

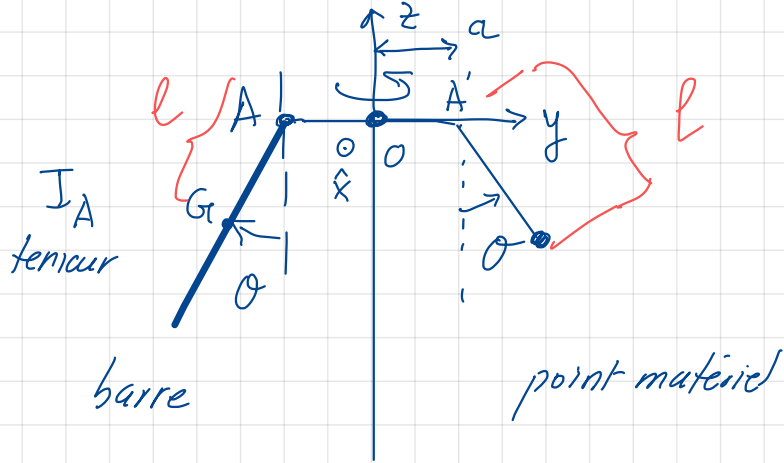
Pendule d'Ansermet

En. cinétique et potentielle.
dérivation des équations par la
méthode de Lagrange

γ

Ph. Willhaup



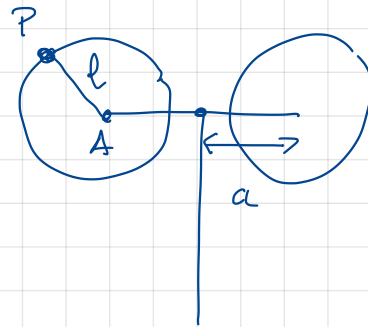
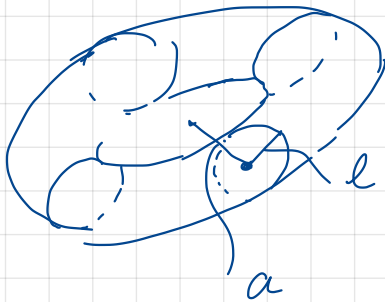


A n'est pas un point fixe

$$\|\vec{OG}\| = l$$

Quel est le type de contrainte si on utilise le repère (O, x, y, z) pour la position de G et de P .

Réponse: c'est un tore:

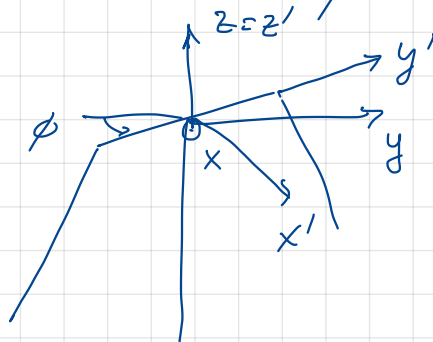


⚠ aux coordonnées sphériques, elles sont cette fois contraintes pour demeurer sur le tore.

$\Rightarrow$ utilisons les coordonnées généralisées

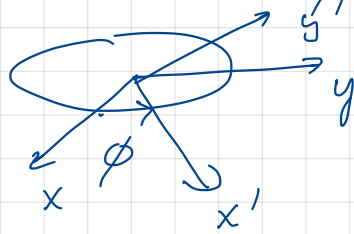
θ et ϕ

(ce ne sont pas les coordonnées θ et ϕ des sphériques!)



Energie cinétique du point matériel

Ecrivons sa position



position du point A

$$x_A = a \cos(\phi) = a \cos(\Omega t)$$

$$y_A = a \sin(\phi) = a \sin(\Omega t)$$

$$z_A = 0. \quad \vec{OA} = a(\cos \Omega t \hat{x} + \sin \Omega t \hat{y}) = a \hat{x}' + \sin \Omega t \hat{y}'$$

$$\vec{AP}' = -l \cos \theta \hat{z}' + l \sin \theta \hat{x}'$$

$$\begin{aligned} \vec{OP}' &= \vec{OA} + \vec{AP}' \\ &= a \hat{x}' + l \sin \theta \hat{x}' - l \cos \theta \hat{z}' \\ &= (a + l \sin \theta) \hat{x}' - l \cos \theta \hat{z}' \end{aligned}$$

$$T = E_{\text{cin}} = \frac{1}{2} m \vec{V}_P \cdot \vec{V}_P$$

$$\begin{aligned} \vec{V}_P &= \dot{\vec{OP}} = \dot{\vec{OA}} + \dot{\vec{AP}} \\ &= a \dot{\hat{x}}' + l \dot{\theta} \cos \theta \hat{x}' + l \sin \theta \dot{\hat{x}}' \\ &\quad + l \sin \theta \dot{\theta} \hat{z}' \end{aligned}$$

$$\dot{\hat{x}}' = \vec{\omega} \wedge \hat{x}' = \Omega \hat{z} \wedge \hat{x}' = \Omega \hat{y}'$$

$$\vec{\omega}' = \Omega \hat{z} \nearrow$$

$$\begin{aligned} \vec{V}_P &= a \Omega \hat{y}' + l \dot{\theta} \cos \theta \hat{x}' + \Omega (l \sin \theta) \hat{y}' \\ &\quad + l \sin \theta \dot{\theta} \hat{z}' \end{aligned}$$

$$\vec{V}_P = \Omega (a + l \sin \theta) \hat{y}' + l \cos \theta \dot{\theta} \hat{x}' + l \sin \theta \dot{\theta} \hat{z}'$$

$$\begin{aligned} \vec{V}_P \cdot \vec{V}_P &= \Omega^2 (a + l \sin \theta)^2 + l^2 \cos^2 \theta \dot{\theta}^2 + l^2 \sin^2 \theta \dot{\theta}^2 \\ &= \Omega^2 (a + l \sin \theta)^2 + l^2 \dot{\theta}^2 \end{aligned}$$

$$E_{cin} = \frac{1}{2} m \Omega^2 (a + l \sin \theta)^2 + \frac{1}{2} m l^2 \dot{\theta}^2$$

$$E_{pot} = mgh = mg(l - l \cos \theta)$$

$$L = E_{cin} - E_{pot} \quad \text{1 d° de l'axe } \theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = m \Omega^2 l^2 (a + l \sin \theta) \cos \theta - mgl \sin \theta.$$

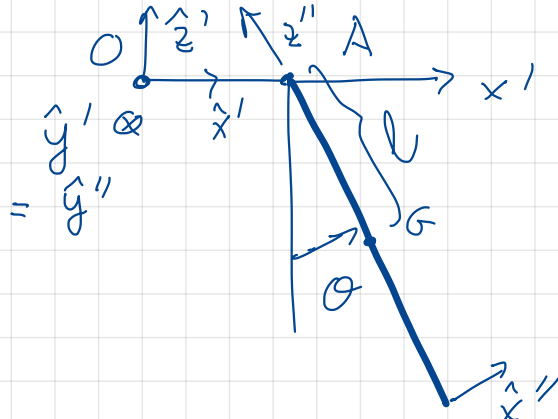
$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = m l^2 \ddot{\theta}$$

Equations différentielles de la dynamique:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = m l^2 \ddot{\theta} - m \Omega^2 l^2 \sin \theta \cos \theta + mgl \sin \theta = 0$$

$$\ddot{\theta} = \Omega^2 \left(\frac{a}{l} + \sin \theta \right) \cos \theta - g/l \sin \theta$$

le système se comporte comme la glissière hémisphérique que lorsque $a = 0$, sinon c'est différent.



Energie cinétique du solide
point arbitraire A

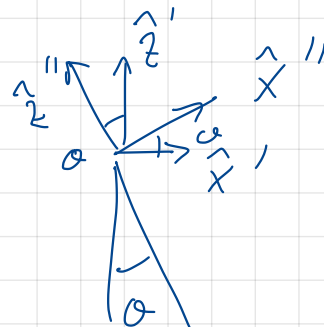
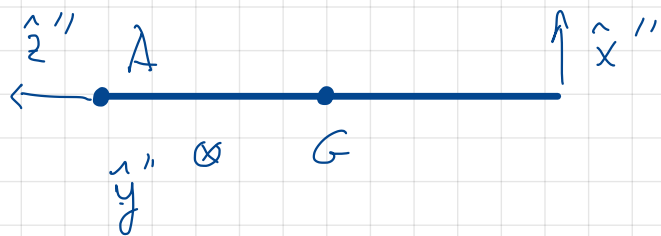
$$E_{\text{cin}} = \frac{1}{2} m \vec{V}_A \cdot \vec{V}_A + \mathcal{M} (\vec{\omega} \wedge \vec{AG}) \cdot \vec{V}_A + \frac{1}{2} \vec{\omega} \cdot \underline{I}_A \vec{\omega}$$

calculer

$$\rightarrow \underline{I}_A = \underline{I}_G + m l^2$$

$$(\text{Steiner}) = \frac{1}{12} m (2l)^2 + m l^2 = \frac{1}{3} m l^2 + m l^2 = \frac{4}{3} m l^2$$

$$\vec{\omega} = -\dot{\theta} \hat{y}' + \Omega \hat{z} \quad \leftarrow \text{du solide}$$



$$\hat{z}' = \hat{z}'' \cos \theta + \hat{x}'' \sin \theta$$

$$\vec{\omega}' = -\dot{\theta} \hat{y}'' + \Omega \cos \theta \hat{z}'' + \Omega \sin \theta \hat{x}''$$

$$E_{\text{cin}} = \frac{1}{2} m \vec{V}_A \cdot \vec{V}_A + \mathcal{M} (\vec{\omega}' \wedge \vec{AG}) \cdot \vec{V}_A + \frac{1}{2} \vec{\omega}' \cdot \underline{I}_A \vec{\omega}'$$

$$\underline{I}_A \vec{\omega}' = -I_A \dot{\theta} \hat{y}'' + I_A \Omega \sin \theta \hat{x}''$$

$$\frac{1}{2} \vec{\omega}' \cdot \underline{I}_A \vec{\omega}' = \frac{1}{2} I_A \dot{\theta}^2 + \frac{1}{2} I_A \Omega^2 \sin^2 \theta$$

$$\vec{V}_A = \Omega a \hat{y}''$$

$$\frac{1}{2} m \vec{V}_A \cdot \vec{V}_A = \frac{1}{2} m \Omega^2 a^2$$

$$\vec{\omega}' \wedge \vec{AG}$$

$$\vec{AG} = -l \hat{z}''$$

$$\vec{\omega} = -\dot{\theta} \hat{y}'' + \Omega \cos\theta \hat{z}'' + \Omega \sin\theta \hat{x}''$$

$$\vec{\omega} \wedge \vec{AG} = (-\dot{\theta} \hat{y}'' + \Omega \sin\theta \hat{x}'') \wedge (-l \hat{z}'')$$

$$= +l \dot{\theta} \hat{x}'' + l \Omega \sin\theta \hat{y}''$$

$$m (\vec{\omega} \wedge \vec{AG}) \cdot \vec{V}_A = m l \Omega^2 a \sin\theta$$

$$E_{\text{cin}} = \frac{1}{2} m \vec{V}_A \cdot \vec{V}_A + m (\vec{\omega} \wedge \vec{AG}) \cdot \vec{V}_A + \frac{1}{2} \vec{\omega} \cdot I_A \vec{\omega}$$

$$\frac{1}{2} m \Omega^2 a^2 + m l \Omega^2 a \sin\theta$$

$$+ \frac{1}{2} I_A \dot{\theta}^2 + \frac{1}{2} I_A \sin^2\theta \Omega^2$$

$$E_{\text{pot}} = mg(l - l \cos\theta)$$

$$\mathcal{L} = \frac{1}{2} m \Omega^2 a^2 + m l \Omega^2 a \sin\theta$$

$$+ \frac{1}{2} I_A (\dot{\theta}^2 + \sin^2\theta \Omega^2) - mg(l - l \cos\theta)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = I_A \dot{\theta}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = m l \Omega^2 a \cos\theta + I_A \sin\theta \cos\theta \Omega^2 - mgl \sin\theta$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \quad I_A \ddot{\theta} - I_A \sin\theta \cos\theta \Omega^2 - m l a \Omega^2 \cos\theta + mgl \sin\theta = 0$$

$$\ddot{\theta} = \Omega^2 \left(\frac{m l a}{I_A} + \sin\theta \right) \cos\theta - \frac{mgl}{I_A} \sin\theta = 0$$

Si $I_A = m \ell^2$ (point matériel)

à comparer avec

$$\ddot{\theta} = \Omega^2 \left(\frac{a}{\ell} + \sin \theta \right) \cos \theta - \frac{g}{\ell} \sin \theta$$


Application numérique $\ell = 16 \text{ [cm]}$ $a = 5 \text{ [cm]}$

$$\Omega = 75 \text{ [t/min]}$$

$$= \frac{75 \cdot 2\pi}{60} \text{ [rad/s]}$$

Angle d'équilibre (dynamique) du pendule simple.

$$\tan \theta = + \frac{\Omega^2}{g} (a + \ell \sin \theta)$$

$$\tan \theta = \left(\frac{75 \cdot 2\pi}{60} \right)^2 \frac{1}{9.81} (0.05 + 0.16 \cdot \sin \theta)$$

$$\theta = 46,1072$$

$$\Omega^2 \left(\frac{m \ell a}{I_A} + \sin \theta \right) \cos \theta - \frac{m g \ell}{I_A} \sin \theta = 0$$

À l'équilibre
dynamique

$$I_A = \frac{4}{3} m \ell^2$$

$$\Omega^2 \left(\frac{3}{4} \frac{a}{\ell} + \sin \theta \right) \cos \theta - \frac{3}{4} \frac{g}{\ell} \sin \theta = 0$$

$$a = 0.05$$

$$\ell = 0,16$$

$$g = 9.81$$

$$\Omega = \left(\frac{75 \cdot 2\pi}{60} \right)$$

$$\Rightarrow \theta = 34,61826025^\circ$$