

Exercises Set 3 - Solution

1 Bonus round

a) The dice can show any number between $X = 1$ and $X = 6$. The minimal gain thus is $100CHF$ and the maximum is $6^2 * 100CHF = 3600 CHF$.

b) The expectation value is computed as $\mathbb{E}[X] = \sum_{i=1}^6 gain(X_i)p(X_i) = \frac{100}{6} \sum_{i=1}^6 i^2$. This can be directly computed, or the Gauss rule can be used: $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$.

This leads to $\mathbb{E}[X] = \frac{100}{6} * 91 \approx 1517 CHF$. It ends up below half of the maximum, because the slow increase of x^2 biases the expectation value towards lower numbers.

c) The average of the square of two numbers is not the same as the square of the average. For example, already for the numbers 1 and 2, the average would be 1.5 and the average squared is $[(1+2)/2]^2 = 2.25$. On the other hand, the average of the squares is $(1^2 + 2^2)/2 = 5/2 = 2.5$.

In general, when you have a *nonlinear* function $f(X)$ then $\mathbb{E}[f(X)]$ is usually equal $f(\mathbb{E}[X])$ (but it can happen to be equal sometimes, and will be equal if all the possible outcomes of X are the same ("degenerate distribution"))

2 Bernoulli distribution and coin game

a)

$$P(10 \text{ "heads"}) = \frac{1}{2}^{10} = 0.098\%$$

$$P(10 \text{ "heads" or } 10 \text{ "tails"}) = P(10 \text{ "heads"}) + P(10 \text{ "tails"}) = \frac{1}{2}^{10} + \frac{1}{2}^{10} = 0.195\%$$

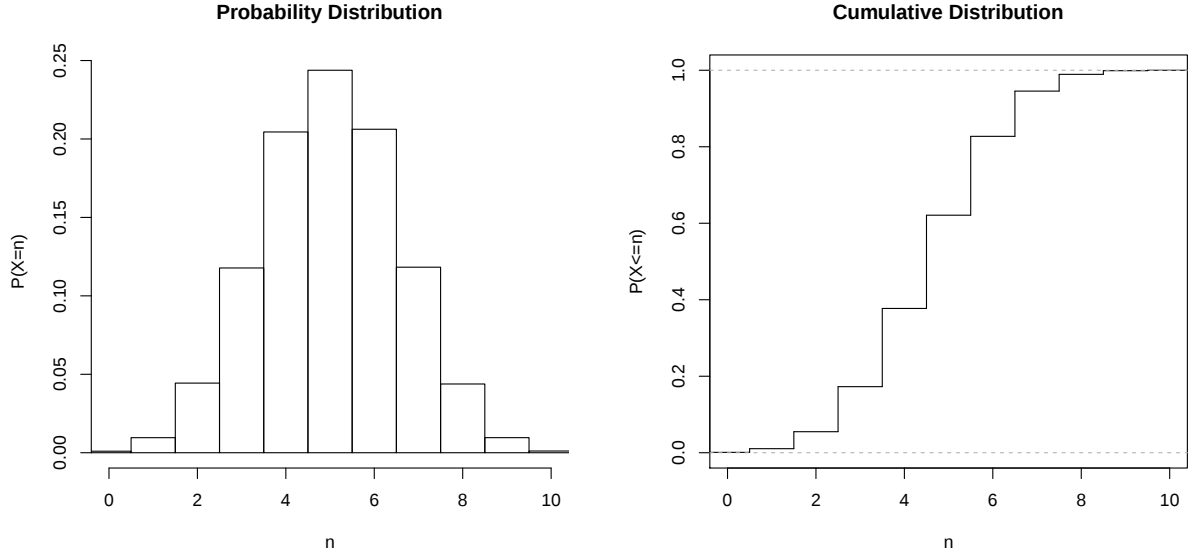
$$P(5 \text{ "heads" and } 5 \text{ "tails"}) = \binom{10}{5} \cdot \frac{1}{2}^5 \cdot \frac{1}{2}^5 = \frac{10!}{5! \cdot 5!} \cdot \frac{1}{2}^{10} = 24.6\%$$

b) We use the Bernoulli formula, with $N = 10$ the number of throws, n the number of "heads", $p = 0.5$ the probability a single throw gives "heads" and $q = 1 - p$ the probability it gives "tails".

$$P(X = n) = \binom{N}{n} p^n q^{N-n} \qquad P(X \leq n) = \sum_{i=0}^n \binom{N}{i} p^i q^{N-i}$$

The expectation value is the sum of the possible values for X , weighted by $P(X = n)$.

$$\begin{aligned} \mathbb{E}(X) &= \sum_{n=0}^N n \cdot P(X = n) = \sum_{n=0}^N n \cdot \frac{N!}{(N-n)!n!} p^n (1-p)^{N-n} \\ &= Np \cdot \sum_{n=0}^N n \cdot \frac{(N-1)!}{(N-n)!n!} p^{n-1} (1-p)^{N-n} \\ &= Np \cdot \sum_{n=1}^N \frac{(N-1)!}{((N-1)-(n-1))!(n-1)!} p^{n-1} (1-p)^{(N-1)-(n-1)} \end{aligned}$$



$$\begin{aligned}
&= Np \cdot \sum_{m=0}^{N-1} \binom{N-1}{m} p^m (1-p)^{(N-1)-m} = Np \cdot \sum_{m=0}^M \binom{M}{m} p^m (1-p)^{M-m} \\
&= Np \cdot P(\tilde{X} \leq M) = Np = 5
\end{aligned}$$

The standard deviation can be found using the following formulas:

$$\begin{aligned}
\sigma &= \sqrt{\text{Var}(X)} \\
\text{Var}(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\
\mathbb{E}(X^2) &= \sum_{n=0}^N n^2 \cdot P(X=n) = \sum_{n=0}^N n^2 \cdot \frac{N!}{(N-n)!n!} p^n (1-p)^{N-n} \\
&= Np \cdot \sum_{n=1}^N n \cdot \frac{(N-1)!}{((N-1)-(n-1))!(n-1)!} p^{n-1} (1-p)^{(N-1)-(n-1)} \\
&= Np \cdot \sum_{m=0}^M (1+m) \cdot \binom{M}{m} p^m (1-p)^{M-m} \\
&= Np + Np \cdot Mp \cdot \sum_{m=1}^M \frac{(M-1)!}{((M-1)-(m-1))!(m-1)!} p^{m-1} (1-p)^{(M-1)-(m-1)} \\
&= Np + N(N-1)p^2 \cdot \sum_{k=0}^K \binom{K}{k} p^k (1-p)^{K-k} = Np + N(N-1)p^2 \\
\text{Var}(X) &= Np + N(N-1)p^2 - (Np)^2 = Np - Np^2 \\
\sigma &= \sqrt{Np(1-p)} = 1.58
\end{aligned}$$

Indeed, σ is zero for $p = 0$ and $p = 1$. The standard deviation measures a degree of variability or randomness. If it is zero, there is no variability and the results are predetermined and identical. Indeed, this is the case. A coin with $p = 0$ will always show "tails" and a coin with $p = 1$ always "heads". Only if $0 < p < 1$, this is a random process.

If $p = \frac{1}{6}$, "heads" will appear every 6 throws, in average. The same result is obtained with the expectancy calculus showed below. The random variable Y is the number of throws until the first

"heads".

$$\begin{aligned}
 P(Y = k) &= \frac{5^{k-1}}{6} \cdot \frac{1}{6} \\
 \mathbb{E}(Y) &= \sum_{k=1}^{\infty} P(Y = k) \cdot k = \frac{1}{6} \sum_{k=1}^{\infty} k \frac{5^{k-1}}{6} \\
 &= \frac{1}{6} \sum_{k=1}^{\infty} k q^{k-1} = \frac{1}{6} \cdot \frac{d}{dq} \sum_{k=1}^{\infty} q^k = \frac{1}{6} \cdot \frac{d}{dq} \frac{1}{1-q} \\
 &= \frac{1}{6} \frac{1}{(1-q)^2} = \frac{1}{6} \frac{36}{(6-5)^2} = 6
 \end{aligned}$$

3 The dice probability distribution

a) If the dice is fair, you can obtain each number with a probability $\frac{1}{6}$. The mean, the variance and the standard deviation are:

$$\begin{aligned}
 \mu = \mathbb{E}(X) &= \sum_{n=1}^6 P(X = n) \cdot n = \sum_{n=1}^6 \frac{n}{6} = 3.5 \\
 \sigma^2 = \text{Var}(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \sum_{n=1}^6 \frac{n^2}{6} - \mathbb{E}(X)^2 = 2.92 \\
 \sigma &= \sqrt{\text{Var}(X)} = 1.71
 \end{aligned}$$

b) Unlike X_1 and X_2 the probability distribution for each dice, S isn't uniform, as many dice combinations give 8, but only one gives 2.

$S = n$	2	3	4	5	6	7	8	9	10	11	12
Probability $P(S = n)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$
CDF $P(S \leq n)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{15}{36}$	$\frac{21}{36}$	$\frac{26}{36}$	$\frac{30}{36}$	$\frac{33}{36}$	$\frac{35}{36}$	$\frac{36}{36}$

The mean, the variance and the standard deviation are:

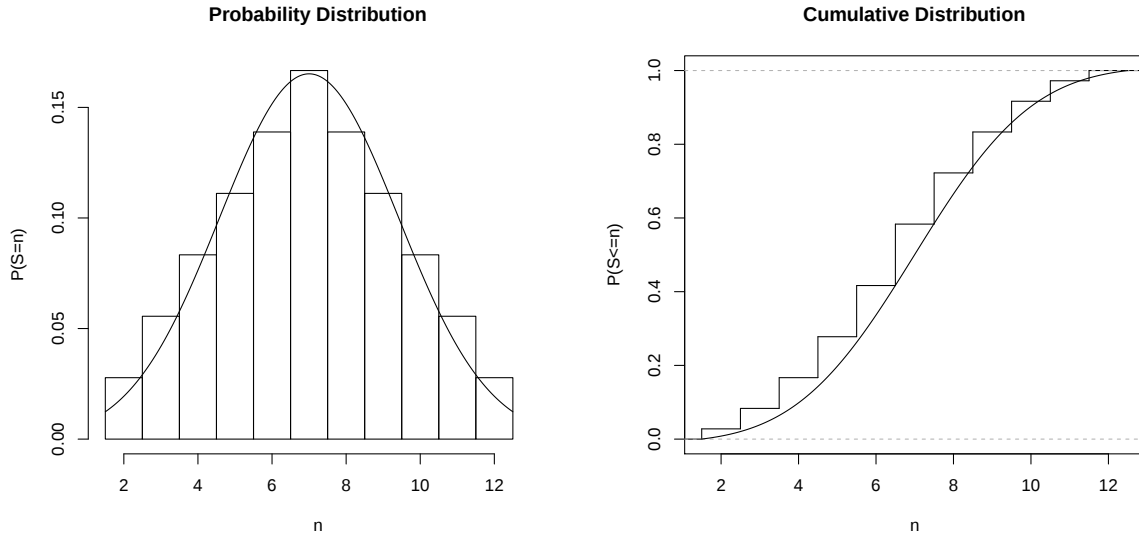
$$\begin{aligned}
 \mu_s = \mathbb{E}(S) &= \sum_{n=2}^{12} P(S = n) \cdot n = 7 = \mathbb{E}(X_1) + \mathbb{E}(X_2) \\
 \sigma_s^2 = \text{Var}(S) &= \mathbb{E}(S^2) - \mathbb{E}(S)^2 = \sum_{n=2}^{12} P(S = n) \cdot n^2 - \mathbb{E}(S)^2 = 5.83 \\
 \sigma_s &= \sqrt{\text{Var}(S)} = 2.42
 \end{aligned}$$

c) The lines on the graphs below represent the Gaussian distribution. The distribution S looks similar.

d) To decide if the game is fair ($\mathbb{E}(G) = 0$), you have to compute the expected gain. Let G be the random variable, which indicate your gain or loss (+2 or -1 CHF).

$$\mathbb{E}(G) = 2 \cdot P(S \geq 10) - 1 \cdot P(S < 10) = 2 \cdot (1 - P(S \leq 9)) - P(S \leq 9) = -0.5 \text{ CHF}$$

So, in average, you will lose money with this game.



$X = n$	0	1	2	3	4	5
Probability $P(X = n)$	0.098%	1.465%	8.789%	26.367%	39.551%	23.730%

4 The Binomial distribution and the grocer

a) X follows the binomial distribution with $N = 5$ and $p = 0.75$: $P(X = n) = \binom{N}{n} p^n (1 - p)^{N-n}$

b) A client complains if $X \leq 3$, so the probability it happen is

$$p_{complain} = P(X \leq 3) = 1 - P(X = 4) - P(X = 5) = 36.719\%$$

c) If 100 clients buy the package, on average 36.719 will complain.

d) For a package sold without complaint, the grocer gains 1CHF. For a complaint, there is a loss of 2CHF (two packages bought, but no income).

The mean grocer's gain is $\mu_{gain} = (1 - p_{complain}) - 2 \cdot p_{complain} = -0.102$ CHF, so the grocer loses money *on average* although there is of course a certain probability that the grocer does still earn money for a finite number of clients.