Final exam
PHYS-101(en)
19 January 2024

Problem booklet

Problems

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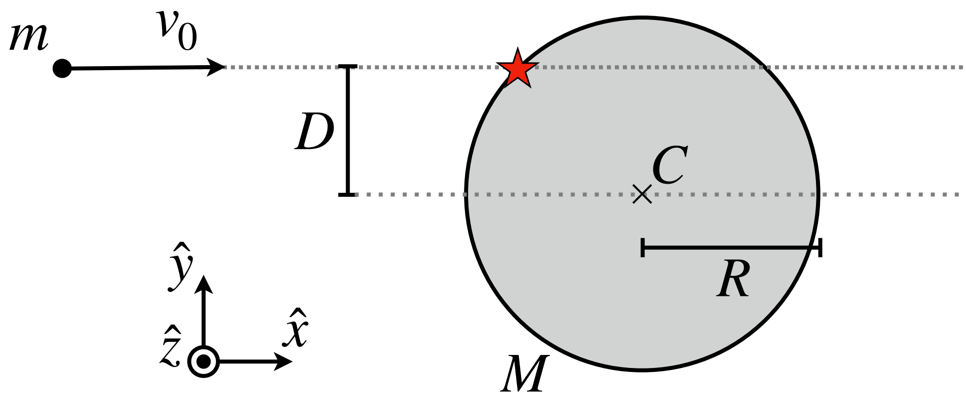
Do NOT open this booklet before the start of the exam

1. Asteroid impact (10 points)



A small asteroid of mass m collides with a planet in outer space. The planet has a radius R, a mass M, and a moment of inertia I_{CM} about any axis passing through to its center of mass (indicated by point C in the figure above). Initially the planet is stationary, meaning that the velocity of the center of mass C is zero and the angular velocity of the planet about C is zero. The asteroid moves in a straight line with speed $\vec{v}_0 = v_0 \hat{x}$. Its trajectory is offset from the center of the planet such that it has an impact parameter of D. The collision is perfectly inelastic such that the asteroid remains stuck to the surface of the planet at the point of impact.

You may assume that

- the gravitational force can be neglected,
- the asteroid may be modeled as a point mass, and
- the planet may be modeled as a rigid sphere with a uniform mass distribution.

A Cartesian coordinate system is shown above, which we will use for this problem. It is inertial and defined such that the planet is at rest before it is hit by the asteroid. The unit vector $\hat{x}$ points in the direction of the initial velocity of the asteroid, $\hat{y}$ points towards the top of the page, and $\hat{z}$ points out of the page.

All answers below (*except for part h*) should be expressed in terms of m, R, M, I_{CM}, v_0, D, $\hat{x}$, $\hat{y}$, $\hat{z}$, and/or any quantities specified in the individual question.

We will define the "planet+asteroid system" to include the planet and the asteroid and let the point G be the center of mass of this system.

a. Consider the total kinetic energy, total momentum, and total angular momentum of the planet+asteroid system. For each of these three quantities, indicate whether it is conserved or not during the collision.

b. Calculate the total kinetic energy K_i of the planet+asteroid system *before* the collision.

c. Calculate the total momentum $\vec{p}_i$ of the planet+asteroid system *before* the collision.

d. Calculate the distance d_{CG} between the center of the planet C and the center of mass of the planet+asteroid system G *after* the collision.

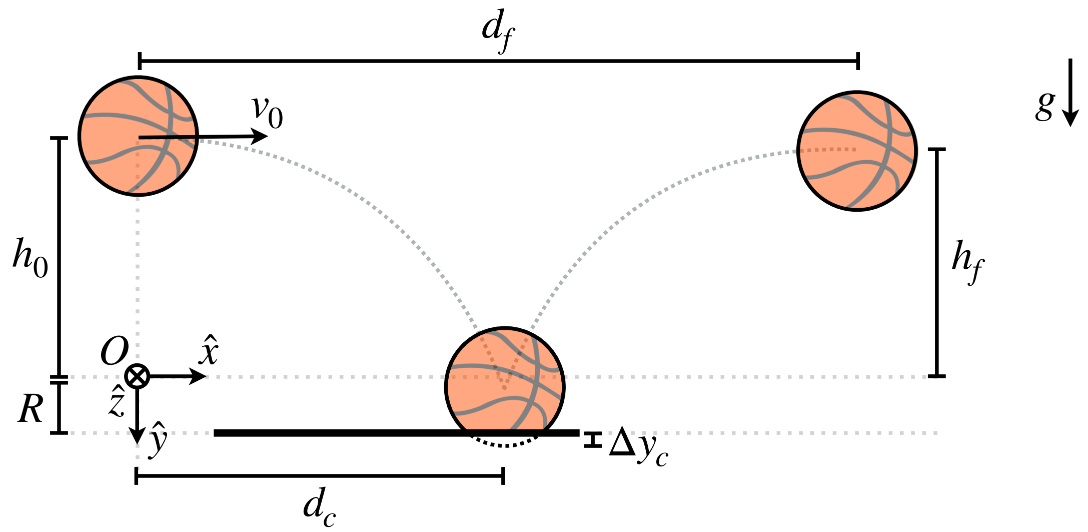
e. Calculate the total angular momentum $\vec{L}_{Gi}$ of the planet+asteroid system about G *before* the collision.

f. Calculate the moment of inertia I_G of the planet+asteroid system about G *after* the collision.

g. Calculate the angular velocity $\vec{\omega}_f$ of the planet+asteroid system about G *after* the collision. Note that you may include K_i, $\vec{p}_i$, d_{CG}, $\vec{L}_{Gi}$, and I_G (as well as their magnitudes) in your answer.

h. Express the total work W done by the internal forces within the planet+asteroid system as a result of the collision in terms of the total mass $m_{tot} = m + M$ and your solutions to parts b through g. Thus, your answer may contain m_{tot}, K_i, $\vec{p}_i$, d_{CG}, $\vec{L}_{Gi}$, I_G, and $\vec{\omega}_f$ (as well as their magnitudes), but it should *not* contain m, R, M, I_{CM}, v_0, nor D.

2. Bouncing (15 points)



In this problem we will investigate the process of a basketball bouncing in great detail. The ball has a radius of R and a mass of m. As shown in the figure above, the ball is thrown purely horizontally from a height h_0 with an initial speed of v_0. It falls under the influence of gravity and impacts the ground, which has a coefficient of static and kinetic friction μ. As a result of colliding with the ground, the ball temporarily deforms by flattening an *unknown* vertical distance of Δy_c that changes with time while in contact with the ground. Given that the elasticity of the ball resists such a deformation, we can model the normal force of the ball on the ground as an ideal spring with a spring constant k and an equilibrium length $\Delta y_c = 0$. The collision with the ground lasts for an *unknown* duration Δt_c after which the ball bounces upwards and reaches an *unknown* maximum final height of h_f at an *unknown* horizontal distance d_f from its starting point.

You may assume that

- the ground is flat and horizontal,
- the ball is released without any rotation about its center of mass,
- the deformation is small compared to the radius of the ball (i.e. $\Delta y_c \ll R$) so that the ball can be approximated as a sphere at all times,
- the radius of the ball is very small compared to the height it is thrown from (i.e. $R \ll h_0$),
- the collision time Δt_c is very short compared to the time in free fall,
- the coefficients of static and kinetic friction between the ball and ground are both equal to μ,
- the ball has a uniform mass distribution,
- air drag is negligible, and
- the acceleration due to gravity is $g\hat{y}$.

A Cartesian coordinate system is shown above, which you may use if you wish. The origin is defined to be directly below the initial location of the center of mass of the basketball, at a distance of R above the level of the ground. The unit vector $\hat{x}$ points in the direction of the initial velocity of the ball, $\hat{y}$ points down, and $\hat{z}$ points into the page.

All answers below should be expressed in terms of R, m, h_0, v_0, μ, k, g, $\hat{x}$, $\hat{y}$, $\hat{z}$, and/or any quantities specified in the individual question.

a. Calculate the horizontal distance traveled by the ball when it first touches the ground d_c.

b. Calculate the velocity of the ball when it first touches the ground $\vec{v}_{ci}$.

c. Draw a free body diagram for the basketball while it is in contact with the ground, including an indication of the point of application of each force (in order to eventually calculate torques). This should include only the normal force $\vec{N}$ from the ground and the friction force $\vec{F}_f$ (and *not* the gravitational force, as it can be neglected by using the impulse approximation).

We will first neglect the effect of friction by letting $\mu = 0$ in order to consider that the ball slides along the ground as it bounces. Remember that we can model the normal force of the ball on the ground as an ideal spring proportional to the vertical distance by which the ball is deformed Δy_c.

d. Find the maximum value of Δy_c experienced by the ball during its collision with the ground, as well as the duration of the collision Δt_c. Note that you may find the trigonometric identities $\cos(\theta - \pi/2) = \sin\theta$, $\cos(\theta + \pi/2) = -\sin\theta$, $\sin(\theta - \pi/2) = -\cos\theta$, and $\sin(\theta + \pi/2) = \cos\theta$ useful.

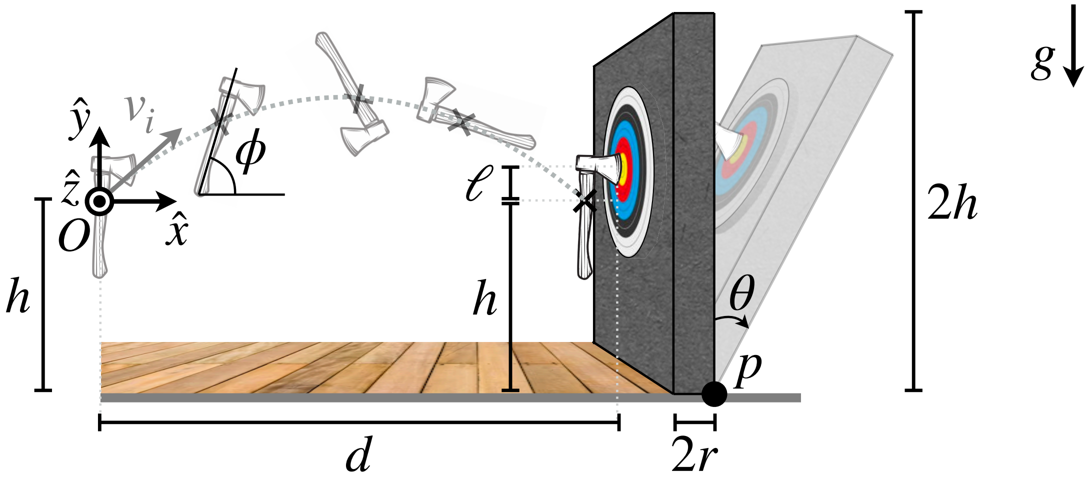
e. Calculate the maximum height h_f that the ball reaches after bouncing and the horizontal distance d_f at which the maximum occurs. Remember that Δt_c is very short compared to the time in free fall.

Next we will include a small, but non-zero value of μ. The value of μ is sufficiently small that the ball slides (i.e. experiences non-zero *kinetic* friction) the entire time that it is in contact with the ground.

f. Calculate the maximum height h_f that the ball reaches after bouncing and the horizontal distance d_f at which the maximum occurs.

g. Calculate the largest value of μ for which the the ball slides (i.e. experiences non-zero *kinetic* friction) the entire time that it is in contact with the ground.

3. Axe throwing (15 points)



In this problem we will model an axe being thrown at a target in order to tip the target over. The axe is composed of a handle with length 4ℓ and mass m and a small metal blade of equal mass m that is fixed to one end of the handle. You throw the axe at the target, releasing it such that its center of mass has a horizontal velocity of v_{xi} and a vertical velocity of v_{yi}. You release the axe such that it also rotates around its center of mass with an initial angular velocity of $\omega_i \hat{z}$. It leaves your hand with the handle of the axe vertical and the blade at the top end of the handle (as shown in the figure above).

You want to hit the target with the blade of the axe. At the moment you release the axe, the target is a horizontal distance d and a vertical distance ℓ above the center of mass of the axe. The target is painted on a board of height $2h$ and thickness $2r$. It is just sitting on the ground and, hence, is free to pivot about an axis in the $\hat{z}$ direction passing though p by changing the angle θ. After the collision, the board and axe together have a total mass M. The center of mass of the combined object is at the geometric center of the board and the combined moment of inertia about p is I_p.

You may assume that

- the axe can be modeled as a thin uniform straight rod of mass m with a point mass (representing the blade) of mass m at one end,
- the angular position of the axe about its center of mass can be quantified by ϕ (i.e. the angle from the $+\hat{x}$ vector to the vector running along the axe handle towards the blade),
- the axe is released upright (i.e. the initial angular position is $\phi_i = \pi/2$),
- the axe is released rotating clockwise (i.e. $\omega_i < 0$),
- the duration of the collision between the axe and the board is very short,
- the board is free to rotate about p and does not translate,
- air drag is negligible, and
- the acceleration due to gravity is $-g\hat{y}$.

A Cartesian coordinate system is shown above, which you may use if you wish. The origin is defined to be the location of the center of mass of the axe at the moment it is released. The unit vector $\hat{x}$ points in the direction of the horizontal velocity of the axe, $\hat{y}$ points up, and $\hat{z}$ points out of the page.

All answers below should be expressed in terms of ℓ, m, v_{xi}, d, h, r, M, I_p, g, $\hat{x}$, $\hat{y}$, $\hat{z}$, and/or any quantities specified in the individual question.

a. Determine the horizontal $x_b(t)$ and vertical $y_b(t)$ positions of the *blade* of the axe as a function of time t. Note that you may use v_{yi} and ω_i in your answer (in addition to the variables stated on the previous page). You may find the trigonometric identities $\sin(\beta + \pi/2) = \cos\beta$ and $\cos(\beta + \pi/2) = -\sin\beta$ useful.

b. Calculate the magnitude of the force $F_h(t)$ that the axe handle exerts on the blade at their point of contact as they both fly through the air together. You may use v_{yi} and ω_i in your answer.

For the remainder of the problem, we will consider the case that the *blade* exactly hits the target with the same orientation as it was released (i.e. with the handle of the axe vertical and the blade at the top end of the handle as shown in the figure above). You are no longer permitted to use v_{yi} and ω_i in your answers.

c. If the axe is released with known v_{xi}, calculate the allowed values of v_{yi} and ω_i.

Next we will consider the dynamics of the target after being hit by the axe to determine if it tips over or not. Assume that, after being hit by the axe, the target has an initial angular speed about p of ω_{p0} in the $-\hat{z}$ direction.

d. Derive the equation of motion for the angular position of the target $\theta(t)$. Note that you may assume that the small angle approximation of $\theta \ll \pi$ is valid. You may find the trigonometric identities $\sin(-\beta + \pi/2) = \cos\beta$ and $\cos(-\beta + \pi/2) = \sin\beta$ useful.

e. Solve the equation of motion to find the general solution for $\theta(t)$. Note that you may include undetermined integration constants in your answer.

f. Use the general solution together with the initial conditions to find the minimum value of ω_{p0} for which the target will be tipping over in the limit of $t \to \infty$.

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