Image Processing 1, Exercise 8

1 Connectivity

[basic] A review of segmentation terminology.

Let the finite support of an image be of size (3×3).

(a) What is the city-block distance between two opposite corners of the image?

> **Solution:** Take the top-left corner pixel $\boldsymbol{a}$ and the bottom-right corner pixel $\boldsymbol{b}$ of the image, use the definition of the city-block distance: $D_4(\boldsymbol{a}, \boldsymbol{b}) = |3-1| + |3-1| = 4$.

(b) What is the chessboard distance between them?

> **Solution:** $D_8(\boldsymbol{a}, \boldsymbol{b}) = max(|3-1|, |3-1|) = 2$.

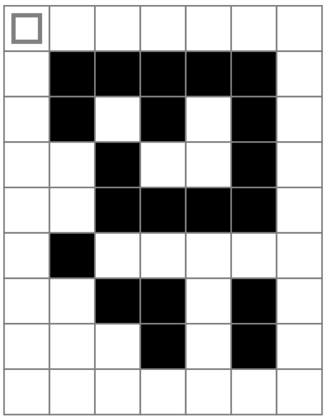
(c) What would be the city-block and chessboard distances between opposite corners of a (4×4) image?

> **Solution:** City-block: $D_4(\boldsymbol{a}, \boldsymbol{b}) = |4-1| + |4-1| = 6$.
> Chessboard: $D_8(\boldsymbol{a}, \boldsymbol{b}) = max(|4-1|, |4-1|) = 3$.

2 Connected Component Labeling

[intermediate] Labeling connected components is an important step in many image analysis pipelines and is provided in commands like Matlab's `bwlabel` and OpenCV's `connectedComponents`.

Let a portion of a binary set $\mathbb{F}$ be given in pictorial representation below, where white indicates background (i.e., □ $\notin \mathbb{F}$) and black indicates foreground (i.e., ■ $\in \mathbb{F}$.)



$\mathbb{F}$

(a) Apply the 4-connected blob-coloring algorithm to $\mathbb{F}$, and report a numeric image that contains the raw indices of the (perhaps equivalent) colors built by the algorithm. Moreover, report the table of color equivalences.

Solution:
$$\begin{bmatrix} \boxed{0} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 2 & 2 & 2 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 4 & 0 & 5 & 0 \\ 0 & 0 & 0 & 4 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \{(1,1),(2,2),(1,2),(3,3),(4,4),(5,5)\}$$

(b) Of how many 4-connected components is $\mathbb{F}$ made?

Solution: In the previous part, we observed that the 4-connected blob-coloring algorithm colored 4-connected regions with 5 different colors. However, due to the equivalency of color (1) and (2), we deduce that this image has four 4-connected components colored as follows (we use ($1'$) to denote the equivalent color for (1) and (2)):

$$\begin{bmatrix} \boxed{0} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1' & 1' & 1' & 1' & 1' & 0 \\ 0 & 1' & 0 & 1' & 0 & 1' & 0 \\ 0 & 0 & 1' & 0 & 0 & 1' & 0 \\ 0 & 0 & 1' & 1' & 1' & 1' & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 4 & 0 & 5 & 0 \\ 0 & 0 & 0 & 4 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(c) Of how many 8-connected components is $\mathbb{F}$ made?

Solution: By checking the 8-connected components in the previous part, we realize that ($1'$), (3), and (4) are equivalent and so there are only two 8-connected components (we use ($1''$) to denote the equivalent color of ($1'$), (3) and (4)):

$$\begin{bmatrix} \boxed{0} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1'' & 1'' & 1'' & 1'' & 1'' & 0 \\ 0 & 1'' & 0 & 1'' & 0 & 1'' & 0 \\ 0 & 0 & 1'' & 0 & 0 & 1'' & 0 \\ 0 & 0 & 1'' & 1'' & 1'' & 1'' & 0 \\ 0 & 1'' & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1'' & 1'' & 0 & 5 & 0 \\ 0 & 0 & 0 & 1'' & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3 Edge Detection

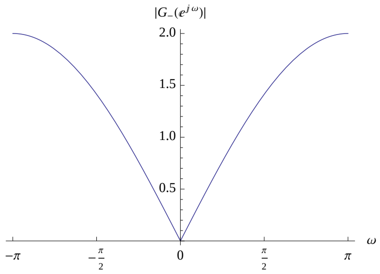
[intermediate] Discrete differentiation is useful in edge detection. There are several ways to design discrete spatial differentiation operators. The purpose of this exercise is to theoretically study their difference.

We want to analyze the characteristics of three popular edge detectors based on discrete data. Their purpose is to approximate the derivative of a function $\forall x \in \mathbb{R} : x \to f(x)$ from the sequence of its samples $\forall k \in \mathbb{Z} : k \to f[k]$. Conversely, the system $\mathcal{D}$ is such that $\mathcal{D}\{f\} = \dot{f}$ corresponds to a true differentiation instead of a finite difference.

(a) Let the backward-difference discrete operator Δ_- be described by $\forall k \in \mathbb{Z} : \Delta_-\{f\}[k] = f[k] - f[k-1]$. Determine the modulus of the frequency response $\hat{g}_-$ and plot it in the main frequency domain. Always label and graduate all axes.

Solution:

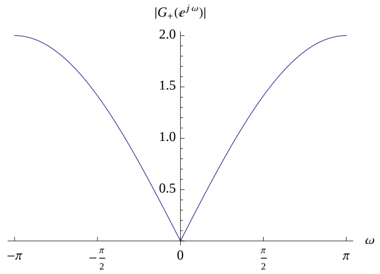
$$\begin{aligned} |\hat{g}_-(\omega)| &= \left|G_-(\mathrm{e}^{\mathrm{j}\,\omega})\right| \\ &= \left|1 - \mathrm{e}^{-\mathrm{j}\,\omega}\right| \\ &= \left|\mathrm{e}^{-\mathrm{j}\,\frac{\omega}{2}}\left(\mathrm{e}^{\mathrm{j}\,\frac{\omega}{2}} - \mathrm{e}^{-\mathrm{j}\,\frac{\omega}{2}}\right)\right| \\ &= \left|2\,\mathrm{j}\,\sin(\frac{\omega}{2})\right| \end{aligned}$$



(b) Let the forward-difference discrete operator Δ_+ be described by $\forall k \in \mathbb{Z} : \Delta_+\{f\}[k] = f[k+1] - f[k]$. Determine the modulus of the frequency response $\hat{g}_+$ and plot it in the main frequency domain. Always label and graduate all axes.

Solution:

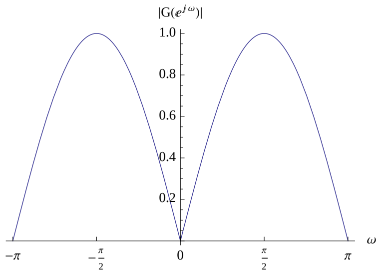
$$\begin{aligned} |\hat{g}_+(\omega)| &= \left|G_+(\mathrm{e}^{\mathrm{j}\,\omega})\right| \\ &= \left|\mathrm{e}^{\mathrm{j}\,\omega} - 1\right| \\ &= \left|\mathrm{e}^{\mathrm{j}\,\frac{\omega}{2}}\left(\mathrm{e}^{\mathrm{j}\,\frac{\omega}{2}} - \mathrm{e}^{-\mathrm{j}\,\frac{\omega}{2}}\right)\right| \\ &= \left|2\,\mathrm{j}\,\sin(\frac{\omega}{2})\right| \end{aligned}$$



(c) Let the centered-difference discrete operator Δ be described by $\forall k \in \mathbb{Z} : \Delta\{f\}[k] = \frac{1}{2}\,(f[k+1] - f[k-1])$. Determine the modulus of the frequency response $\hat{g}$ and plot it in the main frequency domain. Always label and graduate all axes.

Solution:

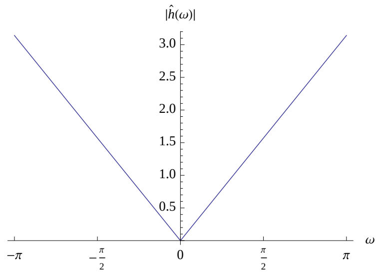
$$\begin{aligned} |\hat{g}(\omega)| &= \left|G(\mathrm{e}^{\mathrm{j}\,\omega})\right| \\ &= \left|\frac{1}{2}\left(\mathrm{e}^{\mathrm{j}\,\omega} - \mathrm{e}^{-\mathrm{j}\,\omega}\right)\right| \\ &= \left|\mathrm{j}\,\sin(\omega)\right| \end{aligned}$$



(d) Determine the frequency response $\hat{h}$ of the system $\mathcal{D}$ in the continuous domain and plot it for $\omega \in [-\pi, \pi]$. Always label and graduate all axes.

Solution:

$$\begin{aligned}\left|\hat{h}(\omega)\right| &= |\mathcal{F}\{\mathcal{D}\{\delta\}\}(\omega)| \\ &= |\mathrm{j}\,\omega|\end{aligned}$$



(e) There are two ways to compute the modulus of the error: the modulus of the difference, or the absolute value of the difference of the moduli. Which of the two is the most representative measure of the committed error? Motivate your answer in words.

Solution: The modulus of the difference is the most representative because it takes the phase into account.

(f) Report in a table the two error quantities (the modulus of the difference and the absolute value of the difference of the moduli) when approximating the continuous differentiation by finite differences, at the origin $\omega = 0$, at the medium frequencies $\omega = -\frac{\pi}{2}$ and $\omega = \frac{\pi}{2}$, and at the high frequencies $\omega = -\pi$ and $\omega = \pi$.

Solution:

	$\omega = -\pi$	$\omega = -\frac{\pi}{2}$	$\omega = 0$	$\omega = \frac{\pi}{2}$	$\omega = \pi$
$\lvert\hat{h}(\omega) - \hat{g}_-(\omega)\rvert$	$\sqrt{\pi^2+4}$	$\frac{1}{2}\sqrt{\pi^2 - 4\,\pi + 8}$	0	$\frac{1}{2}\sqrt{\pi^2 - 4\,\pi + 8}$	$\sqrt{\pi^2+4}$
$\lvert\hat{h}(\omega) - \hat{g}_+(\omega)\rvert$	$\sqrt{\pi^2+4}$	$\frac{1}{2}\sqrt{\pi^2 - 4\,\pi + 8}$	0	$\frac{1}{2}\sqrt{\pi^2 - 4\,\pi + 8}$	$\sqrt{\pi^2+4}$
$\lvert\hat{h}(\omega) - \hat{g}(\omega)\rvert$	π	$\frac{1}{2}\,(\pi - 2)$	0	$\frac{1}{2}\,(\pi - 2)$	π
$\lvert\lvert\hat{h}(\omega)\rvert - \lvert\hat{g}_-(\omega)\rvert\rvert$	$\pi - 2$	$\frac{1}{2}\,(\pi - \sqrt{8})$	0	$\frac{1}{2}\,(\pi - \sqrt{8})$	$\pi - 2$
$\lvert\lvert\hat{h}(\omega)\rvert - \lvert\hat{g}_+(\omega)\rvert\rvert$	$\pi - 2$	$\frac{1}{2}\,(\pi - \sqrt{8})$	0	$\frac{1}{2}\,(\pi - \sqrt{8})$	$\pi - 2$
$\lvert\lvert\hat{h}(\omega)\rvert - \lvert\hat{g}(\omega)\rvert\rvert$	π	$\frac{1}{2}\,(\pi - 2)$	0	$\frac{1}{2}\,(\pi - 2)$	π

(g) [advanced (optional)] Express the second-order Taylor expansions of $\hat{g}_-$, $\hat{g}_+$, and $\hat{g}$ around $\omega = 0$, ignoring cubic and higher-order terms. Also, do the same around $\omega = \pi$. (Hint: use Euler's formula)

Solution:

$$\begin{aligned}
\hat{g}_-(\omega) &\approx \left(1 - \mathrm{e}^{-\mathrm{j}\,\Omega}\right)\big|_{\Omega=0} + \left(-(-\mathrm{j})\;\mathrm{e}^{-\mathrm{j}\,\Omega}\right)\big|_{\Omega=0}\,\omega + \frac{1}{2}\,\left(-(-\mathrm{j})^2\;\mathrm{e}^{-\mathrm{j}\,\Omega}\right)\Big|_{\Omega=0}\,\omega^2 \\
&= \mathrm{j}\,\omega + \frac{1}{2}\,\omega^2 \\
\hat{g}_+(\omega) &\approx \left(\mathrm{e}^{\mathrm{j}\,\Omega} - 1\right)\big|_{\Omega=0} + \left(\mathrm{j}\,\mathrm{e}^{\mathrm{j}\,\Omega}\right)\big|_{\Omega=0}\,\omega + \frac{1}{2}\,\left(\mathrm{j}^2\,\mathrm{e}^{\mathrm{j}\,\Omega}\right)\big|_{\Omega=0}\,\omega^2 \\
&= \mathrm{j}\,\omega - \frac{1}{2}\,\omega^2 \\
\hat{g}(\omega) &\approx (\mathrm{j}\,\sin(\Omega))|_{\Omega=0} + (\mathrm{j}\,\cos(\Omega))|_{\Omega=0}\,\omega + \frac{1}{2}\,(\mathrm{j}\,(-\sin(\Omega)))|_{\Omega=0}\,\omega^2 \\
&= \mathrm{j}\,\omega
\end{aligned}$$

$$
\begin{aligned}
\hat{g}_-(\omega) &\approx \left(1-\mathrm{e}^{-\mathrm{j}\,\Omega}\right)\Big|_{\Omega=\pi} + \left(-(-\mathrm{j})\ \mathrm{e}^{-\mathrm{j}\,\Omega}\right)\Big|_{\Omega=\pi}\ (\omega-\pi) + \frac{1}{2}\ \left(-(-\mathrm{j})^2\ \mathrm{e}^{-\mathrm{j}\,\Omega}\right)\Big|_{\Omega=\pi}\ (\omega-\pi)^2 \\
&= 2-\mathrm{j}\ (\omega-\pi) - \frac{1}{2}\ (\omega-\pi)^2 \\
\hat{g}_+(\omega) &\approx \left(\mathrm{e}^{\mathrm{j}\,\Omega}-1\right)\Big|_{\Omega=\pi} + \left(\mathrm{j}\,\mathrm{e}^{\mathrm{j}\,\Omega}\right)\Big|_{\Omega=\pi}\ (\omega-\pi) + \frac{1}{2}\ \left(\mathrm{j}^2\,\mathrm{e}^{\mathrm{j}\,\Omega}\right)\Big|_{\Omega=\pi}\ (\omega-\pi)^2 \\
&= -2-\mathrm{j}\ (\omega-\pi) + \frac{1}{2}\ (\omega-\pi)^2 \\
\hat{g}(\omega) &\approx (\mathrm{j}\ \sin(\Omega))\big|_{\Omega=\pi} + (\mathrm{j}\ \cos(\Omega))\big|_{\Omega=\pi}\ (\omega-\pi) + \frac{1}{2}\ (\mathrm{j}\ (-\sin(\Omega)))\big|_{\Omega=\pi}\ (\omega-\pi)^2 \\
&= -\mathrm{j}\ (\omega-\pi)
\end{aligned}
$$

(h) [advanced (optional)] For natural images, the energy is concentrated towards the origin of the Fourier domain. Which of the three discrete systems $\{\Delta_-, \Delta_+, \Delta\}$ is the best approximation of the continuous derivative $\mathcal{D}$ in this image-processing context? Motivate your answer based on your previous plots, the table in part (f) and the Taylor expansion in part (g).

Solution: For imaging, the Taylor considerations that apply are those that correspond to $\omega = 0$, because the energy of images is concentrated towards the origin of the Fourier domain, and it is there that it is desirable to enforce the accuracy of the discrete approximations of continuously defined filters. Therefore, g approximates $\mathcal{D}$ better than g_- or g_+. It follows that differentiation is better approximated by a centered finite difference than by a backward or forward difference.