Biomicroscopy I - Solutions Exercise Sheet 9

November 12, 2024

1 Fourier transform with a lens

A. The mathematical representation of the input function $f(x)$ of the apertures in the object plane:

   i. $f_{\mathrm{i}}(x) = \mathrm{rect}\left(\frac{x}{a}\right)$
   ii. $f_{\mathrm{ii}}(x) = f_1(x+b) + f_1(x-b) = \mathrm{rect}\left(\frac{x+b}{a}\right) + \mathrm{rect}\left(\frac{x-b}{a}\right)$
   iii. $f_3(x) = f_1(x) + f_1(x+b) + f_1(x-b) = \mathrm{rect}\left(\frac{x}{a}\right) + \mathrm{rect}\left(\frac{x+b}{a}\right) + \mathrm{rect}\left(\frac{x-b}{a}\right)$

B. The image formated at Fourier plane is a Fourier transform of an entry object.

   Reminder:

   - The Fourier transform is a linear operator:
   $$\mathcal{F}\{f(x) + g(x)\} = \mathcal{F}\{f(x)\} + \mathcal{F}\{g(x)\}$$
   - The Fourier transform of the rectangular function rect is:
   $$f(x) = \mathrm{rect}\left(\frac{x}{a}\right) \overset{\mathcal{F}}{\Longleftrightarrow} F(p_x) = a \cdot \mathrm{sinc}(ap_x),$$
   where
   $$\mathrm{sinc}(p_x) = \frac{\sin(\pi p_x)}{\pi p_x}$$
   - Translation property of the Fourier transform:
   $$f(x - x_0) \overset{\mathcal{F}}{\Longleftrightarrow} e^{-i2\pi p_x x_0} \cdot F(p_x)$$
   - The definition of $\cos(z)$ for complex numbers:
   $$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

   Therefore, the Fourier transforms of the appertures would be:

   i.
   $$F_1(p_x) = a \cdot \mathrm{sinc}(ap_x)$$
   ii.
   $$\begin{aligned} F_2(p_x) &= a \cdot \mathrm{sinc}(ap_x)e^{i2\pi p_x b} + a \cdot \mathrm{sinc}(ap_x)e^{-i2\pi p_x b} \\ &= 2a \cdot \mathrm{sinc}(ap_x) \cdot \cos(2\pi p_x b) \end{aligned}$$
   iii.
   $$\begin{aligned} F_3(p_x) &= a \cdot \mathrm{sinc}(ap_x) + a \cdot \mathrm{sinc}(ap_x)e^{i2\pi p_x b} + a \cdot \mathrm{sinc}(ap_x)e^{-i2\pi p_x b} \\ &= a \cdot \mathrm{sinc}(ap_x) \cdot (1 + 2\cos(2\pi p_x b)) \end{aligned}$$

   The images formed at the Fourier plane of the lens for $a = 1$ and $b = 2$ are shown below in Figure 1.

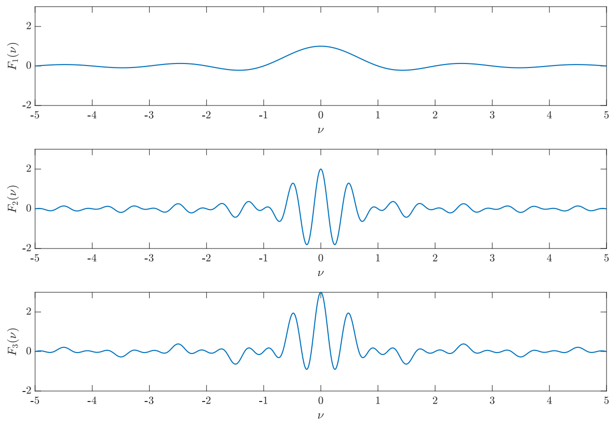

Figure 1: Images of the appertures formed in the Fourier plane.

2 Resolution in $4f$ scheme

In order to define the resolution of the system (or the minimum distance between two monochromatic sources) we will use Rayleigh d^R_{min} and Abbe d^A_{min} criteria:

$$d^R_{min} = 0.61 \cdot \frac{\lambda}{\mathrm{NA}} = 0.61 \cdot \frac{\lambda}{n \sin \alpha}; \qquad d^A_{min} = 0.5 \cdot \frac{\lambda}{\mathrm{NA}} = 0.5 \cdot \frac{\lambda}{n \sin \alpha},$$

where α is the half-angle of the total angular opening of radiation coming from the source.

A. In the presented $4f$-scheme the numerical aperture is defined by the height of lenses:

$$\mathrm{NA} = n \sin \alpha = n \cdot \frac{(h_1/2)}{\sqrt{(h_1/2)^2 + f^2}} \approx 0.3$$

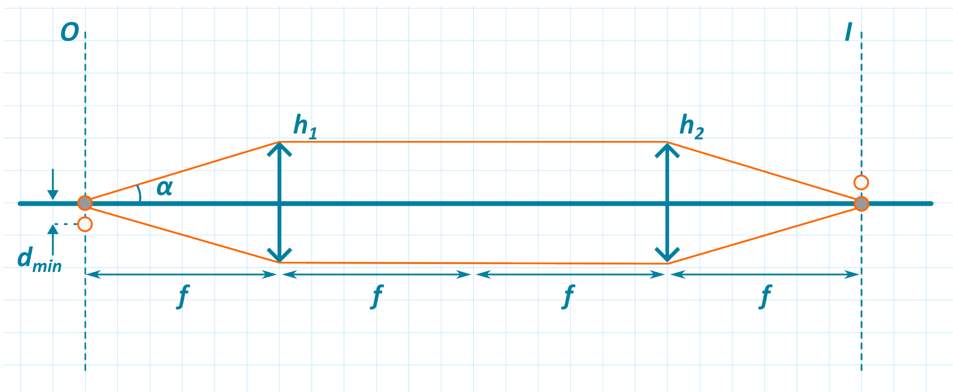

Figure 2: The $4f$ scheme consisting of two thin lenses with marginal rays (in orange) defining the angular opening of the system.

Therefore, the resulting resolution for both criteria could be found:

$$d^R_{min} = 0.61 \cdot \frac{\lambda}{\mathrm{NA}} = 1220 \text{ nm} = 1.22\ \mu\text{m}; \qquad d^A_{min} = 0.5 \cdot \frac{\lambda}{\mathrm{NA}} = 1000 \text{ nm} = 1\ \mu\text{m}.$$

B. When the refractive index of the lenses change from n to $\tilde{n}$, the numerical aperture changes as $\tilde{\mathrm{NA}} = \frac{\tilde{n}}{n}\mathrm{NA}$. In this case, $n = 1$, so $\tilde{\mathrm{NA}} = \tilde{n}\,\mathrm{NA} = 0.45$. The resulting resolution then increases as well (the distance between the resolved sources is smaller):

$$\tilde{d}^R_{min} = 0.61 \cdot \frac{\lambda}{\tilde{\mathrm{NA}}} \approx 813 \text{ nm}; \qquad \tilde{d}^A_{min} = 0.5 \cdot \frac{\lambda}{\tilde{\mathrm{NA}}} \approx 667 \text{ nm}.$$

For changing wavelength $\tilde{\lambda}$ you can directly insert its value in the resolution limit criteria and also increase the resolution by decreasing wavelength:

$$\tilde{d}^R_{min} = 0.61 \cdot \frac{\tilde{\lambda}}{\mathrm{NA}} \approx 976 \text{ nm}; \qquad \tilde{d}^A_{min} = 0.5 \cdot \frac{\tilde{\lambda}}{\mathrm{NA}} \approx 800 \text{ nm}.$$

C. Since the full height $h'_2 = 12$ cm $> h_1$ then the total angular opening should not be influenced by changing this lens to a bigger one and, therefore, the resolution stays the same.

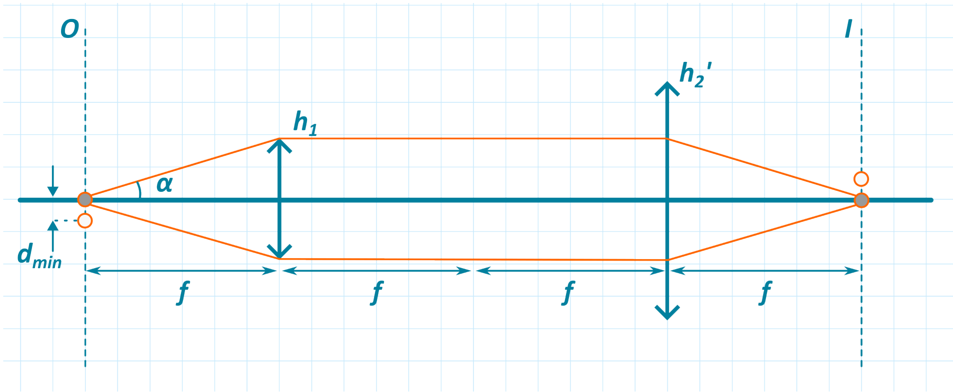

Figure 3: The $4f$ scheme consisting of two thin lenses. The second lens has larger height then the first.

In second case, the full-height $h''_2 = 4$ cm $< h_1$ and, therefore, the angular opening is now defined by the second lens:

$$\mathrm{NA}'' = n \sin \alpha'' = n \cdot \frac{(h_2/2)}{\sqrt{(h_2/2)^2 + f^2}} \approx 0.205$$

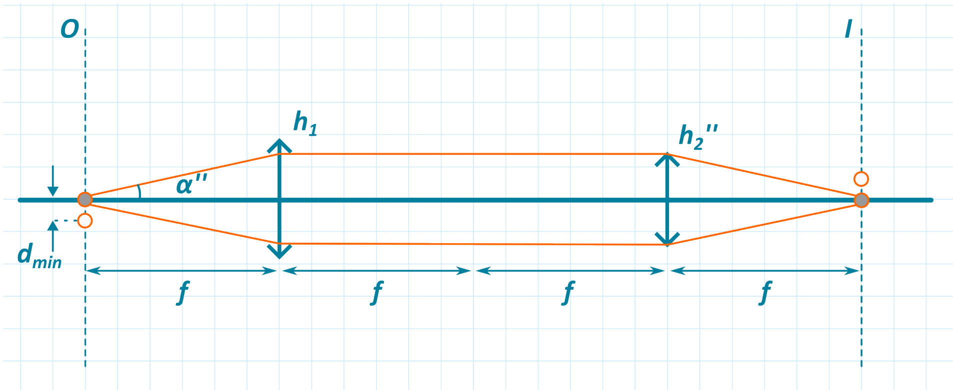

Figure 4: The $4f$ scheme consisting of two thin lenses. The second lens has smaller height then the first.

And corresponding resolution is therefore:

$$d^R_{min} = 0.61 \cdot \frac{\lambda}{\mathrm{NA}''} \approx 1.785 \ \mu\text{m}; \qquad d^A_{min} = 0.5 \cdot \frac{\lambda}{\mathrm{NA}''} \approx 1.463 \ \mu\text{m}.$$

3 Resolution of a human eye

A. A beam of a laser light focused with a lens of focal length f, is propagating through a circular aperture of diameter D. The beam can be approximated by a plane wave. Therefore, its intensity after the propagation through the lens at the plane at a distance of one focal length f is calculated as:

$$I(x,y) = I_0 \left[\frac{2J_1\left(\frac{\pi D\rho}{\lambda f}\right)}{\frac{\pi D\rho}{\lambda f}} \right]^2 \quad \text{where } \rho = \sqrt{x^2+y^2}$$

I_0 is the peak intensity and J_1 is a Bessel function of the first kind. From this formula, the diffraction limited spot size is measured at the radius where this function becomes zero. The radius value is then:

$$\rho = 1.22 \cdot \frac{\lambda \cdot f}{D}$$

which is given by the Rayleigh's Criteria. Using this, we can compute the diameter of the diffraction limited spot:

$$\text{Spot diameter } = 2\rho_s = 2.44\frac{\lambda f}{D} = 2.44\frac{500 \text{ nm} \times 20 \text{ mm}}{2 \text{ mm}} = 12.2\mu\text{m}$$

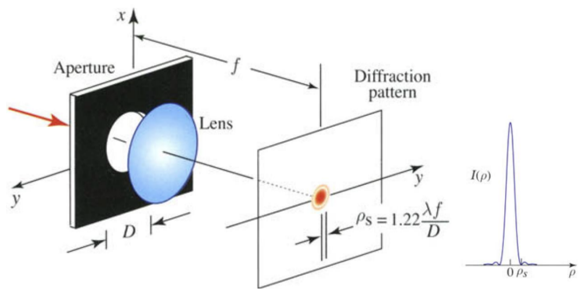

Figure 5: Diffraction pattern at focal length f after a lens with circular aperture.

B. For a square shaped aperture on a lens with focal length f, the intensity on the plane at the distance of one focal length f can be calculated from the propagation through the rectangular aperture:

$$I(x,y) = I_0 \operatorname{sinc}^2 \frac{D_x x}{\lambda f} \operatorname{sinc}^2 \frac{D_y y}{\lambda f}$$

where D_x and D_y are the aperture width and height, respectively; and $\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$. For a square aperture, D_x is equal to D_y. The spot size from this equation is measured at the point where this function becomes zero for the first time. This corresponds to:

$$x_0 = \frac{\lambda f}{D}$$

which gives the half size of the spot along x-axis. The diameter of the diffraction limited spot will be:

$$2x_0 = 2\frac{\lambda f}{D} = 2 \cdot \frac{500 \text{ nm} \cdot 20 \text{ mm}}{2 \text{ mm}} = 10\mu\text{m along the } x\text{-axis}$$

For the y-axis, the spot diameter is the same (10μm), since the aperture is square. Since the feature size is smaller for this case, we would have had better resolution in case we had a square iris. Well... We should enjoy what we are given!

P.S. Please **do not** try to focus a laser beam on the back focal plane of your eye:)