Solution to the Final Exam

19 January 2024

PHYS-101(en)

1. Asteroid impact

a. Since the collision is inelastic, the total kinetic energy is **not conserved**. Since there are no external forces on the system and matter does not enter or leave the system, the total momentum is **conserved**. Since there are no external torques on the system and matter does not enter or leave the system, the total angular momentum is **conserved**.

b. As specified in the problem statement, we are using the inertial Cartesian coordinate system given, which is defined such that the planet is has no translational velocity before the collision. Additionally, we are told that the planet is not rotating initially and that the asteroid can be treated as a point mass. Thus, before the collision the only kinetic energy in the system is the translational kinetic energy of the asteroid, which is given by

$$K_i = \frac{m}{2} v_0^2. \tag{1}$$

c. From part b, we know that the only object that is moving is the asteroid. It carries a momentum of

$$\vec{p}_i = m v_0 \hat{x}, \tag{2}$$

where we must remember that momentum is a vector.

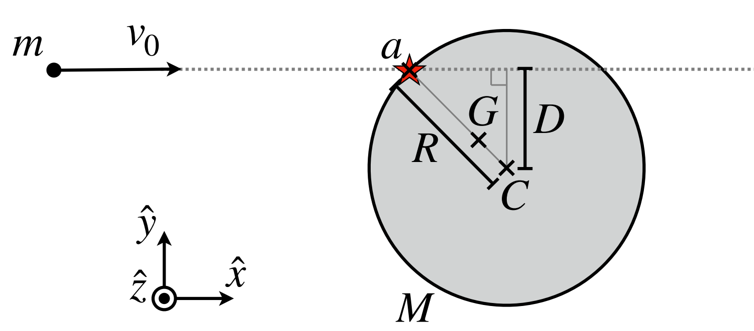
d. To calculate the center of mass of the planet+asteroid system we must consider the displacement vectors between three different points (indicated in the figure below): the geometric center of the planet C, the location of the asteroid impact a, and the center of mass of the system G. To keep track of them, we will be careful and use the notation given by the problem. Each position vector will have two subscripts – the first indicating the start point of the vector and the second indicating the end point. We will let the origin of the coordinate system (which is not specified in the problem statement) be the geometric center of the planet C. Thus, we see that the location of the center of mass of the planet on its own is

$$\vec{r}_p = \vec{r}_{CC} = 0. \tag{3}$$

Just after the collision, the location of the asteroid is

$$\vec{r}_a = \vec{r}_{Ca} = -\sqrt{R^2 - D^2}\,\hat{x} + D\hat{y}, \tag{4}$$

where we have used the Pythagorean theorem to determine the $\hat{x}$ component.



To find the center of mass we take its definition and substitute equations (3) and (4) to find

$$\vec{R}_{CM} = \vec{r}_{CG} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} = \frac{m\vec{r}_a + M\vec{r}_p}{m+M} = \frac{m}{m+M}\left(-\sqrt{R^2-D^2}\hat{x} + D\hat{y}\right). \tag{5}$$

The distance between C and G is simply the magnitude of this vector, which is

$$d_{CG} = |\vec{r}_{CG}| = \frac{m}{m+M}\sqrt{R^2 - D^2 + D^2} = \frac{m}{m+M}R. \tag{6}$$

Note that, since the planet+asteroid forms a rigid body, d_{CG} will stay constant with time after the collision as the object rotates and translates (despite the fact that $\vec{r}_{CG}$ will change direction).

e. The total angular momentum of the system is the sum of the angular momenta of the asteroid and the planet

$$\vec{L}_G = \vec{L}_{Ga} + \vec{L}_{Gp}. \tag{7}$$

The asteroid is modeled as a point mass, so it has an angular momentum about G of

$$\vec{L}_{Ga} = \vec{r}_{Ga} \times m\vec{v}_{Ga}, \tag{8}$$

where $\vec{r}_{Ga}$ is the position vector from G to the asteroid and $\vec{v}_{Ga}$ is the velocity of the asteroid in the frame of reference of G. The planet is a rigid body, so it has an angular momentum due to the translational motion of its center of mass about G and its rotation about its center of mass. This is given by

$$\vec{L}_{Gp} = \vec{r}_{GC} \times M\vec{v}_{GC} + I_G\vec{\omega}_p, \tag{9}$$

where $\vec{r}_{GC}$ is the position vector from G to the center of mass of the planet C, $\vec{v}_{GC}$ is the velocity of the center of mass of the planet C in the frame of reference of G, I_G is the moment of inertia of the planet+asteroid system about G, and $\vec{\omega}_p$ is the angular velocity of the planet about its center of mass. Combining equations (7) through (9), we see that the total angular momentum is

$$\vec{L}_G = \vec{r}_{Ga} \times m\vec{v}_{Ga} + \vec{r}_{GC} \times M\vec{v}_{GC} + I_G\vec{\omega}_p. \tag{10}$$

From part a, we know that the total angular momentum will stay constant in time, so we will choose to calculate all quantities at the moment just before the collision. At this time, the planet is not rotating, so $\vec{\omega}_p = 0$ and we do not need to calculate I_G (yet). The position vector $\vec{r}_{GC}$ is closely related to what we have already calculated in equation (5) since

$$\vec{r}_{GC} = -\vec{r}_{CG} = -\frac{m}{m+M}\left(-\sqrt{R^2-D^2}\hat{x} + D\hat{y}\right). \tag{11}$$

The position vector $\vec{r}_{Ga}$ can be found using the vector addition relationship

$$\vec{r}_{Ca} = \vec{r}_{CG} + \vec{r}_{Ga}, \tag{12}$$

meaning the displacement from C to the asteroid is equal to the displacement from C to G plus the displacement from G to the asteroid. Rearranging this and substituting equations (4) and (5) gives

$$\begin{aligned}\vec{r}_{Ga} = \vec{r}_{Ca} - \vec{r}_{CG} &= \left(-\sqrt{R^2-D^2}\hat{x} + D\hat{y}\right) - \frac{m}{m+M}\left(-\sqrt{R^2-D^2}\hat{x} + D\hat{y}\right)\\ &= \frac{M}{m+M}\left(-\sqrt{R^2-D^2}\hat{x} + D\hat{y}\right).\end{aligned} \tag{13}$$

The velocity vectors are more challenging. Strictly speaking we need to calculate them in the frame of reference moving with G, which is the center of mass reference frame. However, since there are no external forces on the asteroid+planet system, the center of mass velocity $\vec{v}_{CM}$ (i.e. the velocity of G

as seen in the reference frame given in the problem) will be constant in time. Since the only initial motion is in the $\hat{x}$ direction, this means that $\vec{v}_{CM} = v_{CM}\hat{x}$ is also only in the $\hat{x}$ direction. Therefore, the velocities in the frame of reference moving with G can be expressed as $\vec{v}_{Ga} = (v_0 - v_{CM})\,\hat{x}$ and $\vec{v}_{GC} = -v_{CM}\hat{x}$. While we still do not know v_{CM}, we will see that it cancels and does not appear in the final answer. Substituting all of these results into equation (10) and simplifying shows that the total initial angular momentum about G is

$$\begin{aligned}\vec{L}_{Gi} &= \frac{M}{m+M}\left(-\sqrt{R^2-D^2}\hat{x} + D\hat{y}\right) \times m\,(v_0 - v_{CM})\,\hat{x} - \frac{m}{m+M}\left(-\sqrt{R^2-D^2}\hat{x} + D\hat{y}\right) \times M\,(-v_{CM}\hat{x}) \\ &= \frac{M}{m+M}Dm\,(v_0 - v_{CM})\,\hat{y}\times\hat{x} + \frac{m}{m+M}DMv_{CM}\hat{y}\times\hat{x} \\ &= \left(\frac{mM}{m+M}D\,(v_0 - v_{CM}) + \frac{mM}{m+M}Dv_{CM}\right)(-\hat{z}) = -\frac{mM}{m+M}Dv_0\hat{z}. \end{aligned} \tag{14}$$

Note that this result is identical to the answer if you had calculated the total angular momentum in the reference frame of the problem statement (as opposed to changing to the reference frame moving with G). This is not a coincidence. When we calculate the angular momentum of a system about its center of mass, we can adopt any inertial reference frame and will always find the same answer. This is a shortcut that can be used to solve this problem more quickly.

f. The moment of inertia is defined as

$$I_G = \int_{sys} \rho^2 dm, \tag{15}$$

where ρ is the distance from an axis passing through G in the $\hat{z}$ direction and the integral is performed over the entire asteroid+planet system. Since an integral is just a sum, we can separate the contributions from the asteroid I_a and the planet I_p according to

$$I_G = \int_{asteroid} \rho^2 dm + \int_{planet} \rho^2 dm = I_a + I_p. \tag{16}$$

Since the asteroid is a point mass, its contribution to the moment of inertia is simply

$$I_a = md_{Ga}^2 = m|\vec{r}_{Ga}|^2 = \frac{mM}{(m+M)^2}MR^2. \tag{17}$$

Here d_{Ga} is the distance from G to the asteroid, which was found by taking the magnitude of equation (13).

The planet is a rigid uniform sphere, which has a momentum of inertia of I_{CM} around any axis passing through its center of mass. However, we need to calculate its momentum of inertia around G, which is a distance d_{CG} away from its center of mass. Thus, we must use the parallel axis theorem to find

$$I_p = I_{CM} + Md_{CG}^2 = I_{CM} + M\left(\frac{m}{m+M}R\right)^2 = I_{CM} + \frac{mM}{(m+M)^2}mR^2, \tag{18}$$

where we have used equation (6). Substituting equations (17) and (18) into equation (16) gives the final answer of

$$I_G = \frac{mM}{(m+M)^2}MR^2 + I_{CM} + \frac{mM}{(m+M)^2}mR^2 = I_{CM} + \frac{mM}{m+M}R^2. \tag{19}$$

g. In part a, we showed that total angular momentum is a conserved quantity, which is expressed as

$$\vec{L}_{Gi} = \vec{L}_{Gf}. \tag{20}$$

In part e, we calculated the total angular momentum before the collision $\vec{L}_{Gi}$, which is given by equation (14). However, we don't actually need this information as we are permitted to use $\vec{L}_{Gi}$ in our solution. All we need to do is can express the final total angular momentum as

$$\vec{L}_{Gf} = I_G \vec{\omega}_f \tag{21}$$

and combine equations (20) and (21) to find

$$\vec{L}_{Gi} = I_G \vec{\omega}_f \quad \Rightarrow \quad \vec{\omega}_f = \frac{\vec{L}_{Gi}}{I_G}. \tag{22}$$

h. In part a, we found that kinetic energy is not conserved in the collision. The work-kinetic energy theorem

$$W = \Delta K = K_f - K_i \tag{23}$$

tells us that this change in kinetic energy is equal to the total work done by internal forces in the inelastic collision. Importantly, we are only permitted to express our answer in terms of the total mass as well as our solutions to parts b through g. In part b we calculated the total initial kinetic energy K_i, so the challenge is to determine the final kinetic energy of the system. After the collision the asteroid and planet are moving together as one combined object, which translates and rotates about its center of mass G. Thus, equation (23) can be written as

$$W = K_f^{trans} + K_f^{rot} - K_i. \tag{24}$$

The translational kinetic energy of the center of mass is

$$K_f^{trans} = \frac{m_{tot}}{2} V_{CM}^2, \tag{25}$$

where V_{CM} is the magnitude of the center of mass velocity

$$\vec{V}_{CM} = \frac{\sum_i m_i \vec{v}_i}{\sum_i m_i} = \frac{m\vec{v}_a + M\vec{v}_p}{m_{tot}}. \tag{26}$$

The quantity in the numerator is the total momentum of the system, which is a conserved quantity. Thus, the center of mass velocity can be written as $\vec{V}_{CM} = \vec{p}_i / m_{tot}$, which can be substituted into equation (25) to see that the final translational kinetic energy is

$$K_f^{trans} = \frac{p_i^2}{2m_{tot}}. \tag{27}$$

The final rotational kinetic energy about the center of mass is

$$K_f^{rot} = \frac{I_G}{2} \omega_f^2, \tag{28}$$

where both I_G and ω_f have been calculated in previous parts. Thus, substituting equations (27) and (28) into equation (24) gives the final answer of

$$W = \frac{p_i^2}{2m_{tot}} + \frac{I_G}{2} \omega_f^2 - K_i. \tag{29}$$

2. Bouncing

a. In the solutions, we will choose to exclusively use the coordinate system given in the problem statement. It is perfectly fine to define your own coordinate with $\hat{y}$ pointing upwards, but then $\hat{z}$ must then point out of the page (for the coordinate system to be right-handed). This will make the earlier parts of this problem easier, but the last part will be harder (as the angular velocity will be in the $-\hat{z}$ direction).

This part of the problem is simple ballistic motion. We can immediately write down the expressions for the positions and velocities in the horizontal and vertical directions, which are

$$x(t) = v_{x0}t + x_0 \tag{1}$$
$$v_x(t) = v_{x0} \tag{2}$$
$$y(t) = \frac{g}{2}t^2 + v_{y0}t + y_0 \tag{3}$$
$$v_x(t) = gt + v_{y0}. \tag{4}$$

We see that the terms with g are positive (instead of negative as is the case when $\hat{y}$ points upwards). From the problem statement, we know that the initial conditions are $v_{x0} = v_0$, $v_{y0} = 0$, $x_0 = 0$, and $y_0 = -h_0$. Thus, our equations of motion become

$$x(t) = v_0 t \tag{5}$$
$$v_x(t) = v_0 \tag{6}$$
$$y(t) = \frac{g}{2}t^2 - h_0 \tag{7}$$
$$v_x(t) = gt. \tag{8}$$

To find the location at which the ball hits the ground d_c, we must first calculate the time at which the ball hits the ground t_c. This is defined by $y(t_c) = 0$. Using equation (7), we see

$$0 = \frac{g}{2}t_c^2 - h_0 \quad \Rightarrow \quad t_c = \sqrt{\frac{2h_0}{g}}. \tag{9}$$

The horizontal distance traveled in this time is $d_c = x(t_c)$. Using equations (5) and (9), we find the final answer of

$$d_c = v_0\sqrt{\frac{2h_0}{g}}. \tag{10}$$

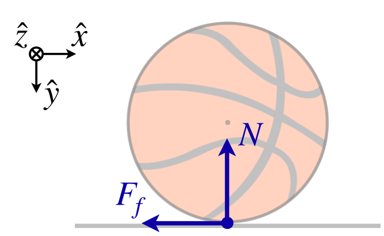
b. This part is also simple ballistic motion. The velocity with which the ball hits the ground is

$$\vec{v}_{ci} = v_x(t_c)\hat{x} + v_y(t_c)\hat{y}. \tag{11}$$

Substituting equations (6), (8), and (9) gives the final answer of

$$\vec{v}_{ci} = v_0\hat{x} + gt_c\hat{y} = v_0\hat{x} + \sqrt{2gh_0}\hat{y}. \tag{12}$$

c. The free body diagram is shown below.



d. To calculate the maximum vertical distance by which the ball flattens, we must analyze the forces on the ball. We start from Newton's second law $\vec{F}_{net} = m\vec{a}$. We only need to consider the $\hat{y}$ component,

$$-N(t) = m\frac{d^2y}{dt^2}, \tag{13}$$

where we have explicitly indicated the fact that the normal force will vary with time. The problem statement tells us that we should model the normal force as a spring with a spring constant k and equilibrium length $\Delta y_c = 0$, meaning that

$$N(t) = k\Delta y_c(t). \tag{14}$$

Substituting this gives

$$-k\Delta y_c(t) = m\frac{d^2y}{dt^2}. \tag{15}$$

Thinking carefully about the geometry of the problem and the coordinate system given to us, we see that $\Delta y_c(t) = y(t)$. For example, when the center of mass of the ball is at $y = 0$, it just barely touches the ground meaning that $\Delta y_c = 0$. Additionally, the more positive y gets, the larger Δy_c. Substituting $\Delta y_c(t) = y(t)$ and rearranging gives

$$\frac{d^2y}{dt^2} + \frac{k}{m}y(t) = 0, \tag{16}$$

which is the standard equation of motion for a harmonic oscillator. It has a general solution of

$$y(t) = A\cos\left(\omega_0 t + \varphi\right), \tag{17}$$

where

$$\omega_0 = \sqrt{\frac{k}{m}} \tag{18}$$

and A and φ are integration constants.

To find A and φ, we must use the initial conditions. To make the math simpler we will define a new time coordinate such that $t = 0$ corresponds to the moment that the ball first touches the ground. This means that $y(0) = 0$, so we have

$$y(0) = 0 = A\cos\left(\varphi\right) \quad \Rightarrow \quad 0 = \cos\left(\varphi\right) \quad \Rightarrow \quad \varphi = \frac{\pi}{2} + n\pi, \tag{19}$$

where $n \in \mathbb{Z}$ can be any integer. Note that we could take $n = 0$, which would make the math simpler and still work. However, we will keep arbitrary n to show that all values of n give the same physical solution. We can substitute equation (19) into equation (17) and use the identity $\cos\left(\theta + \pi/2\right) = -\sin\theta$ given in the problem statement to find

$$y(t) = A\cos\left(\omega_0 t + n\pi + \frac{\pi}{2}\right) = -A\sin\left(\omega_0 t + n\pi\right). \tag{20}$$

Then we can apply $\sin\left(\theta + \pi/2\right) = \cos\theta$ followed by $\cos\left(\theta + \pi/2\right) = -\sin\theta$, both a total of n times, to get

$$y(t) = -(-1)^n A\sin\left(\omega_0 t\right). \tag{21}$$

To determine the amplitude A, we must use an initial condition on the velocity. Fortunately, in part b we already found the vertical component of the velocity when the ball first touches the ground. Thus, we see from equation (12) that $v_y(0) = \sqrt{2gh_0}$. Taking the derivative of equation (21), we see that this initial condition implies that

$$v_y(0) = \sqrt{2gh_0} = -(-1)^n A\omega_0 \cos(0) \quad \Rightarrow \quad A = -\frac{(-1)^n}{\omega_0}\sqrt{2gh_0}. \tag{22}$$

Substituting this back into equation (21) allows us to fully determine the vertical position of the center of mass of the ball throughout the collision to be

$$y(t) = \frac{(-1)^{2n}}{\omega_0}\sqrt{2gh_0}\sin(\omega_0 t) = \frac{1}{\omega_0}\sqrt{2gh_0}\sin(\omega_0 t), \tag{23}$$

where we have used properties of exponentials to show $(-1)^n(-1)^n = (-1)^{2n} = ((-1)^2)^n = (1)^n = 1$.

Analyzing equation (23), we see that the ball will first impact the ground at $t = 0$. Then y will increase with time until $\sin(\omega_0 t) = 1$ when $\omega_0 t = \pi/2$. At this moment, we can use equation (18) and remember that $\Delta y_c(t) = y(t)$ to see that the maximum value of Δy_c is

$$\Delta y_c = \frac{1}{\omega_0}\sqrt{2gh_0} = \sqrt{\frac{2mgh_0}{k}}. \tag{24}$$

After this, y will decrease as the ball rebounds until eventually $y = 0$ again and the ball loses contact with the ground. This occurs when $\omega_0 t = \pi$, which means that the duration of the collision Δt_c is

$$\omega_0 \Delta t_c = \pi \quad \Rightarrow \quad \Delta t_c = \frac{\pi}{\omega_0} = \pi\sqrt{\frac{m}{k}}. \tag{25}$$

e. Since we have neglected the effect of friction and are modeling the normal force as an ideal spring, there are no nonconservative forces acting on the ball. This means that we can apply conservation of mechanical energy between when the ball is released and when it reaches its peak after rebounding off the ground,

$$K_i + U_{gi} = K_f + U_{gf} \quad \Rightarrow \quad \frac{m}{2}v_0^2 + mgh_0 = \frac{m}{2}v_0^2 + mgh_f \quad \Rightarrow \quad h_f = h_0. \tag{26}$$

Note that we have used the fact that there are no forces acting in the $\hat{x}$ direction, so its horizontal velocity never changes throughout the entirety of its motion. Given that it is returning to its original height, its trajectory after bouncing is mirror symmetric to its trajectory before bouncing. Thus, we can say that $d_f = 2d_c$ and use equation (10) to show that

$$d_f = 2v_0\sqrt{\frac{2h_0}{g}}. \tag{27}$$

Note that we have neglected the horizontal distance traveled by the ball while it was in contact with the ground because the problem statement tells us that it is small.

f. This part is identical to parts d and e, except we now include a weak kinetic friction force. Importantly, since the friction force is entirely in the horizontal direction, the vertical dynamics remain unchanged. Hence, the results from our calculation in part d can still be used and we can already say that, as in part e,

$$h_f = h_0. \tag{28}$$

Unfortunately, calculating the horizontal distance traveled is more challenging. We start from Newton's second law in the $\hat{x}$ direction during the collision with the ground, which can be written as

$$-F_f = m\frac{dv_x}{dt}. \tag{29}$$

The magnitude of the kinetic friction force is $F_f = \mu N$. Remembering that $\Delta y_c(t) = y(t)$ and substituting equations (14), (18), and (23) allows us to express the friction force as

$$F_f = \mu k \Delta y_c(t) = \frac{\mu k}{\omega_0}\sqrt{2gh_0}\sin(\omega_0 t) = \mu\sqrt{2mgkh_0}\sin(\omega_0 t)\,. \tag{30}$$

Combining this with equation (29) gives the equation

$$\frac{dv_x}{dt} = -\mu\sqrt{\frac{2gkh_0}{m}}\sin(\omega_0 t)\,. \tag{31}$$

We can directly integrate this once in time and use equation (18) to find

$$v_x(t) = \mu\sqrt{2gh_0}\cos(\omega_0 t) + C, \tag{32}$$

where C is an integration constant. To determine the integration constant, we need an initial condition. We will use the horizontal component of the velocity when the ball first touches the ground. We calculated this in part b, so we can use the horizontal component of equation (12) to find that

$$v_x(0) = v_0 = \mu\sqrt{2gh_0}\cos(0) + C \quad \Rightarrow \quad C = v_0 - \mu\sqrt{2gh_0}. \tag{33}$$

Substituting this back into equation (32) gives

$$v_x(t) = v_0 - \mu\sqrt{2gh_0}\left(1 - \cos(\omega_0 t)\right), \tag{34}$$

the horizontal component of the velocity while the ball is in contact with the ground. Once the ball loses contact with the ground at $t = \Delta t_c$, the friction force no longer acts on the ball, so the horizontal component of the velocity becomes constant again. We are interested in finding this final velocity that the ball departs the ground with. Using equation (25), we see that this is given by

$$v_{xf} = v_x(\Delta t_c) = v_0 - \mu\sqrt{2gh_0}\left(1 - \cos\left(\omega_0\frac{\pi}{\omega_0}\right)\right) = v_0 - 2\mu\sqrt{2gh_0}. \tag{35}$$

Since the horizontal component of the velocity becomes constant after the bounce, the total horizontal distance traveled between when the ball is released and when it reaches its peak height after the bounce is

$$d_f = d_c + v_{xf} t_c. \tag{36}$$

The first term is the horizontal distance traveled before the bounce and the second is the horizontal distance traveled after the bounce. The time between the bounce and the peak after the bounce is still t_c because it is determined by the vertical dynamics, which are the same as in the previous parts. Thus, we can substitute equations (9), (10), and (35) into equation (36) to find the final answer is

$$d_f = v_0\sqrt{\frac{2h_0}{g}} + \left(v_0 - 2\mu\sqrt{2gh_0}\right)\sqrt{\frac{2h_0}{g}} = 2v_0\sqrt{\frac{2h_0}{g}} - 4\mu h_0. \tag{37}$$

Note that we have again neglected the distance traveled while in contact with the ground as it is negligibly short since $\Delta t_c \ll t_c$. Additionally, notice that, if we set $\mu = 0$, we recover the same final answer as in part e.

g. This problem is hard. From our free body diagram in part c, we see that the friction force will create a torque that will cause the ball to start to rotate. The key insight is that the ball will stop sliding if it rotates fast enough to roll without slipping, which is governed by the condition that the horizontal component of the center of mass velocity satisfies $v_x = \omega R$. However, we must also remember that, from our analysis in part f, the ball's horizontal speed is decreasing while in contact with the ground. Since the translational velocity is decreasing with time and the angular velocity is increasing, the ball is more prone to roll without slipping at later times. Thus, kinetic friction will apply the entire time the ball is in contact with the ground only if the final velocity satisfies

$$v_{xf} > \omega_f R, \tag{38}$$

where ω_f is the final angular velocity when the ball loses contact with the ground. We can already use equation (35) to replace v_{xf} and find that

$$v_0 - 2\mu\sqrt{2gh_0} > \omega_f R. \tag{39}$$

The challenge is to find ω_f.

We start with Newton's second law for rotation about the center of mass of the ball

$$\vec{\tau}_{net} = I_{CM}\vec{\alpha}, \tag{40}$$

which relates the net torque $\vec{\tau}_{net}$ to the angular acceleration $\vec{\alpha}$ using the moment of inertia of the ball about its center of mass I_{CM}. While in contact with the ground, the ball experiences two forces, the normal force and kinetic friction force. The torques from each of these has the form $\vec{\tau} = \vec{r} \times \vec{F}$, where $\vec{r}$ is the vector from the center of mass of the ball to the point of application of the force. Thus, we can write equation (40) as

$$\vec{r}_N \times \vec{N} + \vec{r}_f \times \vec{F}_f = I_{CM}\vec{\alpha}. \tag{41}$$

From our free body diagram we see that $\vec{r}_N = R\hat{y}$, $\vec{N} = -N\hat{y}$, $\vec{r}_f = R\hat{y}$, and $\vec{F}_f = -F_f\hat{x}$. Substituting these, Newton's second law for rotation becomes

$$R\hat{y} \times (-N\hat{y}) + R\hat{y} \times (-F_f\hat{x}) = I_{CM}\vec{\alpha} \quad \Rightarrow \quad RF_f\hat{z} = I_{CM}\vec{\alpha}, \tag{42}$$

where we have used the fact that $\hat{y} \times \hat{y} = 0$ and $\hat{y} \times \hat{x} = -\hat{z}$. This implies that the ball will start to rotate only about the $\hat{z}$ axis, so we can write $\vec{\alpha} = d\vec{\omega}/dt = (d\omega/dt)\hat{z}$ and substitute it into the $\hat{z}$ component of equation (42) to find

$$\frac{d\omega}{dt} = \frac{R}{I_{CM}}F_f. \tag{43}$$

Since the problem statement tells us that ball can be approximated as a sphere at all time, we know that neither R nor I_{CM} will change with time. However, the friction force does. Fortunately, the vertical and horizontal translational dynamics are still the same as in previous parts, so we can use our solution for the friction force from part f (i.e. equation (30)). Substituting this into equation (43) gives

$$\frac{d\omega}{dt} = \frac{R}{I_{CM}}\mu\sqrt{2mgkh_0}\sin(\omega_0 t). \tag{44}$$

Integrating once in time and using equation (18) gives

$$\omega(t) = -\frac{R}{I_{CM}}\frac{\mu}{\omega_0}\sqrt{2mgkh_0}\cos(\omega_0 t) + C = -\frac{mR}{I_{CM}}\mu\sqrt{2gh_0}\cos(\omega_0 t) + C, \tag{45}$$

where C is an integration constant. Since the ball is released without any rotation, we can use the initial condition $\omega(0) = 0$ to calculate that the integration constant is

$$\omega(0) = 0 = -\frac{mR}{I_{CM}}\mu\sqrt{2gh_0}\cos(0) + C \quad \Rightarrow \quad C = \frac{mR}{I_{CM}}\mu\sqrt{2gh_0}. \tag{46}$$

Substituting this back into equation (45) gives the angular velocity as a function of time while the ball is in contact with the ground,

$$\omega(t) = \frac{mR}{I_{CM}}\mu\sqrt{2gh_0}\left(1 - \cos\left(\omega_0 t\right)\right). \tag{47}$$

To determine if the ball slips at any time while it is in contact with the ground, we are interested in the final angular velocity $\omega_f = \omega(\Delta t_c)$ as it will be the largest. Using equation (25), we find that

$$\omega_f = \omega(\Delta t_c) = \frac{mR}{I_{CM}}\mu\sqrt{2gh_0}\left(1 - \cos\left(\omega_0\frac{\pi}{\omega_0}\right)\right) = 2\frac{mR}{I_{CM}}\mu\sqrt{2gh_0}. \tag{48}$$

Substituting this into equation (39) allows us to derive

$$\begin{aligned} v_0 - 2\mu\sqrt{2gh_0} > 2\frac{mR^2}{I_{CM}}\mu\sqrt{2gh_0} \quad &\Rightarrow \quad v_0 > 2\mu\sqrt{2gh_0}\left(1 + \frac{mR^2}{I_{CM}}\right) \\ &\Rightarrow \quad \mu < \frac{v_0}{2\sqrt{2gh_0}}\left(1 + \frac{mR^2}{I_{CM}}\right)^{-1}. \end{aligned} \tag{49}$$

This is almost the final answer. However, we must remember that the problem statement does not give I_{CM}. It is perfectly acceptable (and much quicker) to simply copy $I_{CM} = (2/5)mR^2$ from the table presented in class. However, for completeness, we will show how to calculate it below. Before that, we will substitute $I_{CM} = (2/5)mR^2$ into equation (49) in order to find

$$\mu < \frac{v_0}{7\sqrt{2gh_0}}. \tag{50}$$

Thus, the largest value of μ for which the ball will slide the entire time it is in contact with ground is

$$\mu = \frac{v_0}{7\sqrt{2gh_0}}. \tag{51}$$

This is the final answer.

To calculate the moment of inertia of a sphere about its center of mass, we start from the definition of the moment of inertia

$$I_{CM} = \int_m \rho^2 dm, \tag{52}$$

where we must integrate over the entire mass of the ball. Since the ball is three dimensional and a uniform sphere, we will convert from an integral over mass to an integral over space using the volumetric density

$$\rho_V = \frac{m}{V} = \frac{dm}{dV}, \tag{53}$$

where ρ_V is the volumetric density (not the cylindrical radius ρ), V is the total volume, dm is the mass of a differential element, and dV is the volume of a differential element. Since the object is a sphere, it makes sense to use spherical coordinates (r, θ, ϕ). The differential volume in spherical coordinates is $dV = (r\sin\theta d\phi)(rd\theta)(dr)$ and the volume of sphere is $V = (4/3)\pi R^3$, so equation (53) becomes

$$\frac{3m}{4\pi R^3} = \frac{dm}{(r\sin\theta d\phi)(rd\theta)(dr)} \quad \Rightarrow \quad dm = \frac{3m}{4\pi R^3}r^2\sin\theta d\phi d\theta dr. \tag{54}$$

Substituting this into equation (52) gives

$$I_{CM} = \frac{3m}{4\pi R^3}\int_0^R\int_0^\pi\int_0^{2\pi}\rho^2 r^2\sin\theta d\phi d\theta dr, \tag{55}$$

where we have chosen the bounds of integration in order to integrate over the entire sphere. Before we can take the integral, we must convert the cylindrical radius ρ into spherical coordinates. Using trigonometry or looking it up in a table, we can find that $\rho = r\sin\theta$. Thus, the moment of inertia becomes

$$I_{CM} = \frac{3m}{4\pi R^3}\int_0^R\int_0^\pi\int_0^{2\pi} r^4\sin^3\theta d\phi d\theta dr = \frac{3m}{4\pi R^3}\int_0^R r^4\left(\int_0^\pi \sin^3\theta\left(\int_0^{2\pi} d\phi\right)d\theta\right)dr, \tag{56}$$

where we have taken quantities outside of integrals when possible. The innermost integral over ϕ is easy to take, as the argument of the integral is independent of ϕ. Thus, we find

$$I_{CM} = \frac{3m}{4\pi R^3}\int_0^R r^4\left(\int_0^\pi \sin^3\theta\,(2\pi)\,d\theta\right)dr = \frac{3m}{2R^3}\int_0^R r^4\left(\int_0^\pi \sin^3\theta d\theta\right)dr. \tag{57}$$

Taking the integral in θ is more challenging. First we rearrange the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$ and use it to derive

$$\int_0^\pi \sin^3\theta d\theta = \int_0^\pi \sin\theta\sin^2\theta d\theta = \int_0^\pi \sin\theta\left(1-\cos^2\theta\right)d\theta = \int_0^\pi \sin\theta d\theta - \int_0^\pi \sin\theta\cos^2\theta d\theta. \tag{58}$$

The integral in the first term is easy take. For the second term, we will make the substitution $u = \cos\theta$ (and find the bounds for the integral in u by substituting the bounds in θ into this expression). Since $du/d\theta = -\sin\theta$, we know that $d\theta/du = -1/\sin\theta$ and can write

$$\begin{aligned}\int_0^\pi \sin^3\theta d\theta &= \left(-\cos\theta\right]_0^\pi - \int_1^{-1}\sin\theta u^2\frac{d\theta}{du}du = \left(-\cos(\pi)+\cos(0)\right) + \int_1^{-1}u^2du \\ &= 2 + \left(\frac{u^3}{3}\right]_1^{-1} = 2 - \frac{2}{3} = \frac{4}{3}.\end{aligned} \tag{59}$$

Substituting this into equation (57) gives the expected answer of

$$I_{CM} = \frac{3m}{2R^3}\int_0^R r^4\left(\frac{4}{3}\right)dr = \frac{2m}{R^3}\int_0^R r^4 dr = \frac{2m}{R^3}\left(\frac{R^5}{5}\right) = \frac{2}{5}mR^2. \tag{60}$$

3. Axe throwing

a. The axe exhibits two types of motion as it moves through the air: translation of its center of mass and rotation about its center of mass. Since the only force acting on the axe is gravity, the center of mass follows ballistic motion. Thus, as in problem 2.a, we can immediately write down the expressions for the horizontal and vertical positions of the center of mass, which are

$$x_{CM}(t) = v_{xi}t + x_i \tag{1}$$

$$y_{CM}(t) = -\frac{g}{2}t^2 + v_{yi}t + y_i \tag{2}$$

respectively. Note that, because $\hat{y}$ points upwards in this problem, the gravitational term needs to have a negative sign. The coordinate system is defined such that $x_i = 0$ and $y_i = 0$, so we find

$$x_{CM}(t) = v_{xi}t \tag{3}$$

$$y_{CM}(t) = -\frac{g}{2}t^2 + v_{yi}t. \tag{4}$$

Next we must calculate the effect of the rotational motion. First, we must determine the location of center of mass along the handle, to see how far it is from the blade. We will define new coordinate s, which has its origin at the location of the blade and increases as you move along the handle. In this coordinate system, we know that the center of mass of the handle on its own is at $s_h = 2\ell$. This is because the handle is uniform and symmetric, so we know that it is midway along its length and the handle is 4ℓ long. Since we can model the blade as a point mass, its center of mass is at its location of $s_b = 0$. Thus, the center of mass of the entire axe (i.e. handle and blade together) is a distance of

$$s_{CM} = \frac{\sum_i m_i s_i}{\sum_i m_i} = \frac{ms_h + ms_b}{m + m} = \ell \tag{5}$$

away from the blade.

Since there is no air resistance, while the axe is flying through the air, it experiences no torque. This means the blade of the axe undergoes uniform circular motion with its initial angular velocity of $\omega_i\hat{z}$ at a radius of $\rho = s_{CM} = \ell$ away from the center of mass. Given such uniform circular motion, we know that the velocity of the blade of the axe around the center of mass is

$$\vec{v}_b = v_\phi\hat{\phi} = \ell\omega_i\hat{\phi} = \ell\omega_i\left(-\sin\phi\hat{x} + \cos\phi\hat{y}\right), \tag{6}$$

where we have used the cylindrical coordinate unit vector to Cartesian coordinates using the relation $\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y}$. We need to perform this conversion because the problem asks us for the position in Cartesian coordinates. We want to find the position, so, since velocity is the derivative of position, we can write

$$\frac{d\vec{r}_b}{dt} = \ell\omega_i\left(-\sin\left(\phi(t)\right)\hat{x} + \cos\left(\phi(t)\right)\hat{y}\right). \tag{7}$$

We would like to integrate this, but one subtlety, which we have made explicit in this formula, is that the angular position $\phi(t)$ is changing with time. To determine how it varies, we remember that the angular velocity is constant $\omega(t) = \omega_i$ and that $\omega(t) = d\phi/dt$. Combining these two relationships and integrating once in time gives $\phi(t) = \omega_i t + C$. Since the initial condition for the angular position is $\phi(0) = \pi/2$, we see that $C = \pi/2$ and the angular position varies with time according to

$$\phi(t) = \omega_i t + \frac{\pi}{2}. \tag{8}$$

Substituting this into equation (7) gives

$$\frac{d\vec{r}_b}{dt} = \ell\omega_i \left(-\sin\left(\omega_i t + \frac{\pi}{2}\right)\hat{x} + \cos\left(\omega_i t + \frac{\pi}{2}\right)\hat{y}\right) = -\ell\omega_i \left(\cos\left(\omega_i t\right)\hat{x} + \sin\left(\omega_i t\right)\hat{y}\right), \tag{9}$$

where we have made use of the identities given in the question. Now we can directly integrate this expression to find

$$\vec{r}_b(t) = -\ell \sin\left(\omega_i t\right)\hat{x} + \ell\cos\left(\omega_i t\right)\hat{y} + \vec{C}, \tag{10}$$

where $\vec{C}$ is a constant that is a vector (since the equation is a vector equation). To determine it we use the initial condition that the blade is above the center of mass and a distance of ℓ away, so $\vec{r}_b(0) = \ell\hat{y}$. Applying this initial condition we see that $\vec{C} = 0$, so equation (10) becomes

$$\vec{r}_b(t) = -\ell \sin\left(\omega_i t\right)\hat{x} + \ell\cos\left(\omega_i t\right)\hat{y}, \tag{11}$$

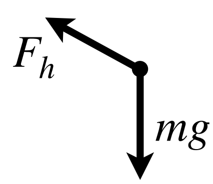
which is the position of the blade relative to the center of mass.

Lastly, we must combine the translational motion (given by equations (3) and (4)) and rotational motion (given by equation (11)) to find the motion of the blade in the laboratory coordinate system defined in the problem statement. This can be accomplished by vector addition as equations (3) and (4) give the position of the center of mass in the laboratory frame and equation (11) gives the position of the blade relative to the center of mass. Thus, we sum the components of equation (11) with equations (3) and (4) to find the final answer

$$x_b(t) = v_{xi}t - \ell\sin\left(\omega_i t\right) \tag{12}$$

$$y_b(t) = -\frac{g}{2}t^2 + v_{yi}t + \ell\cos\left(\omega_i t\right). \tag{13}$$

b. A good starting point to find forces is a free body diagram, which we draw below for the blade of the axe in the laboratory frame. Note that $\vec{F}_h$ is drawn at an arbitrary angle as its direction will vary in a complicated way.



From this we see that Newton's second law is

$$\vec{F}_h - mg\hat{y} = m\vec{a}_b \quad \Rightarrow \quad \vec{F}_h = mg\hat{y} + m\vec{a}_b \tag{14}$$

for the blade, where $\vec{a}_b$ is the acceleration of the blade. Acceleration is just the second derivative of the position, so we can use equations (12) and (13) to write

$$\vec{a}_b = \frac{d^2x_b}{dt^2}\hat{x} + \frac{d^2y_b}{dt^2}\hat{y} = \ell\omega_i^2\sin\left(\omega_i t\right)\hat{x} - g\hat{y} - \ell\omega_i^2\cos\left(\omega_i t\right)\hat{y}. \tag{15}$$

Substituting this into equation (14) gives

$$\vec{F}_h = m\ell\omega_i^2\sin\left(\omega_i t\right)\hat{x} - m\ell\omega_i^2\cos\left(\omega_i t\right)\hat{y} = m\ell\omega_i^2\left(\sin\left(\omega_i t\right)\hat{x} - \cos\left(\omega_i t\right)\hat{y}\right). \tag{16}$$

Taking the magnitude of this gives the final answer of

$$F_h = m\ell\omega_i^2\sqrt{\sin^2(\omega_i t) + \cos^2(\omega_i t)} = m\ell\omega_i^2, \tag{17}$$

where we have used the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$.

The above derivation is performed in the laboratory frame given in the problem statement (which is inertial), but we can perform the calculation more quickly in the center of mass reference frame (which is non-inertial). In the center of mass frame, a fictitious force appears due to the translational acceleration $\vec{A}_{FN}$ of the center of mass in the laboratory frame. This force has a strength of $-m\vec{A}_{FN} = mg\hat{y}$, as the acceleration of the center of mass is simply $-g\hat{y}$. Thus, Newton's second law in the center of mass frame is

$$\vec{F}_h - mg\hat{y} + mg\hat{y} = m\vec{a}_{Nb} \quad \Rightarrow \quad \vec{F}_h = m\vec{a}_{Nb}, \tag{18}$$

where $\vec{a}_{Nb}$ is the acceleration of the blade in the non-inertial center of mass reference frame. This acceleration is easy to calculate as, in the center of mass reference frame, the blade is only undergoing uniform circular motion at a radius of $\rho = \ell$ with an angular velocity of ω_i. Thus, the acceleration of the blade is the centripetal acceleration $\vec{a}_{Nb} = -\ell\omega_i^2\hat{\rho}$. Substituting this into equation (17) and taking the magnitude gives the final answer of

$$\vec{F}_h = -m\ell\omega_i^2\hat{\rho} \quad \Rightarrow \quad F_h = m\ell\omega_i^2, \tag{19}$$

which is identical to equation (17).

c. In order for the axe blade to exactly hit the target, we require that $x_b(t_f) = d$ and $y_b(t_f) = l$ simultaneously at some time t_f. Using equations (12) and (13), these conditions become

$$d = v_{xi}t_f - \ell\sin(\omega_i t_f) \tag{20}$$

$$\ell = -\frac{g}{2}t_f^2 + v_{yi}t_f + \ell\cos(\omega_i t_f). \tag{21}$$

Additionally, we have a constraint on the angular position ϕ: we require the axe to hit the target with the handle vertical such that $\phi(t_f) = \pi/2 - 2\pi n$, where $n \in \mathbb{Z}^+$ can be any positive integer and represents the number of full revolutions the axe makes before hitting the target. Note that we must subtract $2\pi n$ (rather than add it) because the problem statement tells us that $\omega_i < 0$ (so ϕ must be decreasing with time). Using equation (8), this conditions becomes

$$\omega_i t_f + \frac{\pi}{2} = \frac{\pi}{2} - 2\pi n \quad \Rightarrow \quad \omega_i t_f = -2\pi n. \tag{22}$$

This is a constraint on ω_i, but we still need to determine t_f. Since we know v_{xi}, we can find t_f by substituting equation (22) into equation (20) to find

$$d = v_{xi}t_f - \ell\sin(-2\pi n) \quad \Rightarrow \quad t_f = \frac{d}{v_{xi}}. \tag{23}$$

Then we can substitute both this and equation (22) into equation (21) to find that there is one allowed value of v_{yi}, which is

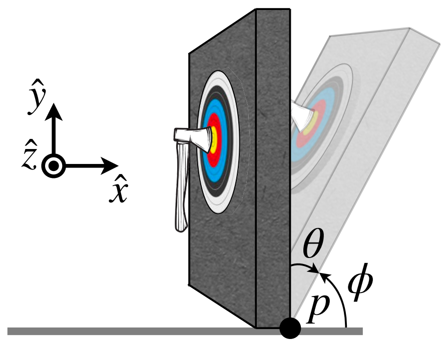
$$\ell = -\frac{g}{2}\left(\frac{d}{v_{xi}}\right)^2 + v_{yi}\left(\frac{d}{v_{xi}}\right) + \ell\cos(-2\pi n) \quad \Rightarrow \quad v_{yi} = \frac{g}{2}\frac{d}{v_{xi}}. \tag{24}$$

This is half of the final answer. The allowed values of ω_i are found by substituting equation (23) into equation (22) to find

$$\omega_i = -2\pi n\frac{v_{xi}}{d}, \tag{25}$$

where $n \in \mathbb{Z}^+$ can be any positive integer. In practice, when you throw an axe it typically rotates once before hitting the target, which would correspond to $n = 1$.

d. This details of this problem are hard. The fact that we need to take the small angle approximation in $\theta \ll \pi$ forces us to express dynamics in terms of θ, but θ is not the traditional cylindrical angle as it is defined to start from the $+\hat{y}$ axis and increase clockwise. Thus, it will be useful to also define the traditional cylindrical angle ϕ (which starts from the $+\hat{x}$ axis and increases counter-clockwise) as shown in the figure below.



To understand the rotation of the board about point p, we will start by writing Newton's second law for rotation

$$\vec{\tau}_{net} = I_p \vec{\alpha}. \tag{26}$$

We will first focus on the right side of the equation. The target is only free to have angular acceleration in the $\pm\hat{z}$ direction, so we can write $\vec{\alpha} = \alpha\hat{z}$. Additionally, we know that $\alpha = d^2\phi/dt^2$ using the traditional cylindrical angle ϕ, which gives

$$\vec{\tau}_{net} = I_p \frac{d^2\phi}{dt^2}\hat{z}. \tag{27}$$

Eventually we must take the small angle approximation in $\theta \ll \pi$, so we need to convert from ϕ to θ. We can do this by studying the above figure and noting that $\theta + \phi = \pi/2$. Rearranging and taking two derivatives in times demonstrates that

$$\omega = \frac{d\phi}{dt} = -\frac{d\theta}{dt} \quad \Rightarrow \quad \alpha = \frac{d^2\phi}{dt^2} = -\frac{d^2\theta}{dt^2}, \tag{28}$$

which allow us to write equation (27) as

$$\vec{\tau}_{net} = -I_p \frac{d^2\theta}{dt^2}\hat{z}. \tag{29}$$

Now we turn to the left side of Newton's second law for rotation. There are several forces acting on the board: gravity, the normal force from the ground, and some sort of friction force from the ground (which is required to keep the board from translating). However, both the normal force and the friction force are applied at the p as the board rotates. Thus, the only torque comes from gravity, so equation (29) becomes

$$\vec{F}_g = -I_p \frac{d^2\theta}{dt^2}\hat{z} \quad \Rightarrow \quad \vec{r}_g \times (-Mg\hat{y}) = -I_p \frac{d^2\theta}{dt^2}\hat{z} \quad \Rightarrow \quad Mg\vec{r}_g \times \hat{y} = I_p \frac{d^2\theta}{dt^2}\hat{z}. \tag{30}$$

When calculating torques, the force from gravity acts at the center of mass of the object, which we are told is at the geometric center of the board. However, one must be careful in expressing $\vec{r}_g$ as it must be valid as the board rotates and changes angle θ. To make it as simple as possible, we will adopt a cylindrical coordinate system with the origin at p. Given these coordinates, we can write the displacement from p to the center of the board as

$$\vec{r}_g = h\hat{\rho} - r\hat{\theta}. \tag{31}$$

Then we can write these unit vectors in Cartesian coordinates using the identities $\hat{\rho} = \sin\theta\hat{x} + \cos\theta\hat{y}$ and $\hat{\theta} = \cos\theta\hat{x} - \sin\theta\hat{y}$. Note that these are *not* the standard conversions between cylindrical and Cartesian unit vectors because the angle θ is defined from the $+\hat{y}$ axis towards the $+\hat{x}$ axis (as opposed to the opposite as is conventional). Instead you have to think about how the unit vectors change as θ is changed and write down the expressions on your own. Substituting these unit vector conversions, equation (31) becomes

$$\vec{r}_g = (h\sin\theta - r\cos\theta)\,\hat{x} + (h\cos\theta + r\sin\theta)\,\hat{y}, \tag{32}$$

which can be used to write equation (30) as

$$\begin{aligned} &Mg\left((h\sin\theta - r\cos\theta)\,\hat{x} + (h\cos\theta + r\sin\theta)\,\hat{y}\right) \times \hat{y} = I_p\frac{d^2\theta}{dt^2}\hat{z} \\ \Rightarrow \quad &Mg\,(h\sin\theta - r\cos\theta)\,\hat{z} = I_p\frac{d^2\theta}{dt^2}\hat{z}. \end{aligned} \tag{33}$$

We can take the $\hat{z}$ component of this equation to find

$$\frac{d^2\theta}{dt^2} - \frac{Mg}{I_p}(h\sin\theta - r\cos\theta) = 0. \tag{34}$$

This is an equation of motion for the angular position of the target, but it is very complicated and difficult to solve. To simplify it, we will take the small angle approximation $\theta \ll \pi$. This will be a good approximation if the board is tall and skinny (i.e. $h \gg r$) because a small tilt angle will be sufficient to move the center of mass of the board to the right of p (which ensures that the board will tip all the way over). Thus, whether or not the board tips is entire determined by the behavior when $\theta \ll \pi$. Assuming $\theta \ll \pi$ allows us to approximate $\cos\theta \approx 1$ and $\sin\theta \approx \theta$. Equation (34) becomes

$$\frac{d^2\theta}{dt^2} - \frac{Mgh}{I_p}\theta = -\frac{Mgr}{I_p}, \tag{35}$$

which is the final answer for the equation of motion governing the rotation of the target.

e. There are two ways to solve this equation. The quicker way is to notice that it has the form of the forced damped harmonic equation from the Math Review document. This form is given by

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega_0^2 x = \frac{F_d}{m}\cos(\omega_d t) + C \tag{36}$$

and the document tells us that it is solved by

$$x(t) = e^{-\lambda t}\left(A_1 e^{t\sqrt{\lambda^2-\omega_0^2}} + A_2 e^{-t\sqrt{\lambda^2-\omega_0^2}}\right) + A_d(\omega_d, F_d)\cos(\omega_d t + \varphi(\omega_d)) + \frac{C}{\omega_0^2}, \tag{37}$$

where A_1 and A_2 are integration constants and

$$A_d(\omega_d, F_d) = \frac{F_d/m}{\sqrt{(2\lambda\omega_d)^2 + (\omega_0^2 - \omega_d^2)^2}} \tag{38}$$

$$\varphi(\omega_d) = \tan^{-1}\left(\frac{2\lambda\omega_d}{\omega_d^2 - \omega_0^2}\right). \tag{39}$$

Thus, if we let $x = \theta$, $\lambda = 0$, $\omega_0^2 = -Mgh/I_p$, $F_d/m = 0$, and $C = -Mgr/I_p$, we find that

$$A_d(\omega_d, F_d) = 0 \tag{40}$$
$$\varphi(\omega_d) = \tan^{-1}(0) = 0 \tag{41}$$

and a solution of

$$\theta(t) = A_1 e^{t\sqrt{Mgh/I_p}} + A_2 e^{-t\sqrt{Mgh/I_p}} + \frac{r}{h}. \tag{42}$$

This is the final answer.

If you didn't write down the solution, there is a second way to solve equation (35). First, we notice that this differential equation has a constant inhomogeneous term (i.e. the term on the right side of the equation). This can be removed by making the substitution

$$\theta(t) = \Theta(t) + \left(-\frac{Mgr}{I_p}\right) / \left(-\frac{Mgh}{I_p}\right) = \Theta(t) + \frac{r}{h} \tag{43}$$

so that equation (35) becomes

$$\frac{d^2\Theta}{dt^2} - \frac{Mgh}{I_p}\Theta(t) = 0. \tag{44}$$

This is a standard technique to eliminate constant inhomogeneous terms, which we used to help solve other differential equations encountered in this course. Studying equation (44), we see that it is quite similar to the harmonic equation, except that the second term has a negative sign instead of a positive sign. We can fix this fact if we use the imaginary unit $i = \sqrt{-1}$ to write it as

$$\frac{d^2\Theta}{dt^2} + \left(i\sqrt{\frac{Mgh}{I_p}}\right)^2 \Theta(t) = 0. \tag{45}$$

Thus, the equation can be cast as a harmonic equation and solved that way. However, we see that $\omega_0 = i\sqrt{Mgh/I_p}$, so we would have i inside the cosine function. To get rid of it we would have to use Euler's formula, thereby converting the cosine into exponential functions. Instead we will use this intuition that the solution is exponentials and guess a form to solve equation (44). We will guess

$$\Theta(t) = A_1 e^{C_1 t}, \tag{46}$$

where we don't know either A_1 or C_1. Substituting this into equation (44) gives

$$A_1 C_1^2 e^{C_1 t} - A_1 \frac{Mgh}{I_p} e^{C_1 t} = 0 \quad \Rightarrow \quad C_1 = \pm\sqrt{\frac{Mgh}{I_p}}. \tag{47}$$

We see that if C_1 has either of these two possible values, then the form we guessed will solve the differential equation. Consequently, the most general possible solution is a linear combination of the two possible solutions,

$$\Theta(t) = A_1 e^{t\sqrt{Mgh/I_p}} + A_2 e^{-t\sqrt{Mgh/I_p}}, \tag{48}$$

where A_1 and A_2 are integration constants. Substituting this back into equation (43) to get the solution for $\theta(t)$ confirms the result we obtained from the Math Review document (e.g. equation (42)).

f. We have just obtained the general solution given by equation (42). To determine the integration constants, we must use the initial conditions. We will define a new time coordinate such that the axe hits the target at $t = 0$. At this time, we know that the board has an angular position of $\theta(0) = 0$. Substituting equation (42) into this yields

$$\theta(0) = 0 = A_1 + A_2 + \frac{r}{h}. \tag{49}$$

Unfortunately, this is just one equation and we have two unknown integration constants. We must also consider the initial angular velocity, which the question tells us is $\omega(0) = -\omega_{p0}$. However, we must remember that the angular velocity is $\omega = d\phi/dt$, rather than $d\theta/dt$. This means we must use equation (28) to determine that the initial value of $d\theta/dt$ is actually $+\omega_{p0}$. Taking the derivative of equation (42), evaluating it at $t = 0$, and setting the result equal to $+\omega_{p0}$ gives

$$\omega_{p0} = A_1\sqrt{\frac{Mgh}{I_p}} - A_2\sqrt{\frac{Mgh}{I_p}} \quad \Rightarrow \quad 0 = A_1 - A_2 - \omega_{p0}\sqrt{\frac{I_p}{Mgh}}. \tag{50}$$

If we sum equations (49) and (50), we can determine one of the integration constants,

$$0 = 2A_1 + \frac{r}{h} - \omega_{p0}\sqrt{\frac{I_p}{Mgh}} \quad \Rightarrow \quad A_1 = \frac{1}{2}\left(\omega_{p0}\sqrt{\frac{I_p}{Mgh}} - \frac{r}{h}\right). \tag{51}$$

Substituting this back into equation (49) gives the other,

$$A_2 = -\frac{1}{2}\left(\omega_{p0}\sqrt{\frac{I_p}{Mgh}} + \frac{r}{h}\right). \tag{52}$$

Thus, we can substitute equations (51) and (52) into equation (42) to find that the full solution to the differential equation is

$$\theta(t) = \frac{1}{2}\left(\omega_{p0}\sqrt{\frac{I_p}{Mgh}} - \frac{r}{h}\right)e^{t\sqrt{Mgh/I_p}} - \frac{1}{2}\left(\omega_{p0}\sqrt{\frac{I_p}{Mgh}} + \frac{r}{h}\right)e^{-t\sqrt{Mgh/I_p}} + \frac{r}{h}. \tag{53}$$

Studying the solution, we see that the first term is the important one in determining if the board tips over. In the limit of long time $t \to \infty$, this term will dominate as it is growing exponentially, while the other terms are exponentially decaying or constant. If the prefactor of the first term is positive, then θ will increase as $t \to \infty$ and the board will tip over forwards about p. If the prefactor is negative, the board may initially rotate forwards, but in the long time limit θ will eventually start to decrease with time and the board will return to $\theta = 0$. At this point it would be back to its initial angular position and it would start rotating about a point that is *not* p (at which time our analysis would break down). From this logic, we have deduced that the critical quantity is the prefactor of the first term. As long as it is positive, i.e.

$$0 < \frac{1}{2}\left(\omega_{p0}\sqrt{\frac{I_p}{Mgh}} - \frac{r}{h}\right) \quad \Rightarrow \quad \sqrt{\frac{g}{h}\frac{Mr^2}{I_p}} < \omega_{p0}, \tag{54}$$

the board will tip over forwards in the limit of $t \to \infty$. Thus, the minimum value of ω_{p0} for which tipping still occurs is

$$\omega_{p0} = \sqrt{\frac{g}{h}\frac{Mr^2}{I_p}}. \tag{55}$$