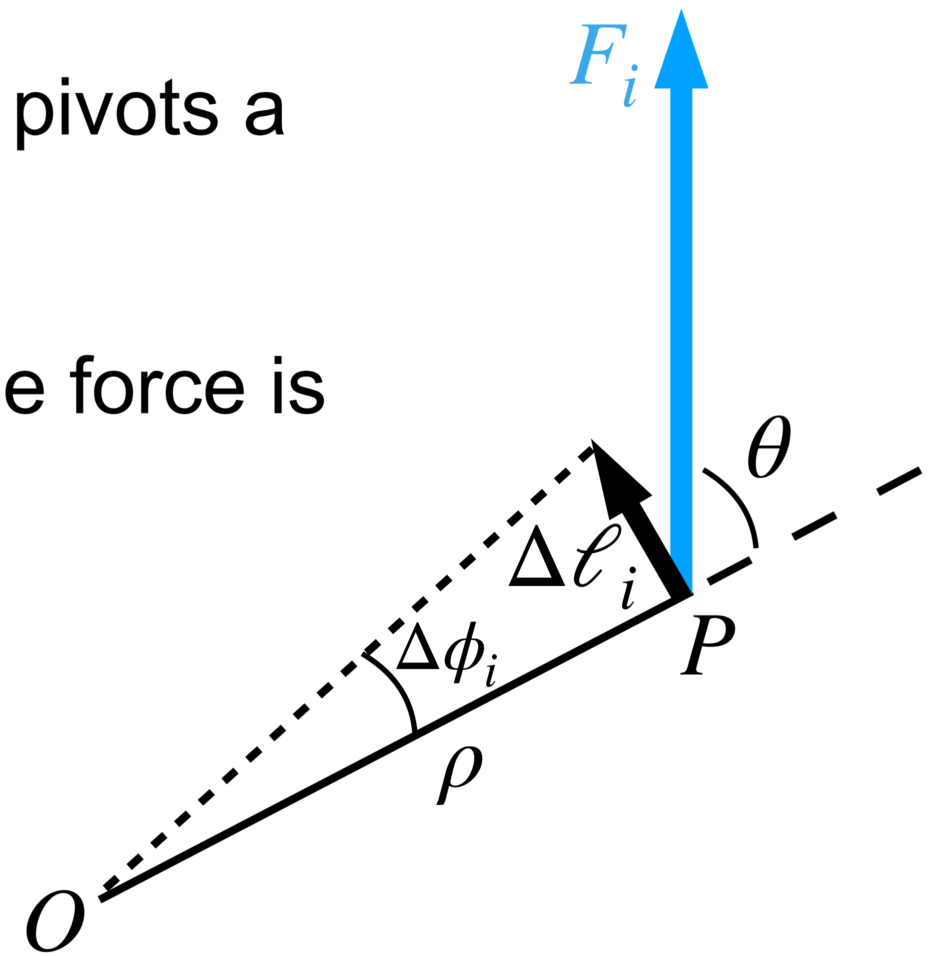

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- Total work is the sum over differential changes in angle

$$W = \lim_{\Delta\phi_i \to 0} \sum_i \tau_i \Delta\phi_i = \int_{\phi_a}^{\phi_b} \tau \, d\phi$$

- Thus, the power is $P = \dfrac{dW}{dt} = \boxed{\tau\,\omega}$

Work-kinetic energy theorem for rotation

- Reminder: This tells us how the kinetic energy of an object changes due to the work performed on it

$$K^{rot} = \frac{1}{2} I \omega^2$$

$$\frac{dK^{rot}}{dt} = \frac{d}{dt}\left[\frac{1}{2} I \omega^2\right] = \frac{1}{2} I \frac{d}{dt}(\omega^2) = \frac{1}{2} I \left(2\omega \overbrace{\frac{d\omega}{dt}}^{=\alpha}\right) = \omega \overbrace{I\alpha}^{\tau_{net}}$$

$$= \omega \tau_{net} \qquad \omega = \frac{d\phi}{dt}$$

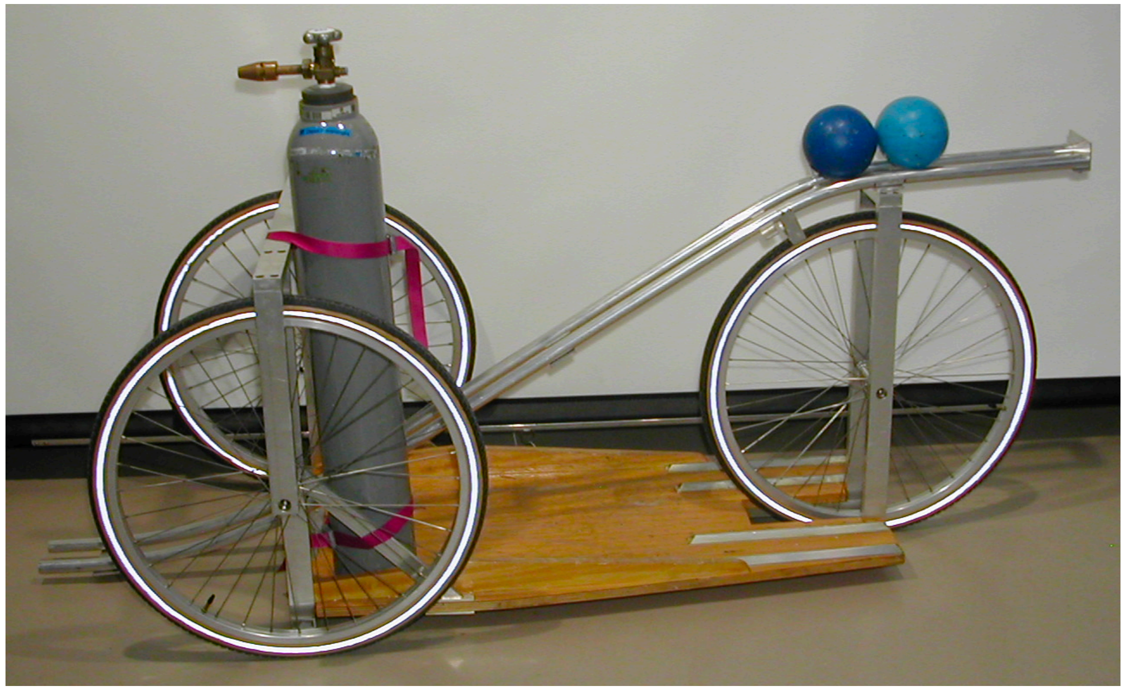
$$= \frac{d\phi}{dt} \tau_{net}$$

$$\underbrace{\int_{t_i}^{t_f} \frac{dK^{rot}}{dt} dt}_{K^{rot}(t_f) - K^{rot}(t_i) = \Delta K^{rot}} = \int_{t_i}^{t_f} \tau_{net} \frac{d\phi}{dt} dt = \int_{\phi_i}^{\phi_f} \tau_{net} \, d\phi = W^{rot}$$

$$\boxed{\Delta K^{rot} = W^{rot}}$$

DEMO (15)

Action-reaction **disk**



Angular momentum

- As you may suspect, **angular momentum** $\overrightarrow{L}$ is the rotational analogue of *linear* momentum $\vec{p} = m\vec{v}$ (lecture 6)

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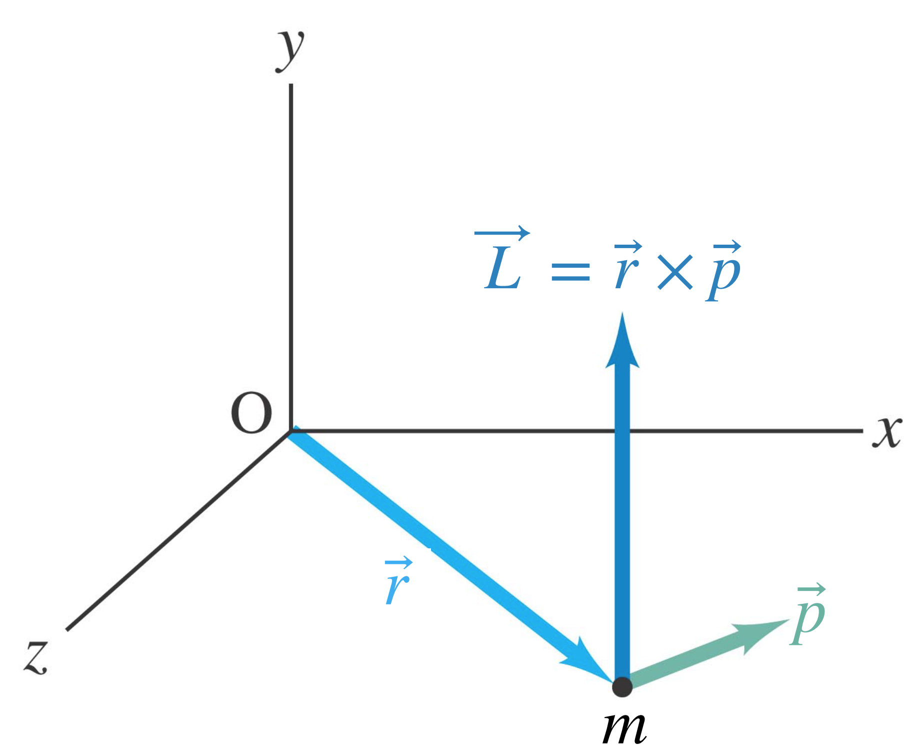
- It is the *moment* of momentum, defined as

$$\overrightarrow{L} = \vec{r} \times \vec{p}$$

where $\vec{r}$ is the position vector from a pivot point to the object



Angular momentum

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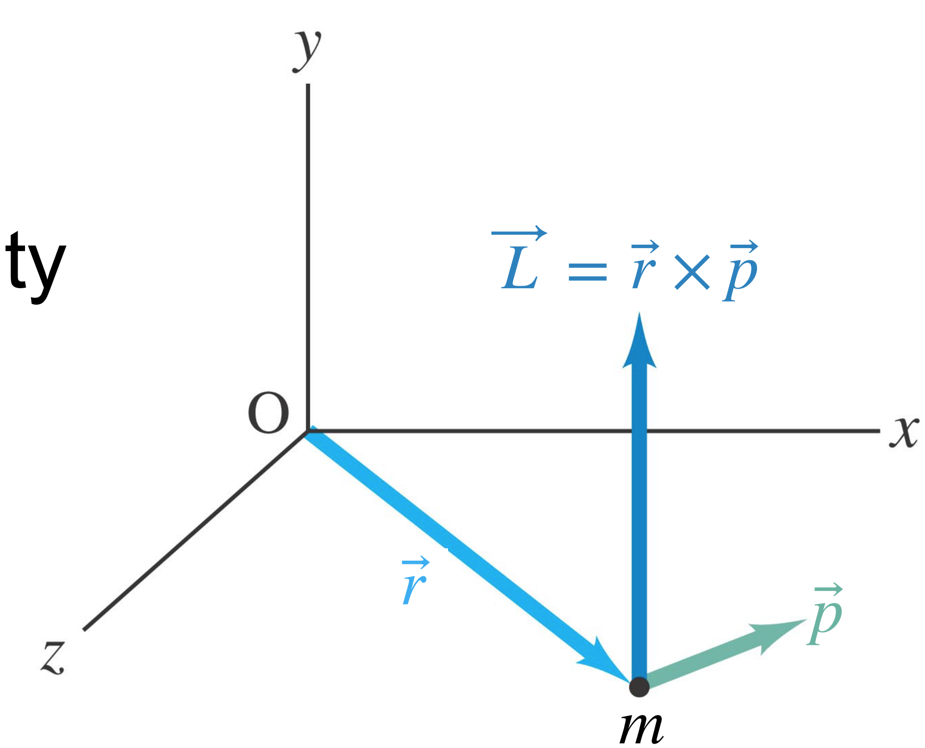
- It has units of [kg·m^2/s]

- Like linear momentum, it is a vector quantity and will be conserved under certain conditions



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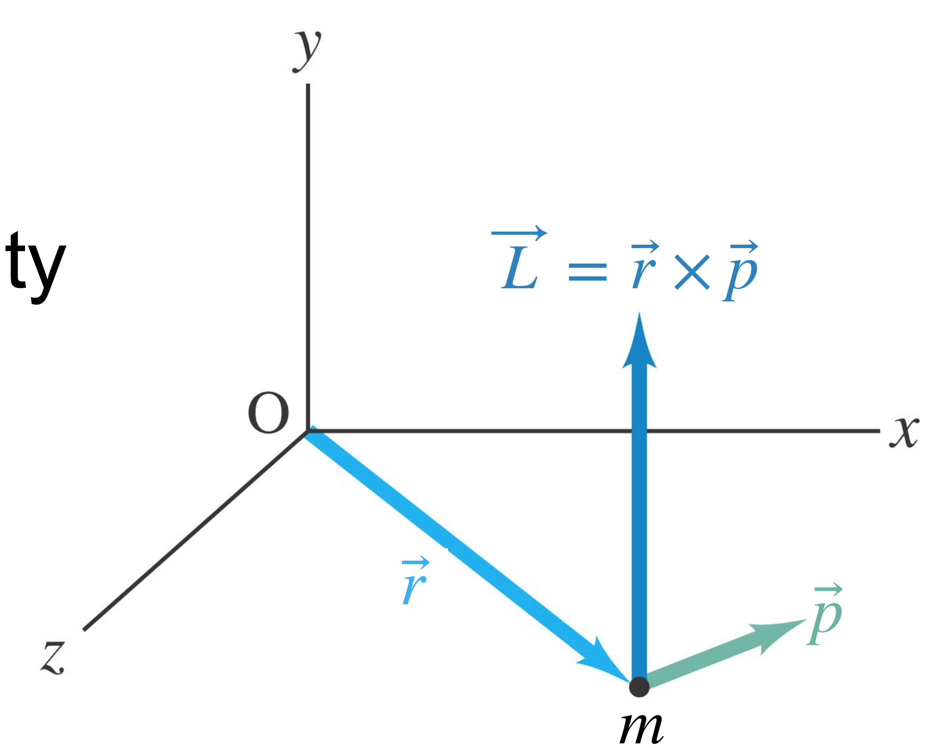
- It has units of [kg·m^2/s]

- Like linear momentum, it is a vector quantity and will be conserved under certain conditions

- Unlike linear momentum, it depends on where the pivot is chosen



Conceptual question

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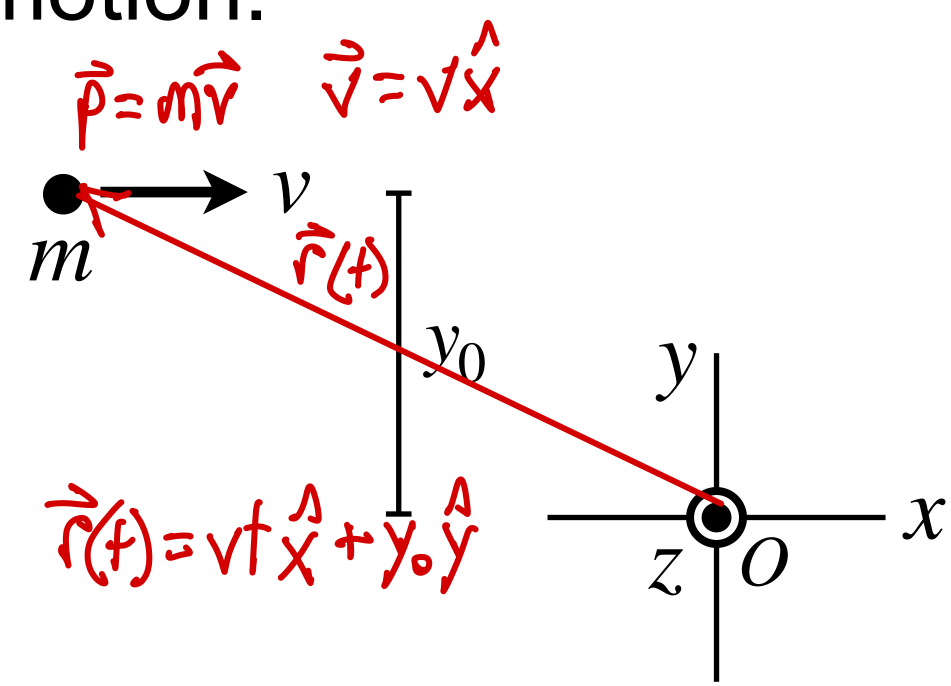
A particle is moving in the x-y plane with a constant velocity and constant height y_0 (as shown below). The magnitude of the angular momentum L_O about the origin…

A. is zero because this is not circular motion.
B. decreases, then increases.
C. increases, then decreases.
D. is constant.



$$\vec{L}_0 = \vec{r} \times \vec{p}$$
$$= (vt\hat{x} + y_0\hat{y}) \times (mv\hat{x})$$
$$= mv^2t\,\cancel{\hat{x}\times\hat{x}}^{\,0} + mvy_0\,\underbrace{\hat{y}\times\hat{x}}_{-\hat{z}}$$
$$= -mvy_0\hat{z}$$
$$|\vec{L}_0| = mvy_0$$

Conceptual question

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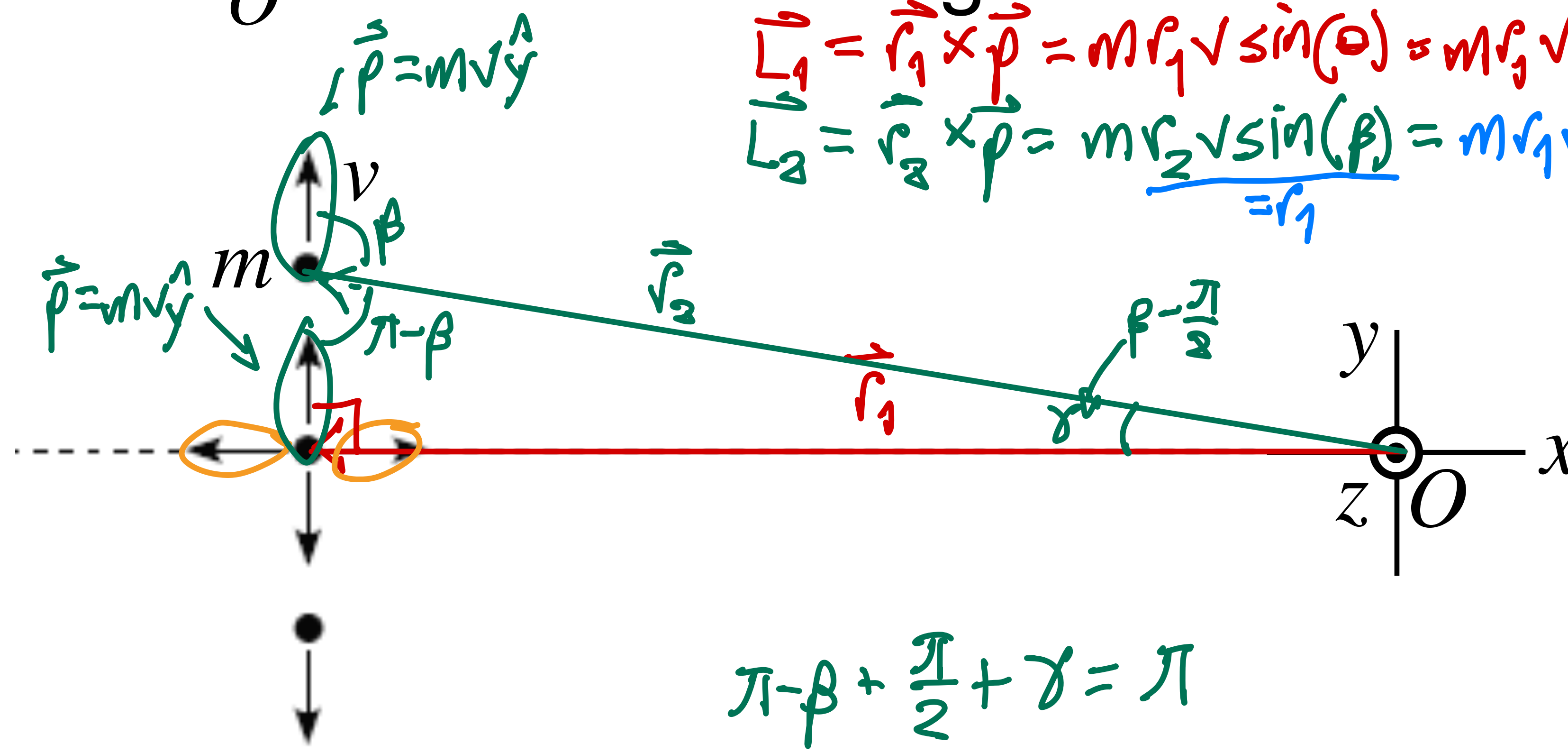
The diagram shows six possible combinations of position and velocity for a particle of mass m and speed v moving in the x-y plane. How many **distinct** values of the angular momentum vector $\vec{L}_O$ relative to the origin does this represent?

A. 1
B. 2
C. 3
D. 4
E. 5
F. 6

$\vec{L}_1 = \vec{r}_1 \times \vec{p} = m r_1 v \sin(\theta) = m r_1 v$

$\vec{L}_2 = \vec{r}_2 \times \vec{p} = m \underbrace{r_2 v \sin(\beta)}_{= r_1} = m r_1 v$



$\pi - \beta + \frac{\pi}{2} + \gamma = \pi$

$\gamma = \beta - \frac{\pi}{2}$

$r_1 = r_2 \cos(\gamma) = r_2 \cos(\beta - \frac{\pi}{2}) = r_2 \sin(\beta)$

Example: Point particle angular momentum

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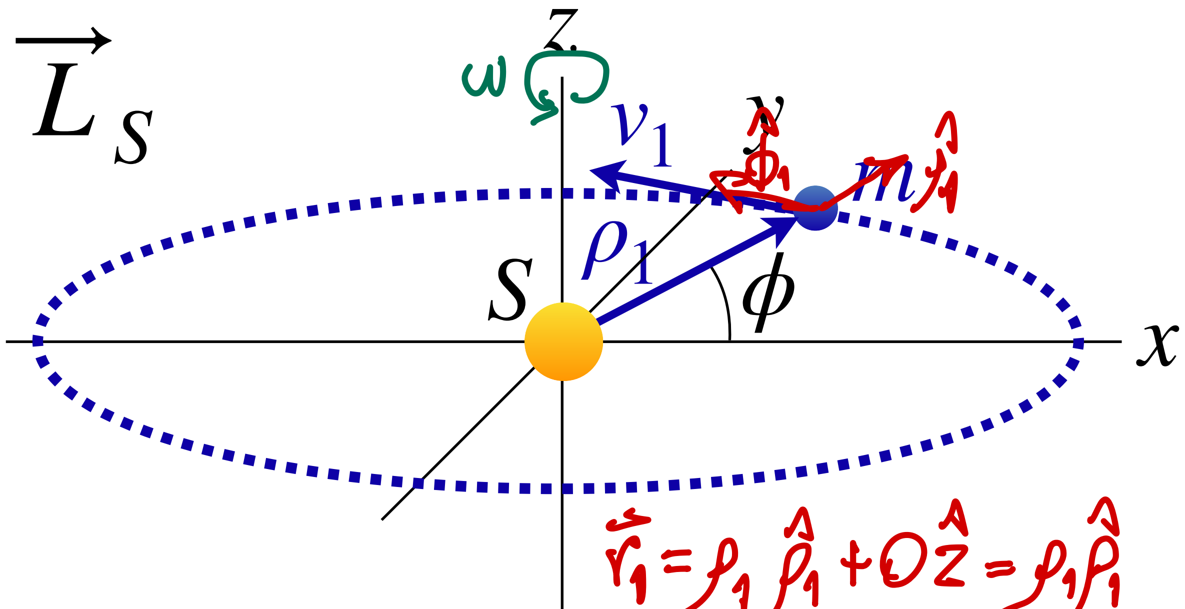
A point of mass m (lets say a planet) is executing uniform circular motion with $\vec{\omega}$ around point S (lets say a star).

A. What is its angular momentum $\vec{L}_S$ about S?



$$\vec{r}_1 = \rho_1 \hat{\rho}_1 + 0\hat{z} = \rho_1 \hat{\rho}_1$$

$$\vec{v}_1 = v_1 \hat{\phi}_1$$

$$\vec{L}_S = \vec{r}_1 \times \vec{p}_1 = \vec{r}_1 \times (m\vec{v}_1) = m\,\vec{r}_1 \times \vec{v}_1$$

$$= m(\rho_1 \hat{\rho}_1) \times (v_1 \hat{\phi}_1)$$

$$= m\rho_1 v_1 (\hat{\rho}_1 \times \hat{\phi}_1) = m\rho_1 v_1 \hat{z}$$

$$= m\rho_1(\rho_1 \omega)\hat{z}$$

$$= \underbrace{m\rho_1^2}_{I_0} \underbrace{\omega\hat{z}}_{\vec{\omega}}$$

$$= \boxed{I_0 \vec{\omega}}$$

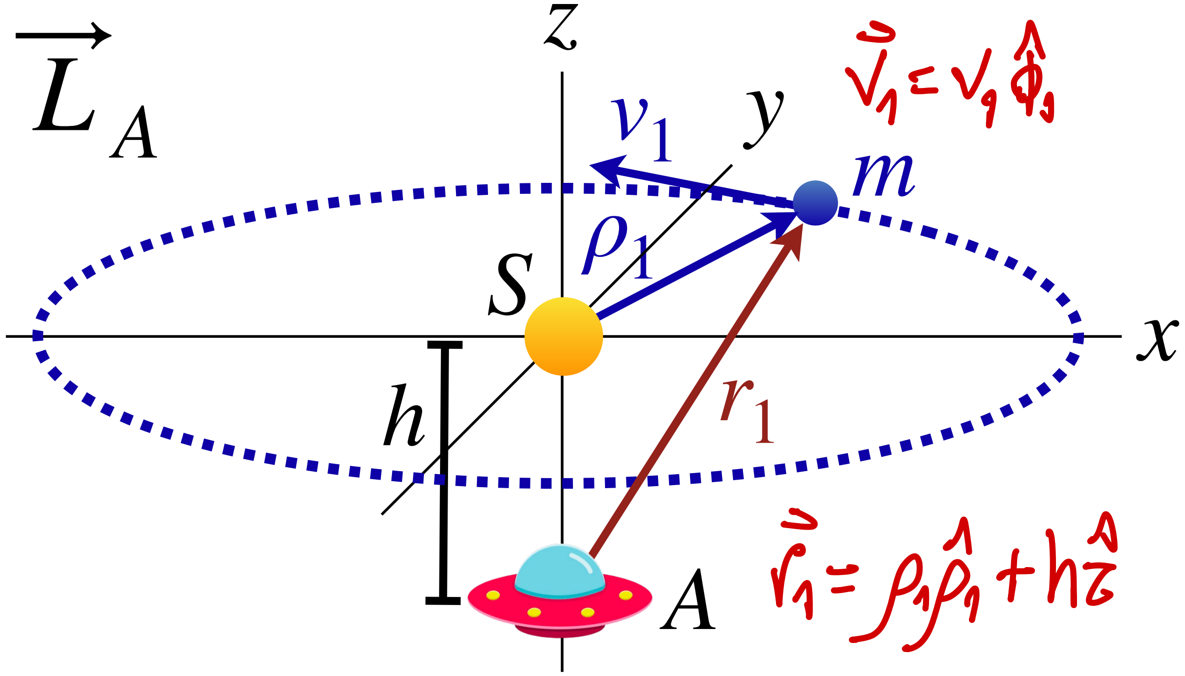
$$v_1 = \rho_1 \omega$$

$$I_0 = \sum_i m_i \rho_i^2 = m\rho_1^2 = I_0$$

Example: Point particle angular momentum

A point of mass m (lets say a planet) is executing uniform circular motion with $\vec{\omega}$ around point S (lets say a star).

B. What is its angular momentum $\vec{L}_A$ about a lower point A?



$$\vec{L}_A = \vec{r}_1 \times \vec{p} = m\,\vec{r}_1 \times \vec{v}_1$$

$$= m(\rho_1 \hat{\rho}_1 + h\hat{z}) \times (v_1 \hat{\phi}_1)$$

$$= \underbrace{m(\rho_1 \hat{\rho}_1) \times (v_1 \hat{\phi}_1)}_{= \vec{L}_S} + m h v_1\, \hat{z} \times \hat{\phi}_1$$

$$= \vec{L}_S + m h v_1 (-\hat{\rho}_1)$$

$$= \boxed{\vec{L}_S - m h \rho_1 \omega \hat{\rho}_1}$$

$\hat{\rho}_1 = \hat{\rho}_1(t)$

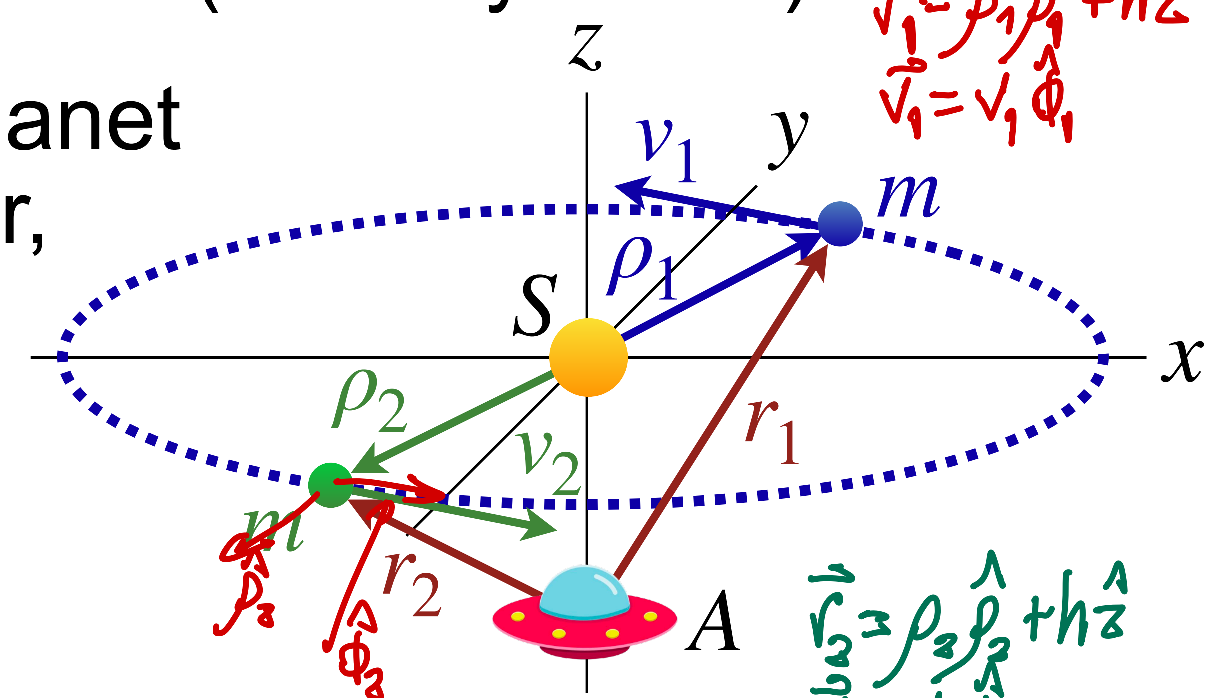
Example: Point particle angular momentum

A point of mass m (lets say a planet) is executing uniform circular motion with $\vec{\omega}$ around point S (lets say a star).

$\vec{r}_1 = \rho_1 \hat{\rho}_1 + h\hat{z}$

$\vec{v}_1 = v_1 \hat{\phi}_1$

C. If we add a second identical planet on the opposite side of the star, what is $\vec{L}_{sys}$ of the system?



$\vec{r}_2 = \rho_2 \hat{\rho}_2 + h\hat{z}$

$\vec{v}_2 = v_2 \hat{\phi}_2$

We have here:

$\rho_1 = \rho_2 \qquad \hat{\rho}_2 = -\hat{\rho}_1$

$\omega_1 = \omega_2 = \omega$

$v_1 = v_2 \qquad \hat{\phi}_2 = -\hat{\phi}_1$

$$\vec{L}_1 = m\rho_1^2\omega_1\hat{z} - mh\rho_1\omega_1\hat{\rho}_1$$

$$\vec{L}_2 = \vec{r}_2 \times \vec{p}_2 = m\rho_2^2\omega_2\hat{z} - mh\rho_2\omega_2\hat{\rho}_2 = m\rho_1^2\omega_1\hat{z} - mh\rho_1\omega_1(-\hat{\rho}_1)$$

$$\vec{L}_{sys} = \vec{L}_1 + \vec{L}_2 = \underbrace{2m\rho_1^2}_{I_A}\overbrace{\omega_1\hat{z}}^{\vec{\omega}} + 0\cdot\hat{\rho}_1 = I_A\vec{\omega}$$

$$I_A = \sum_i m_i\rho_i^2 = m\rho_1^2 + m\rho_2^2 = 2m\rho_1^2$$

Example: Point particle angular momentum

A point of mass m (lets say a planet) is executing uniform circular motion with $\overrightarrow{\omega}$ around point S (lets say a star).

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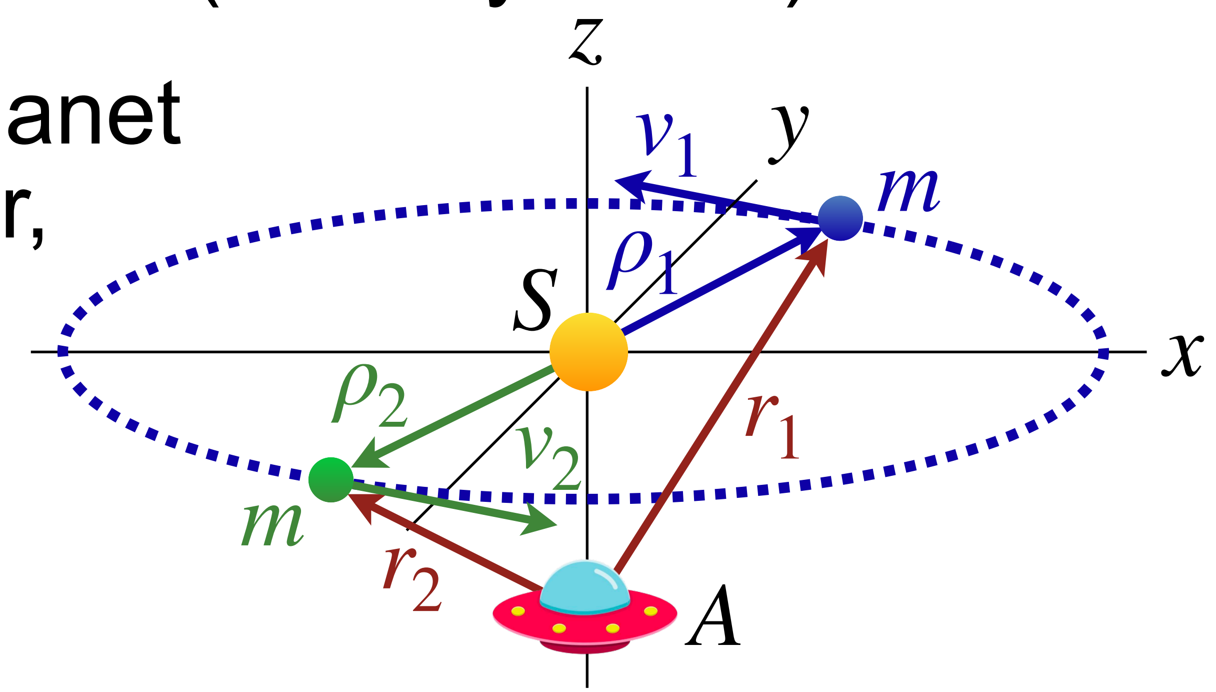

- When $\vec{p}_{sys} = 0$, $\overrightarrow{L}_{sys}$ is independent of the pivot location

Angular momentum of rotating rigid bodies

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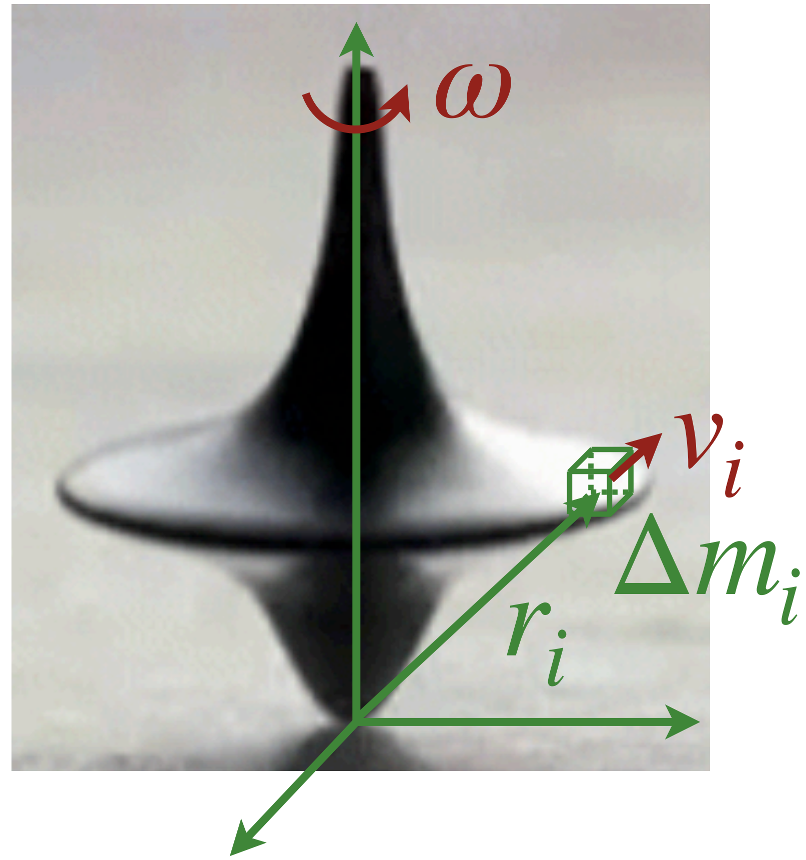
- Imagine the object is composed of many differential elements, labeled $i = 1, 2, 3, \ldots$, with positions $\vec{r}_i$

$$\vec{L} = \sum_i \vec{L}_i$$



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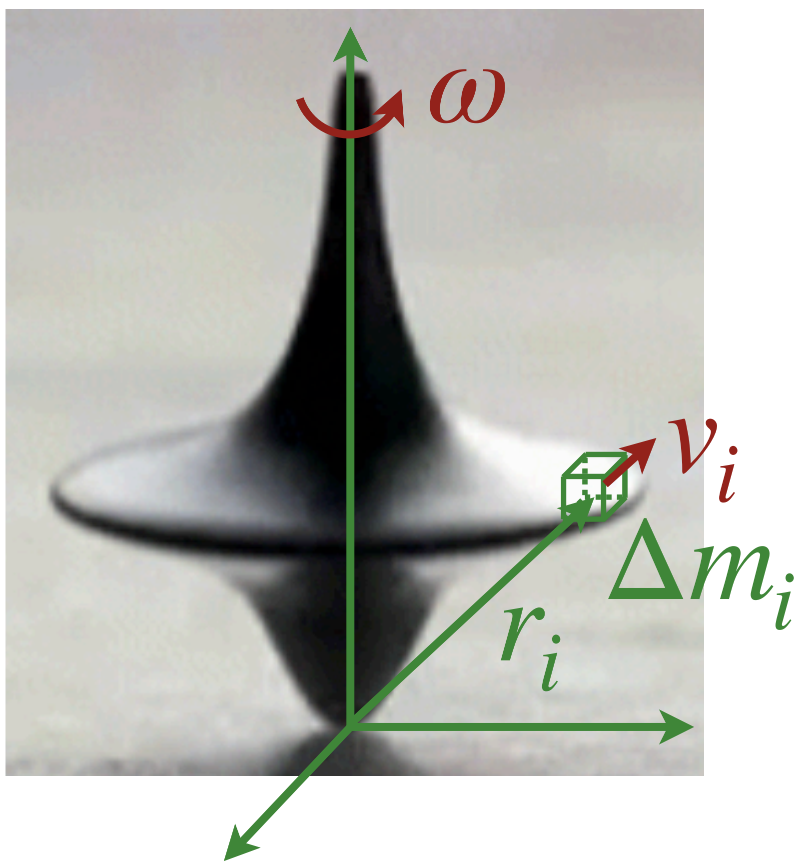
$$\vec{L} = \sum_i \vec{L}_i = \sum_i \vec{r}_i \times \Delta m_i \vec{v}_i$$



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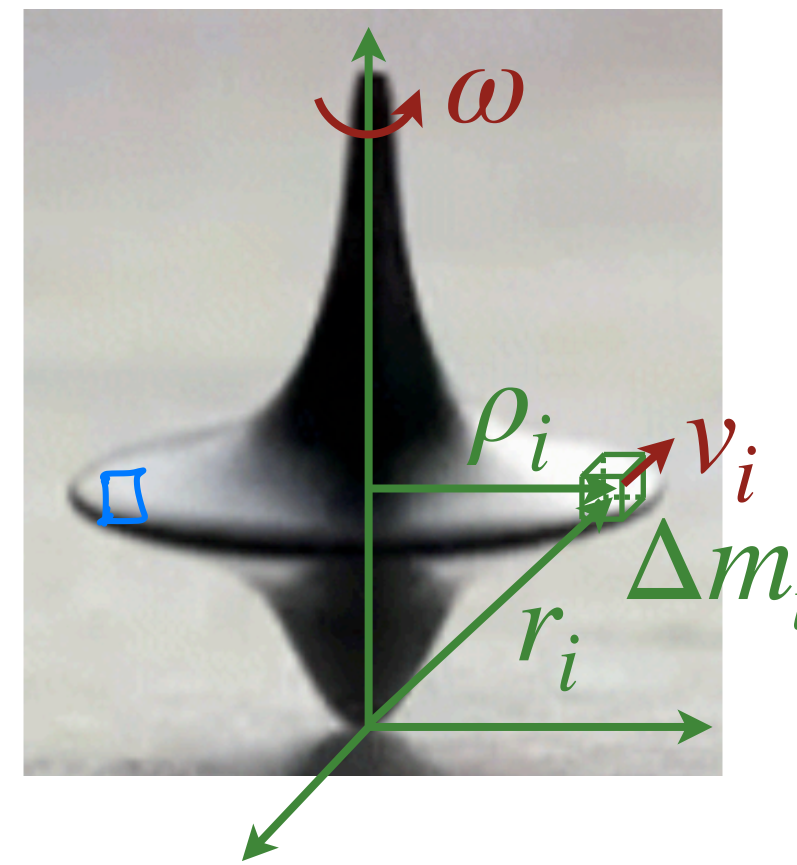
$$\overrightarrow{L} = \sum_i \overrightarrow{L}_i = \sum_i \vec{r}_i \times \Delta m_i \vec{v}_i$$

- For pure rotation of a symmetric object, the $\hat{\rho}$ component of $\overrightarrow{L}$ cancels

$$\vec{L} = \sum_i \Delta m_i (\rho_i \hat{\rho}_i + z_i \hat{z}) \times (v_i \hat{\phi}_i)$$

$$= \sum \Delta m_i (\rho_i \hat{\rho}_i \times v_i \hat{\phi}_i) + \cancel{\sum \Delta m_i (z_i \hat{z} \times v_i \hat{\phi}_i)}$$

$$= \sum \Delta m_i \rho_i v_i \hat{z}$$



Angular momentum of rotating rigid bodies

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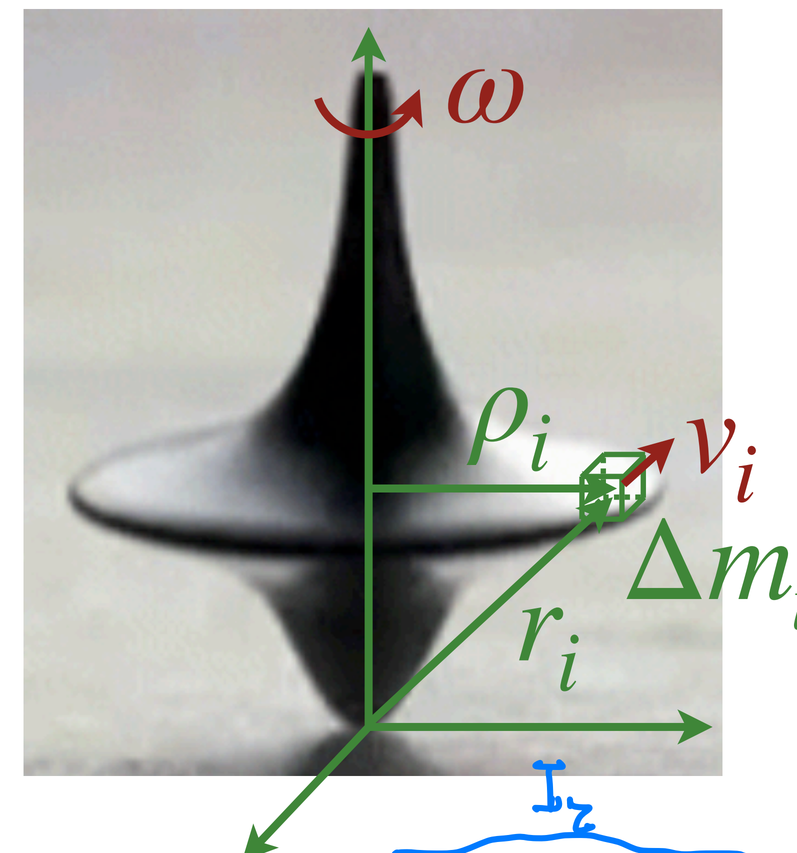


- For pure rotation of a <u>symmetric</u> object, the $\hat{\rho}$ component of $\overrightarrow{L}$ cancels

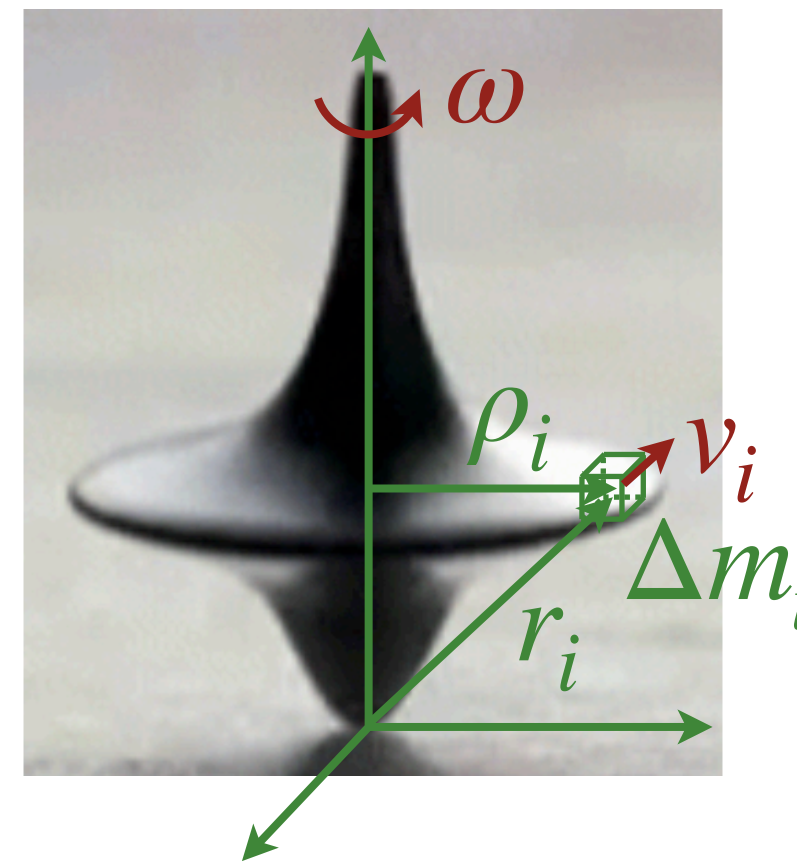
$$\vec{L} = \sum_i \Delta m_i (\rho_i \hat{\rho}_i + z_i \hat{z}) \times (v_i \hat{\phi}_i)$$
$$= \sum \Delta m_i (\rho_i \hat{\rho}_i \times v_i \hat{\phi}_i) + \cancel{\sum \Delta m_i (z_i \hat{z} \times v_i \hat{\phi}_i)}$$
$$= \sum \Delta m_i \rho_i v_i \hat{z}$$

- Since $v_i = \rho_i \omega$, we see that $\overrightarrow{L} = \sum_i \rho_i^2 \Delta m_i \, \omega \hat{z} = \omega \hat{z} \overbrace{\sum_i \rho_i^2 \Delta m_i}^{I_z}$

Angular momentum of rotating rigid bodies

- Imagine the object is composed of many differential elements, labeled $i = 1, 2, 3, \ldots$, with positions $\vec{r}_i$

$$\vec{L} = \sum_i \vec{L}_i = \sum_i \vec{r}_i \times \Delta m_i \vec{v}_i$$



- For pure rotation of a <u>symmetric</u> object, the $\hat{\rho}$ component of $\vec{L}$ cancels

$$\vec{L} = \sum_i \Delta m_i (\rho_i \hat{\rho}_i + z_i \hat{z}) \times (v_i \hat{\phi}_i)$$

$$= \sum \Delta m_i (\rho_i \hat{\rho}_i \times v_i \hat{\phi}_i) + \cancel{\sum \Delta m_i (z_i \hat{z} \times v_i \hat{\phi}_i)}$$

$$= \sum \Delta m_i \rho_i v_i \hat{z}$$

- Since $v_i = \rho_i \omega$, we see that $\vec{L} = \sum_i \rho_i^2 \Delta m_i \, \omega \hat{z}$

- In the limit of $\Delta m_i \to 0$, $\vec{L} = \int_M \rho^2 dm \vec{\omega} \quad \Rightarrow \quad \boxed{\vec{L} = I \vec{\omega}}$

Angular momentum and torque

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- If angular momentum $\vec{L}$ is analogous to momentum and torque $\vec{\tau}$ is analogous to force, what is their relationship?

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i$$

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}\left[\sum_i \vec{r}_i \times \vec{p}_i\right] = \sum_i \frac{d}{dt}\left[\vec{r}_i \times \vec{p}_i\right] = \sum_i \left[\underbrace{\frac{d\vec{r}_i}{dt}}_{=\vec{v}_i} \times \underbrace{\vec{p}_i}_{=m_i\vec{v}_i} + \vec{r}_i \times \overbrace{\frac{d\vec{p}_i}{dt}}^{\vec{F}_i}\right]$$

$$0$$

$$\boxed{\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i = \vec{\tau}_{net}}$$

Conservation of angular momentum

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the total angular momentum does not change with time.

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If the net torque on a system is ***<u>zero</u>***
(and matter is not exchanged),
the total angular momentum does <u>not</u> change with time.

- In other words, if $\vec{\tau}_{net} = 0$ then the angular momentum is conserved:

$$\overrightarrow{L}_i = \overrightarrow{L}_f$$

DEMO (17)

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Swivel stool

Conceptual question

A figure skater stands on one spot on the ice (assumed frictionless) and spins around with her arms extended. When she pulls her arms in, she reduces her moment of inertia and her angular speed increases. Compared to her initial **rotational kinetic energy**, her **rotational kinetic energy** after she has pulled her arms in must be…

A. the same.
B. larger.
C. smaller.

$$\vec{L} = I\vec{\omega}$$

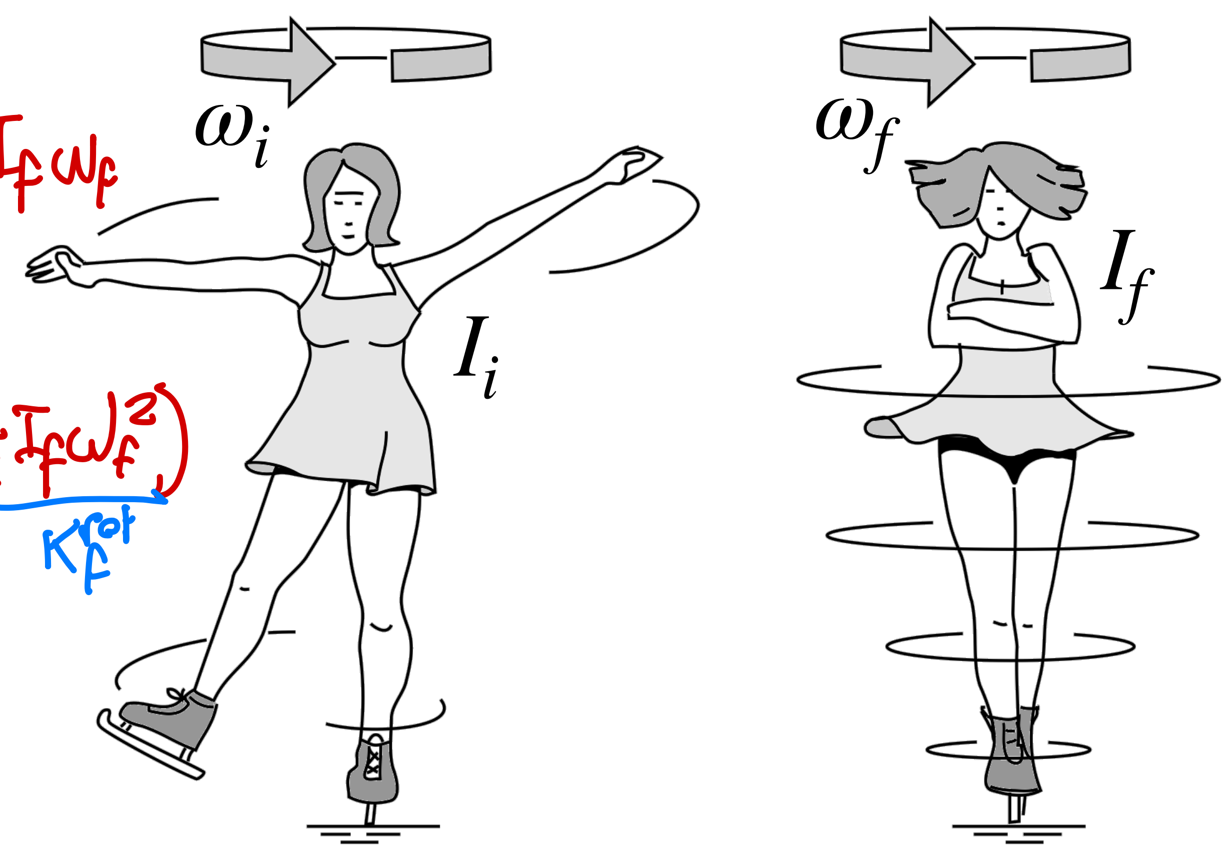
$$\vec{L}_i = \vec{L}_f \Rightarrow I_i\omega_i = I_f\omega_f$$

$$\frac{1}{2} I_i^2\omega_i^2 = \frac{1}{2} I_f^2\omega_f^2$$

$$\Rightarrow I_i\underbrace{\left(\frac{1}{2} I_i\omega_i^2\right)}_{K_i^{rot}} = I_f\underbrace{\left(\frac{1}{2} I_f\omega_f^2\right)}_{K_f^{rot}}$$

$$\Rightarrow I_i K_i^{rot} = I_f K_f^{rot}$$

$$\Rightarrow K_f^{rot} = \frac{I_i}{I_f} K_i^{rot}$$



Summary of rotation and translation

Rotational motion (about a fixed axis)		Translational motion (in one dimension)	
Angular position	ϕ	Position	x
Angular speed	$\omega = d\phi/dt$	Speed	$v = dx/dt$
Angular acceleration	$\alpha = d\omega/dt$	Acceleration	$a = dv/dt$
Moment of inertia	$I = \int \rho^2 dm$	Mass	m
Net torque	$\Sigma\tau_{ext} = I\alpha$	Net force	$\Sigma F_{ext} = ma$
Rotational kinetic energy	$K^{rot} = I\omega^2/2$	Translational kinetic energy	$K^{trans} = mv^2/2$
Work	$W = \int_{\phi_a}^{\phi_b} \tau d\phi$	Work	$W = \int_{x_a}^{x_b} F dx$
Power	$P = \tau\omega$	Power	$P = Fv$
Angular momentum	$L = I\omega$	Momentum	$p = mv$
Net torque	$\Sigma\tau_{ext} = dL/dt$	Net torque	$\Sigma F_{ext} = dp/dt$

Mock exam tomorrow

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Bon courage!

Conceptual question

A 1 kg rock is suspended by a massless string from one end of a 1 m uniform measuring stick with mass m. If the configuration below is in static equilibrium, what is m?

A. 0.25 kg
B. 0.5 kg
C. 1.0 kg
D. 2.0 kg
E. 4.0 kg