Exercises Set 6 - Solution

1 Nanoparticles with a normal distribution

The z-value is $z = (x - \mu)/\sigma$. To get $\Phi(z)$ you can look it up in a z-table ("standard normal table") e.g. on Wikipedia, z-table.net, ztable.io (has nice illustrations)... Or compute it using

$$\Phi(z) = \frac{1}{2}(1 + \text{erf}[z/\sqrt(2)]) = \frac{1}{2}(\text{erfc}[-z/\sqrt(2)])$$

where erf is the "error function" and erfc the "complementary error function" (which some calculators feature). Also keep in mind that $\Phi(-z) = 1 - \Phi(z)$

a) 90% interval means between a cumulative probability ($\Phi(z)$) of 5% and 95%. For $\Phi(z) = 95\%$ the z-value is 1.645, for 5% it is consequently -1.645.

   In method A this corresponds to 359nm to 441nm.

   In method B this corresponds to 374nm to 506nm.

b) $z_A = -2$ and $z_B = -2.25$. In principle this is enough to say that method B is better (lower z-value means lower probability).

   We can still caclulate the probabilities, which are $P_A(x \leq 350\text{nm}) = 0.02275$ and $P_B(x \leq 350\text{nm}) = 0.01222$. So method B is almost twice as good.

c) $z_A = -4$ and $z_B = -3.5$, so now method A is better. $P_A(x \leq 300\text{nm}) = 0.00003$ and $P_B(x \leq 300\text{nm}) = 0.00023$, so it is indeed much better. Note that although the mean of method B is further away from 300nm when measured in nm, it is less far away when measured in standard deviations.

d) The z-value we look for is still the same (1.645). But now the standard deviation *of the mean* for $N = 1'000'000$ nanoparticles is given by $\sigma_{\text{mean}} = \sigma/\sqrt{1'000'000} = \sigma/1'000$ (this you get from the central limit theorem or just from the fact that they are i.i.d.). So for method A we have the range 399.96nm to 400.04nm, and for method B we have 439.93nm to 440.07

e) So in this case we know σ_{mean} and can use it to estimate σ of the population/process. This gives $\sigma = \sigma_{\text{mean}} * \sqrt{N}$, where N is now $10'000$. So $\sigma = 10\text{nm}$ for method C, lower than for any for the other methods.

   The z-value is now -4 for the requirement in b, and -9 for the requirement in c (giving a probability of only about 1 in 10^{19} for particles below 300nm!). So in both cases your method C is the best. Well done!

2 Poisson's law and supernova explosions

a) The random variable X can be any integer equal or greater than zero. The cumulative probability over all the possible value should be equal to 1 (so it is a well-defined probability distribution)..

$$\sum_{k=0}^{\infty} P(X = k) = C \cdot \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = C \cdot e^{\lambda} = 1 \qquad => \qquad C = e^{-\lambda}$$

b) To obtain the mean we have to take a sum over all possible values, weighted by their probabilities.

$$\mathbb{E}(X) = \sum_{k=0}^{\infty} P(X = k) \cdot k = e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \cdot k$$

We can drop the $k = 0$ element of the sum as this one is now 0 (we multiply with 0 there)

$$= e^{-\lambda} \cdot \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \cdot k$$

Then we use the fact that $k/k! = 1/(k-1)!$ and also factor out one λ before the sum.

$$= \lambda e^{-\lambda} \cdot \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$$

We substitute $(k-1)$ for some dummy variable (here called l), so we recognize again the form of the Maclaurin series of the exponential function.

$$= \lambda e^{-\lambda} \cdot \sum_{l=0}^{\infty} \frac{\lambda^l}{l!} = \lambda e^{-\lambda} e^{\lambda} = \lambda$$

c) The value of k is the number of events per century, hence $\lambda = \frac{1}{4}$.

d) Now we can simply evaluate the Poissonian distribution function:

$$\begin{aligned}
P(X = 2) &= e^{-0.25} \cdot \frac{0.25^2}{2} = 0.0243 \\
P(X \geq 2) &= 1 - P(X = 1) - P(X = 0) = 1 - e^{-0.25}(0.25 + 1) = 0.0265 \\
P(X \geq 1) &= 1 - P(X = 0) = 1 - e^{-0.25} = 0.221 \\
P(X = 0) &= e^{-0.25} = 0.779
\end{aligned}$$

3 Maxwell-Boltzmann law and the particle speeds in an ideal gas

a) The speed of the particle v can be any positive real number. As before, the sum of the probability for all possible values should give 1.

$$\int_0^{\infty} P(V = v) dv = \int_0^{\infty} C \cdot v^2 e^{-\frac{m \cdot v^2}{2kT}} dv = -C \cdot \left[v \cdot \frac{kT}{m} e^{-\frac{m \cdot v^2}{2kT}} \right]_0^{\infty} + \int_0^{\infty} \frac{CkT}{m} \cdot e^{-\frac{m \cdot v^2}{2kT}} dv$$

Here we have used "integration by parts" (i.e. $\int fg' = fg - \int f'g$). The first term vanishes since the exponential function decreases much faster than x increases (try typing $10 \cdot \exp[-10]$ and $100 \cdot \exp[-100]$ into your calculator!).

The second term can be found from the scaled Gaussian integral (see last week's exercise) and taking into account the fact that the Gaussian function is symmetric around 0 - so the integral from 0 to ∞ is just half of the total integral.

It follows:

$$\int_0^{\infty} P(V = v) dv = \int_0^{\infty} \frac{CkT}{m} \cdot e^{-\frac{m \cdot v^2}{2kT}} dv = \frac{CkT}{m} \cdot \sqrt{\pi \frac{2kT}{4m}} = 1 \qquad => \qquad C = \frac{\pi}{2} \left(\frac{2m}{\pi kT} \right)^{3/2}$$

b) The average speed is given by. Again we start by integrating by parts, and for the last integral we look at the derivative of $\exp[-ax^2]$ and work backwards.:

$$\begin{aligned}
\mathbb{E}(V) &= \int_0^\infty C \cdot v^3 \cdot e^{-\frac{m \cdot v^2}{2kT}} dv \\
&= -C \cdot \left[v^2 \cdot \frac{kT}{m} e^{-\frac{m \cdot v^2}{2kT}} \right]_0^\infty + \frac{CkT}{m} \cdot \int_0^\infty 2v \cdot e^{-\frac{m \cdot v^2}{2kT}} \\
&= \frac{CkT}{m} \cdot \left[\frac{-2kT}{m} e^{-\frac{m \cdot v^2}{2kT}} \right]_0^\infty \\
&= 2C \cdot \frac{(kT)^2}{m^2} = \sqrt{\frac{8kT}{\pi m}} = 473 \text{ m/s}
\end{aligned}$$

4 Aircraft wing probability of failure

First we compute the probability of unsafe fly for both kind of plane. All engine's failure are independent to each other and a Binomial distribution is considered.

$$\begin{aligned}
P_2 = P(\text{2-engines plane fails}) &= P(\text{2 engines fail}) = p^2 = 10^{-12} \\
P_4 = P(\text{4-engines plane fails}) &= P(\text{2 engines fail on the same side}) + P(\text{3 or 4 engines fail}) \\
&= 2 \cdot p^2(1-p)^2 + \binom{4}{3} p^3(1-p) + p^4 = 2 \cdot p^2(1-p) \cdot (1+p) + p^4 \\
&= 2 \cdot p^2(1-p^2) + p^4 = 2 \cdot p^2 - p^4 = 2 \cdot 10^{-12} - 10^{-24}
\end{aligned}$$

So, the 4-reactors planes is slightly less safe.

Let Y be the random variable which indicates the number of failing planes due to reactor issues per year. Its average is:

$$\mathbb{E}(Y) = 365 \cdot 24'000 \cdot \frac{P_2 + P_4}{2} = 13.14 \cdot 10^{-6}$$

We can model the event a plane fail with a Binomial distribution of parameter P_2 for 2-reactors planes and P_4 for the other ones. The number of trials $n = 0.5 \cdot 365 \cdot 24'000$ is the same for both kinds of planes.

$$\begin{aligned}
\mathrm{Var}(Y) &= \mathrm{Var}(\mathcal{B}(P_2, n) + \mathcal{B}(P_4, n)) = \mathrm{Var}(\mathcal{B}(P_2, n)) + \mathrm{Var}(\mathcal{B}(P_4, n)) \\
&= nP2(1 - P_2) + nP4(1 - P_4) = 13.14 \cdot 10^{-6} \\
\sigma_Y &= \sqrt{\mathrm{Var}(Y)} = 3.62 \cdot 10^{-3}
\end{aligned}$$

Both variances simply add because the event a 2-reactors plane fails is independent to the event a 4-reactors plane fails.

Alternative approach

Since both $n \cdot P_2$ and $n \cdot P_4$ are really small, it is also possible to use Poisson's distribution with parameter $\lambda_2 = n \cdot P_2$ resp. $\lambda_4 = n \cdot P_4$. In fact this law is an approximation of the Binomial distribution for highly improbable events.

The average and standard deviation are:

$$\begin{aligned}
\mathbb{E}(Y) &= \mathbb{E}(\mathcal{P}(\lambda_2) + \mathcal{P}(\lambda_4)) = \lambda_2 + \lambda_4 = 13.14 \cdot 10^{-6} \\
\mathrm{Var}(Y) &= \mathrm{Var}(\mathcal{P}(\lambda_2) + \mathcal{P}(\lambda_4)) = \lambda_2 + \lambda_4 = 13.14 \cdot 10^{-6} \\
\sigma_Y &= \sqrt{\mathrm{Var}(Y)} = 3.62 \cdot 10^{-3}
\end{aligned}$$

Which give the same result as before. We use $\mathbb{E}(\mathcal{P}(\lambda)) = \lambda$ (see exercise 2 above) and $\mathrm{Var}(\mathcal{P}(\lambda)) = \lambda$.

$$
\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\
\mathbb{E}(X^2) &= \sum_{k=0}^{\infty} P(X=k) \cdot k^2 = e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \cdot k^2 \\
&= e^{-\lambda} \cdot \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \cdot k(k-1) + e^{-\lambda} \cdot \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \cdot k \\
&= \lambda e^{-\lambda} \cdot \left(\sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \cdot (k-1) + \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \right) \\
&= \lambda e^{-\lambda} \cdot \left(\lambda \cdot \sum_{k=1}^{\infty} \frac{\lambda^{k-2}}{(k-2)!} + \sum_{l=0}^{\infty} \frac{\lambda^l}{l!} \right) \\
&= \lambda e^{-\lambda} \cdot \left(\lambda e^{\lambda} + e^{\lambda} \right) = \lambda^2 + \lambda \\
\mathrm{Var}(X) &= \lambda^2 + \lambda - \lambda^2 = \lambda
\end{aligned}
$$