Image Processing 1, Exercise 7

1 Nonlinear Filtering

[basic] The median filter is commonly used in practice in spite of being nonlinear. Here we consider whether there might be a separable implementation of the median filter.

Provide an example of data on which a 3×3 median filter is not equivalent to a 1×3 median filter followed by a 3×1 median filter.

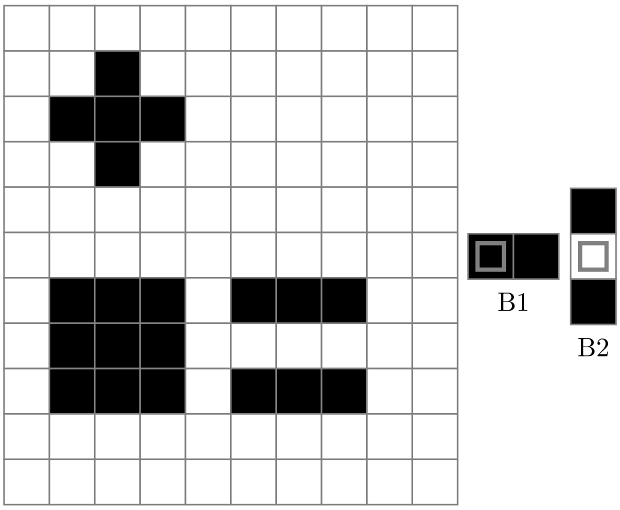
Solution:

Let a (3×3) image be $f = \begin{bmatrix} 1 & 1 & 1 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. On one hand, applying horizontal (1×3) median filters to f results in the (3×1) image $\begin{bmatrix} 1 \\ \boxed{0} \\ 0 \end{bmatrix}$; then, applying a vertical (3×1) median filter results in the (1×1) image $\begin{bmatrix} \boxed{0} \end{bmatrix}$. On the other hand, applying the 2D (3×3) median filters to f results in the (1×1) image $\begin{bmatrix} \boxed{1} \end{bmatrix} \neq \begin{bmatrix} \boxed{0} \end{bmatrix}$.

2 Binary Morphological Operators

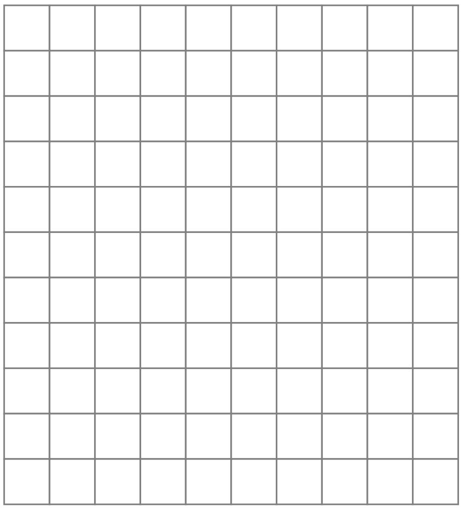
[basic] We apply some binary morphological operators in order to understand their mathematical definitions and how they affect binary images.

We use the convention:  1 and 0

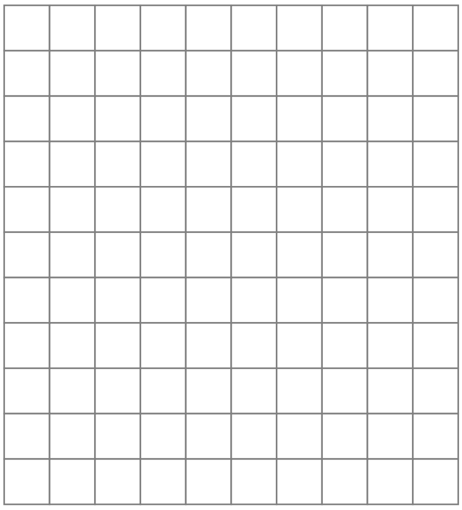


Image

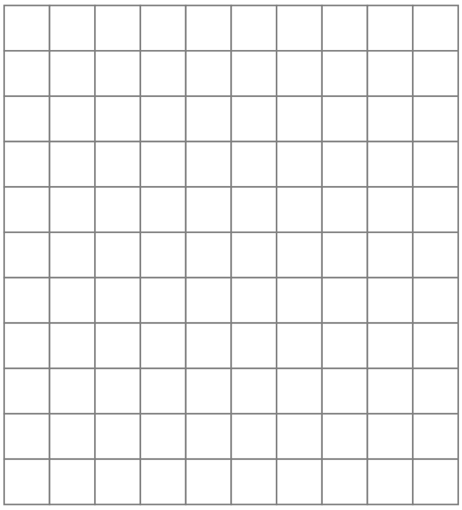
(a) Draw the erosion and the dilation of this image using the two structuring element B1 and B2



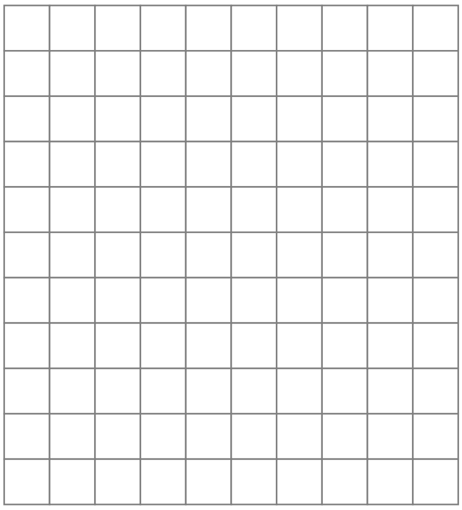
Erosion by B1



Dilation by B1

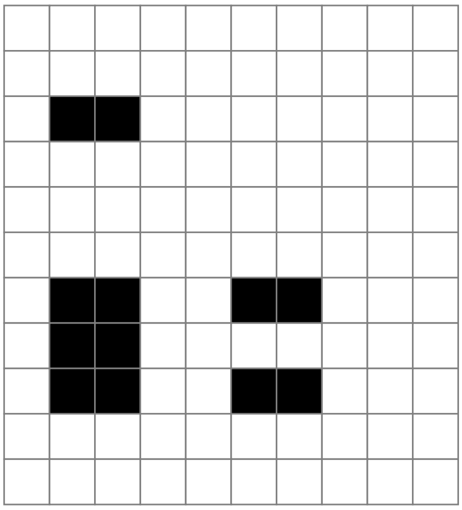


Erosion by B2

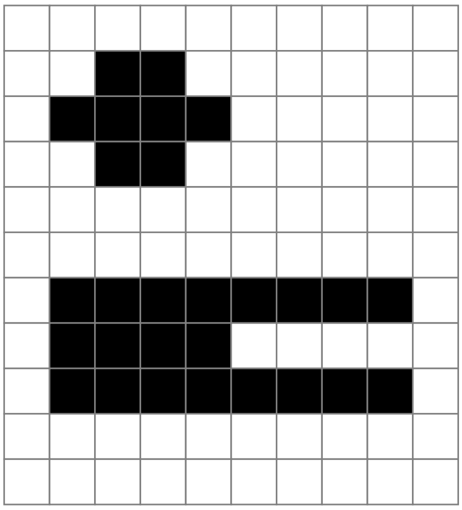


Dilation by B2

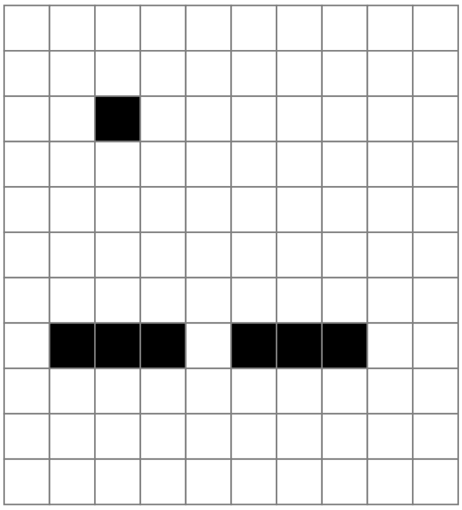
Solution:



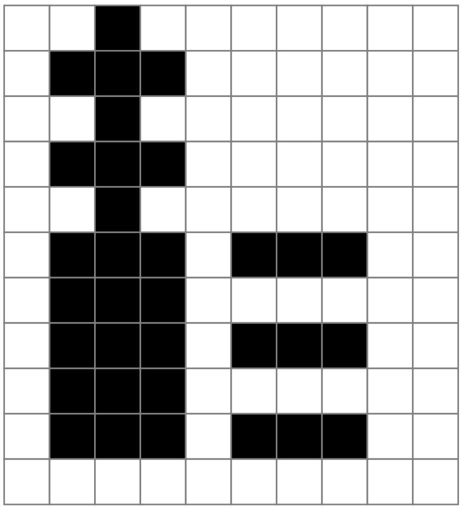
Erosion by B1



Dilation by B1

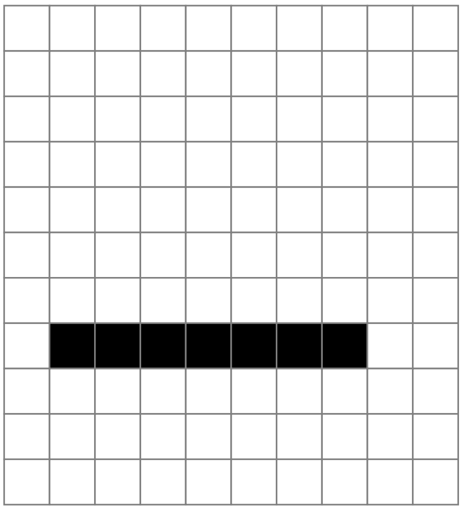


Erosion by B2



Dilation by B2

(b) [intermediate] Find a sequence of operations that results in this image.



Image

Solution: In this part, the cross is removed, while the other structures are modified. Removing the cross means that some erosion operators are necessary, while avoiding to completely erase all the structures. To fill both tasks, dilation operator(s) before eroding can be handful (to connect several structure for instance).

Then, the sequence can be
$f \oplus B1 \ominus B2 \ominus B1 \ominus B1 \oplus B1$
Or
$f \ominus B2 \ominus B1 \oplus B1 \oplus B1 \ominus B1$.

3 Grayscale Morphological Operators

[basic] We apply some morphological operators in order to understand their mathematical definitions and how they affect grayscale images.

Let $a = \begin{bmatrix} \dots & \times & 0 & 3 & 4 & \boxed{-1} & 4 & 6 & \times & \dots \end{bmatrix}$ be some $\times$-padded sequence, where the presence of a symbol $\times$ indicates the absence of valid data at the corresponding location in the sequence. Apply to a the grayscale morphological operations indicated in the table, with the $\times$-padded structuring elements $b_1 = \begin{bmatrix} 0 & \boxed{0} & 0 \end{bmatrix}$, $b_2 = \begin{bmatrix} 0 & \boxed{\times} & 0 \end{bmatrix}$, and $b_3 = \begin{bmatrix} \boxed{0} & 1 \end{bmatrix}$. For structuring elements, the presence of a $\times$ symbol indicates that the contribution of the sample at that location should be ignored.

k	-4	-3	-2	-1	0	1	2	3
a	$\times$	0	3	4	$\boxed{-1}$	4	6	$\times$
Dilation of a by b_1: $a \oplus b_1$								
Closing of a by b_1: $a \bullet b_1 = (a \oplus b_1) \ominus b_1$								
Dark top-hat of a by b_1: $(a \bullet b_1) - a$								
Erosion of a by b_2: $a \ominus b_2$								
Opening of a by b_2: $a \circ b_2 = (a \ominus b_2) \oplus b_2$								
Bright top-hat of a by b_2: $a - (a \circ b_2)$								
Dilation of a by b_3: $a \oplus b_3$								
Erosion of a by b_3: $a \ominus b_3$								
Gradient of a by b_3: $(a \oplus b_3) - (a \ominus b_3)$								

Solution:

k	-4	-3	-2	-1	0	1	2	3
a	$\times$	0	3	4	$\boxed{-1}$	4	6	$\times$
Dilation of a by b_1: $a \oplus b_1$	0	3	4	4	$\boxed{4}$	6	6	6
Closing of a by b_1: $a \bullet b_1 = (a \oplus b_1) \ominus b_1$	0	0	3	4	$\boxed{4}$	4	6	6
Dark top-hat of a by b_1: $(a \bullet b_1) - a$	$\times$	0	0	0	$\boxed{5}$	0	0	$\times$
Erosion of a by b_2: $a \ominus b_2$	0	3	0	-1	$\boxed{4}$	-1	4	6
Opening of a by b_2: $a \circ b_2 = (a \ominus b_2) \oplus b_2$	3	0	3	4	$\boxed{-1}$	4	6	4
Bright top-hat of a by b_2: $a - (a \circ b_2)$	$\times$	0	0	0	$\boxed{0}$	0	0	$\times$
Dilation of a by b_3: $a \oplus b_3$	$\times$	0	3	4	$\boxed{5}$	4	6	7
Erosion of a by b_3: $a \ominus b_3$	-1	0	3	-2	$\boxed{-1}$	4	6	$\times$
Gradient of a by b_3: $(a \oplus b_3) - (a \ominus b_3)$	$\times$	0	0	6	$\boxed{6}$	0	0	$\times$

Explanation:

We provide details on how to perform dilation and erosion with $a \oplus b_3$ and $a \ominus b_3$ as an example.

(a) $a \oplus b_3$

Recall the definition of grayscale dilation:

$$(f \oplus b)[\boldsymbol{k}] = \max_{\boldsymbol{k}_0 \in \Omega_b} \{ f[\boldsymbol{k} - \boldsymbol{k}_0] + b[\boldsymbol{k}_0] | (\boldsymbol{k} - \boldsymbol{k}_0) \in \Omega_f \} .$$

First, we write down the supports Ω_{b_3} and Ω_a:

$$\Omega_{b_3} = \{0, 1\}, \quad \Omega_a = \{-4, -3, -2, -1, 0, 1, 2, 3\} .$$

Then, we evaluate $(a \oplus b_3)$ at each index $k \in \Omega_a$ by three steps:

- check $k_0 \in \Omega_{b_3}$ is satisfied
- check $(k - k_0) \in \Omega_a$ is satisfied
- evaluate $a[k - k_0] + b[k_0]$ then take the maximum of all of them

$k = -4$:

$$\left.\begin{array}{ll} k_0 = 0, (k - k_0 = -4 \in \Omega_a): & a[-4] + b_3[0] = \times + 0 = \times \end{array}\right\} \max = \times$$
$$k_0 = 1, (k - k_0 = -5) \notin \Omega_a$$

$k = -3$:

$$\left.\begin{array}{ll} k_0 = 0, (k - k_0 = -3 \in \Omega_a): & a[-3] + b_3[0] = 0 + 0 = 0 \\ k_0 = 1, (k - k_0 = -4) \in \Omega_a): & a[-4] + b_3[1] = \times + 1 = \times \end{array}\right\} \max = 0$$

$k = -2$:

$$\left.\begin{array}{ll} k_0 = 0, (k - k_0 = -2 \in \Omega_a): & a[-2] + b_3[0] = 3 + 0 = 3 \\ k_0 = 1, (k - k_0 = -3) \in \Omega_a): & a[-3] + b_3[1] = 0 + 1 = 1 \end{array}\right\} \max = 3$$

$k = -1$:

$$\left.\begin{array}{ll} k_0 = 0, (k - k_0 = -1 \in \Omega_a): & a[-1] + b_3[0] = 4 + 0 = 4 \\ k_0 = 1, (k - k_0 = -2) \in \Omega_a): & a[-2] + b_3[1] = 3 + 1 = 4 \end{array}\right\} \max = 4$$

$k = 0$:

$$\left.\begin{array}{ll} k_0 = 0, (k - k_0 = 0 \in \Omega_a): & a[0] + b_3[0] = -1 + 0 = -1 \\ k_0 = 1, (k - k_0 = -1) \in \Omega_a): & a[-1] + b_3[1] = 4 + 1 = 5 \end{array}\right\} \max = 5$$

$k = 1$:

$$\left.\begin{array}{ll} k_0 = 0, (k - k_0 = 1 \in \Omega_a): & a[1] + b_3[0] = 4 + 0 = 4 \\ k_0 = 1, (k - k_0 = 0) \in \Omega_a): & a[0] + b_3[1] = -1 + 1 = 0 \end{array}\right\} \max = 4$$

$k = 2$:

$$\left.\begin{array}{ll} k_0 = 0, (k - k_0 = 2 \in \Omega_a): & a[2] + b_3[0] = 6 + 0 = 6 \\ k_0 = 1, (k - k_0 = 1) \in \Omega_a): & a[1] + b_3[1] = 4 + 1 = 5 \end{array}\right\} \max = 6$$

$k = 3$:

$$\left.\begin{array}{ll} k_0 = 0, (k - k_0 = 3 \in \Omega_a): & a[3] + b_3[0] = \times + 0 = \times \\ k_0 = 1, (k - k_0 = 2) \in \Omega_a): & a[2] + b_3[1] = 6 + 1 = 7 \end{array}\right\} \max = 7$$

(b) $a \ominus b_3$

Recall the definition of grayscale erosion:

$$(f \ominus b)[\boldsymbol{k}] = \min_{-\boldsymbol{k}_0 \in \Omega_b} \{f[\boldsymbol{k} - \boldsymbol{k}_0] - b[-\boldsymbol{k}_0] | (\boldsymbol{k} - \boldsymbol{k}_0) \in \Omega_f\}.$$

First, we write down the supports Ω_{b_3} and Ω_a:

$$\Omega_{b_3} = \{0, 1\}, \quad \Omega_a = \{-4, -3, -2, -1, 0, 1, 2, 3\}.$$

Then, we evaluate $(a \ominus b_3)$ at each index $k \in \Omega_a$ by three steps:

- check $-k_0 \in \Omega_{b_3}$ is satisfied
- check $(k - k_0) \in \Omega_a$ is satisfied
- evaluate $a[k - k_0] - b[-k_0]$ then take the minimum of all of them

$k = -4$:

$$\left.\begin{array}{ll} -k_0 = 0, k_0 = 0, (k - k_0 = -4 \in \Omega_a): & a[-4] - b_3[0] = \times - 0 = \times \\ -k_0 = 1, k_0 = -1, (k - k_0 = -3 \in \Omega_a): & a[-3] - b_3[1] = 0 - 1 = -1 \end{array}\right\} \min = -1$$

$k = -3$:

$$\left.\begin{array}{ll} -k_0 = 0, k_0 = 0, (k - k_0 = -3 \in \Omega_a): & a[-3] - b_3[0] = 0 - 0 = 0 \\ -k_0 = 1, k_0 = -1, (k - k_0 = -2 \in \Omega_a): & a[-2] - b_3[1] = 3 - 1 = 2 \end{array}\right\} \min = 0$$

$k = -2$:

$$\left.\begin{array}{ll} -k_0 = 0, k_0 = 0, (k - k_0 = -2 \in \Omega_a): & a[-2] - b_3[0] = 3 - 0 = 3 \\ -k_0 = 1, k_0 = -1, (k - k_0 = -1 \in \Omega_a): & a[-1] - b_3[1] = 4 - 1 = 3 \end{array}\right\} \min = 3$$

$k=-1$:

$$\left.\begin{array}{ll} -k_0=0, k_0=0, (k-k_0=-1\in\Omega_a): & a[-1]-b_3[0]=4-0=4 \\ -k_0=1, k_0=-1, (k-k_0=0\in\Omega_a): & a[0]-b_3[1]=-1-1=-2 \end{array}\right\}\min=-2$$

$k=0$:

$$\left.\begin{array}{ll} -k_0=0, k_0=0, (k-k_0=0\in\Omega_a): & a[0]-b_3[0]=-1-0=-1 \\ -k_0=1, k_0=-1, (k-k_0=1\in\Omega_a): & a[1]-b_3[1]=4-1=3 \end{array}\right\}\min=-1$$

$k=1$:

$$\left.\begin{array}{ll} -k_0=0, k_0=0, (k-k_0=1\in\Omega_a): & a[1]-b_3[0]=4-0=4 \\ -k_0=1, k_0=-1, (k-k_0=2\in\Omega_a): & a[2]-b_3[1]=6-1=5 \end{array}\right\}\min=4$$

$k=2$:

$$\left.\begin{array}{ll} -k_0=0, k_0=0, (k-k_0=2\in\Omega_a): & a[2]-b_3[0]=6-0=6 \\ -k_0=1, k_0=-1, (k-k_0=3\in\Omega_a): & a[3]-b_3[1]=\times-1=\times \end{array}\right\}\min=6$$

$k=3$:

$$\begin{array}{ll} -k_0=0, k_0=0, (k-k_0=3\in\Omega_a): & a[3]-b_3[0]=\times-0=\times\ \}\min=\times \\ -k_0=1, k_0=-1, (k-k_0=4\notin\Omega_a) & \end{array}$$

Finally, we show how to handle $\times$-padded structuring elements with $a\ominus b_2$ at index -4 as an example.

$$\Omega_{b_2}=\{-1,0,1\}\,.$$

$k=-4$:

$$\begin{array}{ll} -k_0=-1, k_0=1, (k-k_0=-5\notin\Omega_a) & \\ -k_0=0, k_0=0, (k-k_0=-4\in\Omega_a): & a[-4]-b_2[0]\Rightarrow \text{ignored} \\ -k_0=1, k_0=-1, (k-k_0=-3\in\Omega_a): & a[-3]-b_2[-1]=0-0=0\ \}\min=0 \end{array}$$