Image Processing 1, Exercise 9

1 Neural network formalism

[basic] This exercise will investigate the mathematical formalism of convolutional neural networks.

Let $f \in \ell_2(\mathbb{Z})$ and consider the 3-channel convolutional layer with input f:

$$\begin{bmatrix} \sigma\big((h_1 * f)[k]\big) \\ \sigma\big((h_2 * f)[k]\big) \\ \sigma\big((h_3 * f)[k]\big) \end{bmatrix},$$

where h_1, h_2, h_3 are filters of the form

$$h_1 = \begin{bmatrix} 0 & \boxed{1} & 0 \end{bmatrix} \qquad h_2 = \begin{bmatrix} 1 & \boxed{-4} & 1 \end{bmatrix} \qquad h_3 = \begin{bmatrix} 0 & \boxed{1} & 0 \end{bmatrix}$$

and $\sigma : \mathbb{R} \to \mathbb{R}$ is a fixed pointwise nonlinearity. The 3-channel convolutional layer can be implemented as the neural network architecture

$$\boldsymbol{\sigma}(\mathbf{W}\boldsymbol{f}_W[k]) = \begin{bmatrix} \sigma\big((h_1 * f)[k]\big) \\ \sigma\big((h_2 * f)[k]\big) \\ \sigma\big((h_3 * f)[k]\big) \end{bmatrix},$$

where we recall that $\boldsymbol{\sigma}$ applies σ component-wise and

$$f_W[k] = (f[k - k_0])_{k_0 \in W}$$

denotes the patch extraction operator. Let $W = \{-2, -1, 0, 1, 2\}$, so that

$$f_W[k] = \begin{bmatrix} f[k+2] \\ f[k+1] \\ f[k] \\ f[k-1] \\ f[k-2] \end{bmatrix} \in \mathbb{R}^5.$$

Specify the size and the entries of the matrix $\mathbf{W}$, with justification.

2 Stability of convolution layers

[basic] To investigate the stability of convolution layers, you need to have a solid understanding of discrete convolution operators.

The convolution masks used in CNNs are usually of finite length, which implies that they are absolutely summable. Here, we show that the latter property is sufficient for the convolution layers to be stable; that is, Lipschitz continuous.

(a) Prove that the convolution operator $\mathrm{T}_{\mathrm{LSI}} : f \mapsto h * f$ with $h \in \ell_1(\mathbb{Z}^d)$ is Lipschitz continuous. Hint: Derive a simple bound for the frequency response of h.

(b) Lipschitz constant of spatial averagers: Consider the linear averaging operator $\mathrm{T}_{\mathrm{ave}} : f \mapsto g * f$ with an impulse response g such $g[\boldsymbol{k}] \geq 0$ for all $\boldsymbol{k}$ and $\sum_{\boldsymbol{k} \in \mathbb{Z}^d} g[\boldsymbol{k}] = 1$ (lowpass condition). Show that $\mathrm{Lip}(\mathrm{T}_{\mathrm{ave}}) = 1$, which tells us that the insertion of such an operator cannot degrade the overall stability of a neural network.

(c) Composition of filters: Prove that $h_1, h_2 \in \ell_1(\mathbb{Z}^d) \Rightarrow h_2 * h_1 \in \ell_1(\mathbb{Z}^d)$. What happens when h_1 and h_2 are both averagers as in Question (b)?

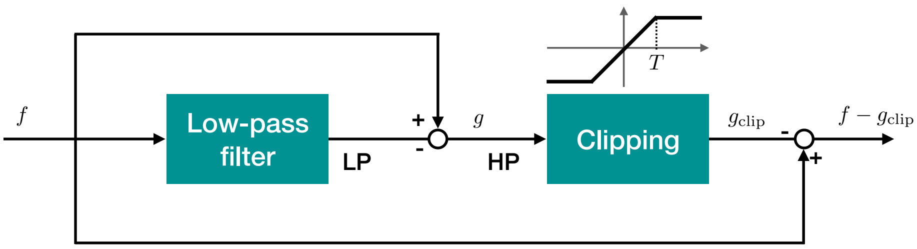

Figure 1: Block diagram of a simple residual denoiser.

3 Analysis of residual denoiser

[intermediate] This exercise will train you to derive stability bounds for simple neuronal architectures that combine convolution operators and pointwise nonlinearities.

The task here is to perform a stability analysis of the residual denoiser in Fig. 1 with $\mathrm{T}_{\mathrm{HP}} : f \mapsto f - h * f$ where h is Gaussian-like smoother. Specifically, h is a lowpass filter whose frequency response is such that $0 \leq H(\mathrm{e}^{\mathrm{j}\boldsymbol{\omega}}) \leq H(\mathrm{e}^{\mathrm{j}\boldsymbol{0}}) = 1$. The pointwise clipping operator is defined as

$$\sigma_T(x) = \begin{cases} T, & x \geq T \\ x, & x \in [-T, T] \\ -T, & x \leq -T \end{cases}$$

(a) Prove that $\mathrm{Lip}(\mathrm{T}_{\mathrm{HP}}) \leq 1$.

(b) Compute $\mathrm{Lip}(\sigma_T)$ with $\sigma_T : \mathbb{R} \to \mathbb{R}$.

(c) Global stability of pointwise operators. Let $\sigma : \mathbb{R} \to \mathbb{R}$ with $\mathrm{Lip}(\sigma) = L$. We then define the image-wide version of the operator $\boldsymbol{\sigma}$ with $\boldsymbol{\sigma}\{f\}[\boldsymbol{k}] = \sigma(f[\boldsymbol{k}])$, for all $\boldsymbol{k} \in \mathbb{Z}^d$. Prove that $\boldsymbol{\sigma} : \ell_2(\mathbb{Z}^d) \to \ell_2(\mathbb{Z}^d)$ with $\mathrm{Lip}(\boldsymbol{\sigma}) = L$. The latter is a basic result that is listed in the course and that can be invoked in exercises for deriving basic stability estimates.

(d) Use the previous results to obtain a simple bound of the Lipschitz constant of the whole system.