Glissière hémisphérique

Lagrange I

γ

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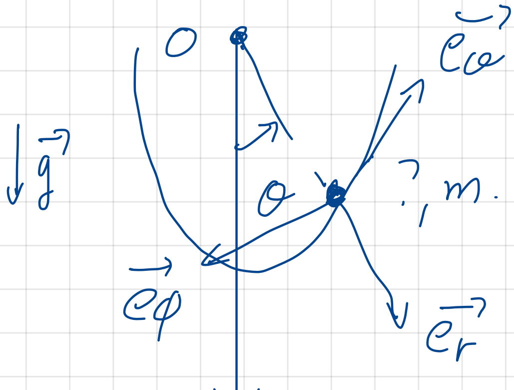
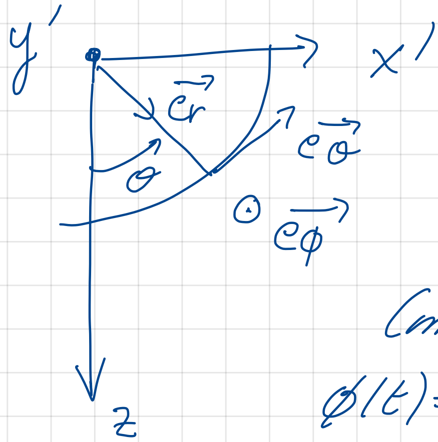
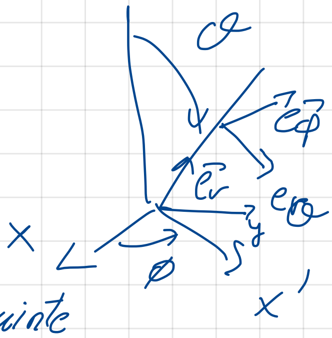


$(P, \vec{e_r}, \vec{e_\theta}, \vec{e_\phi})$





Contrainte

$\phi(t) = \omega t + \phi_0$

avec $\omega = -\Omega$. [rad/s].

$\omega = \dot{\phi}$

Vitesse du point P:

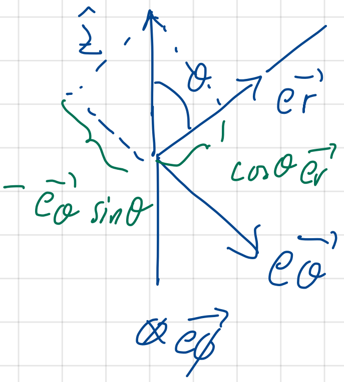
$$\vec{OP} = r\,\vec{e_r}$$

$$\dot{\vec{OP}} = \dot{r}\,\vec{e_r} + r\,\dot{\vec{e_r}}$$

$$= \dot{r}\,\vec{e_r} + r\,\vec{\omega} \wedge \vec{e_r}$$

$\vec{\omega}$ du repère: $\vec{\omega} = \omega\,\hat{z} + \dot{\theta}\,\vec{e_\phi}$

$$= \omega\cos\theta\,\vec{e_r} - \omega\sin\theta\,\vec{e_\theta} + \dot{\theta}\,\vec{e_\phi}$$



$\hat{z} = \boxed{\cos\theta\,\vec{e_r} - \sin\theta\,\vec{e_\theta}}$ (par projections)

$\vec{\omega} \wedge \vec{e_r} =$

$-\omega\sin\theta\,\vec{e_\theta} \wedge \vec{e_r} + \dot{\theta}\,\vec{e_\phi} \wedge \vec{e_r}$

$= \omega\sin\theta\,\vec{e_\phi} + \dot{\theta}\,\vec{e_\theta}$

$$\boxed{\dot{\vec{OP}} = \dot{r}\,\vec{e_r} + r\omega\sin\theta\,\vec{e_\phi} + r\dot{\theta}\,\vec{e_\theta}}$$

On aurait pu poser directement $\dot{\phi} = \omega$ dans les coordonnées sphériques.

Energie cinétique

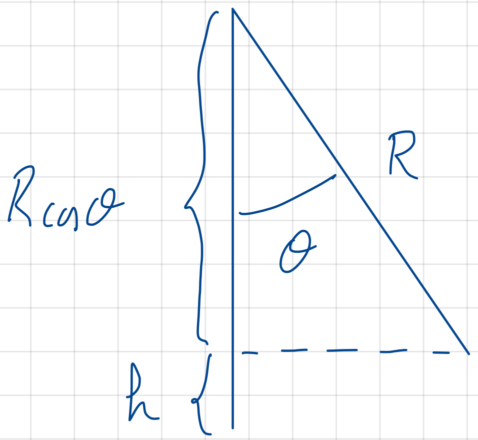
$$T = \frac{1}{2} m\,\vec{V_p} \cdot \vec{V_p} = \frac{1}{2} m\,\dot{\vec{OP}} \cdot \dot{\vec{OP}} =$$

$$\frac{1}{2} m \left(\dot{r}^2 + r^2\omega^2\sin^2\theta + r^2\dot{\theta}^2 \right)$$

contrainte $r = R \implies \dot{r} = 0$

$$T = \frac{1}{2} m \left(R^2\omega^2\sin^2\theta + R^2\dot{\theta}^2 \right)$$

Energie potentielle : $mgh = mg(R - R\cos\theta)$
$$= mgR - mgR\cos\theta$$



$h = R - R\cos\theta$

$$L = T - V = \frac{1}{2}m(R^2\omega^2\sin^2\theta + R^2\dot\theta^2) - mgR + mgR\cos\theta$$

1 coordonnée généralisée θ

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) - \frac{\partial L}{\partial\theta} = 0$$

$$\frac{\partial L}{\partial\theta} = mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta$$

$$\frac{\partial L}{\partial\dot\theta} = mR^2\dot\theta$$

(Remarque c'est un moment cinétique mom. Inertie × vit. angulaire)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) = mR^2\ddot\theta$$

$\Longrightarrow$ dynamique $\cancel{m}R^2\ddot\theta - \cancel{m}R^2\omega^2\sin\theta\cos\theta + \cancel{m}gR\sin\theta = 0$

$$\boxed{\theta'' - \omega^2\sin\theta\cos\theta + \frac{g}{R}\sin\theta = 0}$$

interprétation points d'équilibre : $\ddot\theta = 0$ $\dot\theta = 0$

$$\cancel{m}R^2\omega^2\cancel{\sin}\theta\cos\theta = \cancel{m}gR\cancel{\sin}\theta$$

$$R^{\cancel{2}}\omega^2\cos\theta = g\cancel{R}$$

$$\boxed{\cos\theta = \frac{g}{R\omega^2}}$$