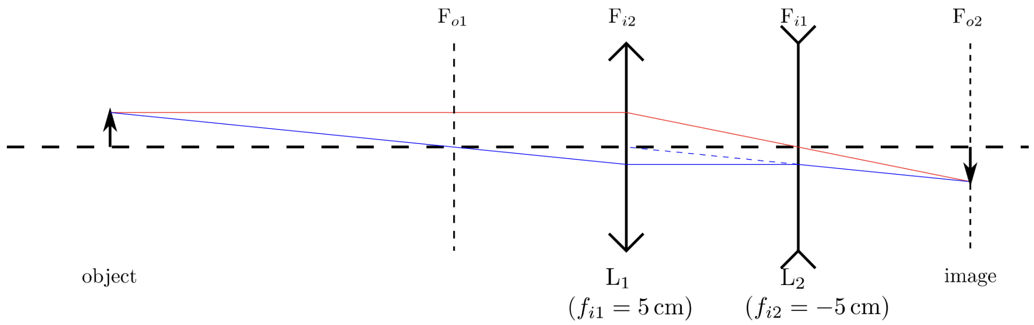
Biomicroscopy I - Solutions Exercise Sheet 2

26 September 2023

1 Thin lenses: Ray tracing

A By intersecting two rays starting at the same point of the object, we get the corresponding image point. The easiest way in this case is to intersect a horizontal ray (red) and a ray passing through the first focal point (blue) starting from the object. The magnification is $m = -1$. The image is 5 cm to the right of the second lens (L_2).



B By proceeding similarly, we find that the magnification $m = -2$. The image is located 10cm to the right of the second lens L_2.

C By proceeding similarly, we find that the magnification $m = -\frac{2}{3}$. The image is located 3.3cm to the right of the second lens L_2.

2 Thin lenses: ABCD matrix

Let us start by writing the transfer matrix $T(L_1L_2)$ from lens L_1 to L_2. We can decompose it into several submatrices:

- The thin lens L_1:

$$L_1 = \begin{bmatrix} 1 & 0 \\ -\Phi_1 & 1 \end{bmatrix} \text{ with } \Phi_1 = \frac{1}{f_1}$$

- The free space propagation T_1 in between:

$$T_1 = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

- And the thin lens L_2:

$$L_2 = \begin{bmatrix} 1 & 0 \\ -\Phi_2 & 1 \end{bmatrix} \text{ with } \Phi_1 = \frac{1}{f_2}$$

The transfer matrix $T(L_1L_2)$ is then:

$$T(L_1L_2) = L_2 \cdot T_1 \cdot L_1 = \begin{bmatrix} 1 & 0 \\ -\Phi_2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -\Phi_1 & 1 \end{bmatrix} =$$
$$= \begin{bmatrix} 1 - d\Phi_1 & d \\ d\Phi_1\Phi_2 - \Phi_1 - \Phi_2 & 1 - d\Phi_2 \end{bmatrix}$$

We now have the transfer matrix $T(L_1L_2)$, which describes the system defined by the two lenses.

3 Consecutive this lenses

We need to prove that a system of two lenses, L_1 and L_2, without any separation ($d = 0$cm), is equivalent to having no lenses if and only if $f_1 = -f_2$. We first notice that using ABCD matrices we can express a lens-free system as the 2×2 identity matrix, $\mathbf{I}_2$.

Let us first prove that if two lenses L_1 and L_2 have focal distances f and $-f$ respectively, then $L_1 \cdot L_2 = \mathbf{I}_2$. By operating the product:

$$L_1 \cdot L_2 = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} + \frac{1}{f} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}_2$$

We now need to prove the other implication: for two arbitrary lenses L_1 and L_2, $\mathbf{I}_2 = L_1L_2$ only when $f_1 = -f_2$.

Let us start with two arbitrary lenses L_1 and L_2, with focal distances f_1 and f_2. First, we operate the product:

$$L_1 \cdot L_2 = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_1} - \frac{1}{f_2} & 1 \end{bmatrix}$$

We now impose $L_1 \cdot L_2$ to be equal to $\mathbf{I}_2$:

$$\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_1} - \frac{1}{f_2} & 1 \end{bmatrix} = L_1 \cdot L_2$$

Therefore, we need $-\frac{1}{f_1} - \frac{1}{f_2}$ in order for L_1L_2 to be the 2×2 identity matrix:

$$-\frac{1}{f_1} - \frac{1}{f_2} = 0 \iff -\frac{1}{f_1} = \frac{1}{f_2} \iff f_1 = -f_2$$

So, L_1L_2 is the identity matrix only when $f_1 = -f_2$.

Having proved both implications of the statement, we conclude the proof.